Homework 5

Algebraic Topology

D. Projection: \( P(x, y, z) = (x, y) \) and \( P^{-1}(x, y) = (x, y, \sqrt{1 - x^2 - y^2}) \).

Affine: \( A(x, y) = R(x - 1/2, y - 1/2) \) and \( A^{-1}(x, y) = (x/R + 1/2, y/R + 1/2) \) for \( R \gg 0 \). If \( r = f(\theta) \) describes the boundary in polar coordinates, scale by \( S(r, \theta) = (r/f(\theta), \theta) \) with continuous inverse \( S^{-1}(r, \theta) = (rf(\theta), \theta) \).

§55 1. Let \( r : B^2 \to A \) be a retract of \( B^2 \) onto \( A \) and \( f : A \to A \) continuous. If \( j : A \hookrightarrow B^2 \) is the inclusion, then the composition \( j \circ f \circ r : B^2 \to B^2 \) is continuous and has a fixed point \( x \), so that \( j(f(r(x))) = x \). Letting \( a = f(r(x)) \in A \) we have \( x = j(a) = a \in A \) and so \( a = f(r(a)) = f(a) \).

2. Since \( h : S^1 \to S^1 \) is nulhomotopic, it extends to a map \( \tilde{h} : B^2 \to S^1 \subset B^2 \) by Lemma 55.3. Compose with inclusion \( j : S^1 \hookrightarrow B^2 \) to get \( j \circ \tilde{h} : B^2 \to B^2 \), hence \( j(\tilde{h}(b)) = b \) for some \( b \in B^2 \) by Brouwer’s theorem. Set \( s = \tilde{h}(b) \in S^1 \), so \( s = b \) and so \( h(s) = s \) since \( \tilde{h} \) extends \( h \).

Let \( \alpha(t) = e^{2\pi it} \) be the generator for \( \pi^1(S^1, 1) \) and \( \beta = h \circ \alpha \in \pi_1(S_1, b_0) \), where \( b_0 = h(\alpha(0)) \). Let \( p : \mathbb{R} \to S^1 \) be the covering \( p(x) = e^{2\pi ix} \) and choose \( e_0 \in \mathbb{R} \) with \( p(e_0) = b_0 \). Since \( p^{-1}(-1) = 1/2 + \mathbb{Z} \subset \mathbb{R} \), there exists \( n \in \mathbb{Z} \) with \( n - 1/2 < e_0 < n + 1/2 \). \( \beta : I \to \mathbb{R} \) be the unique lift of the path \( \beta \) starting at \( e_0 \). Since \( h \) is nulhomotopic, \( \beta(1) = e_0 \). On the other hand, the unique lift \( \tilde{\gamma} \) of \( \gamma = -\alpha(t) \) starting at \( n - 1/2 \) is given by \( \tilde{\gamma}(t) = n - 1/2 + t \), so that \( \tilde{\gamma}(0) = n - 1/2 \) and \( \tilde{\gamma}(1) = n + 1/2 \). Since \( \tilde{\gamma}(t) - \tilde{\beta}(t) \) changes sign, there is \( 0 < T < 1 \) with \( \tilde{\gamma}(T) = \tilde{\beta}(T) \). Composing with \( p \) gives \(-\alpha(T) = \gamma(T) = \beta(T) = h(\alpha(T)) \) so that \( h(x) = -x \) for \( x = \alpha(T)^1 \).

3. Let \( T(x) = A(x)/\|A(x)\| \) for \( x \) in the first octant satisfying \( \|x\| = 1 \). Since \( A \) is nonsingular, \( A(x) \neq 0 \) so that \( T : B \to B \) is well-defined and continuous, where \( B \) is the intersection of the unit sphere and the first octant. Since \( B \) is homeomorphic to \( B^2 \) by Problem D above, \( T \) has a fixed point \( v \), i.e. \( A(v) = v/\|A(v)\| \) so that \( \|A(v)\| \) is a positive eigenvalue.

§58 1. Let \( H : X \times I \to X \) and \( G : A \times I \to A \) be the deformation retracts from \( X \) onto \( A \) and from \( A \) onto \( B \). Take \( F : X \times I \to X \) by \( F(x, t) = H(x, 2t) \) for \( 0 \leq t \leq 1/2 \) and \( F(x, t) = G(H(x, 1), 2t - 1) \) for \( 1/2 \leq t \leq 1 \). Then (a) \( F(x, 0) = x \) for \( x \in X \), (b) \( F \) retracts \( X \) onto \( B \) and \( F(x, 1) = G(H(x, 1), 1) = x \) for \( x \in B \) and (c) \( H(x, 1) = G(H(x, 1), 0) \) when \( t = 1/2 \), since \( H(x, 1) \in A \) and \( G(x, 1) = x \) for \( x \in A \), so \( F \) is continuous.

\(^1\)There is probably a better way to see that \( h(x) = -x \) for some \( x \) using vector fields (see proof of theorems 55.5 and 55.6 in book for example).
2. (a) \( \mathbb{Z} \) (b) \( \infty^2 \) (c) \( \mathbb{Z} \) (d) 0 (e) \( \infty \) (f) 0 (g) 0 (h) 0 (i) \( \mathbb{Z} \) (j) \( \mathbb{Z} \) (k) \( \infty \) (l) 0.

4. Two pictures, I’ll show these in class.

7. Since \( j \circ f \simeq \operatorname{Id}_X \), \( j_* \circ f_* \) is surjective by Corollary 58.5, hence so is \( j_* \).

Thus the issue is whether \( j_* \) is injective or not. (a) If \( f \) is a retract so that \( f(a) = a \) for \( a \in A \), then \( f \circ j : A \to A \) is the identity, hence so is \( f_* \circ j_* \) and \( j_* \) is injective. (b) If \( H(A \times I) \subset A \), then \( H(a, 1) = a \) for \( a \in A \) because \( H(x, 1) = \operatorname{Id}_X \) and \( H(a, 0) = j(f(a)) = f(a) = f(j(a)) \) for \( a \in A \), hence \( H \) gives a homotopy \( f \circ j \simeq \operatorname{Id}_A \). This implies \( f_* \circ j_* \) 1-1 and \( j_* \) also. (c) Take \( A = S^1 \subset B^2 = X \) and \( f : X \to A \) with \( f(x) = a \in A \) for some constant \( a \in A \). Then \( j \circ f \) is homotopic to the identity because \( X \) is contractible.

\(^2\)The figure eight