Homework 3

2. For paths $\alpha, \beta$ and $\gamma = \alpha \ast \beta$, observe that

$$\gamma(s) = \begin{cases} 
\alpha(2s) & s \leq 1/2 \\
\beta(2s - 1) & s \geq 1/2 
\end{cases} \Rightarrow \overline{\gamma}(s) = \gamma(1 - s) = \begin{cases} 
\alpha(2 - 2s) & s \geq 1/2 \\
\beta(1 - 2s) & s \leq 1/2 
\end{cases}$$

Since $\overline{\beta}(2s) = \beta(1 - 2s)$ and $\overline{\alpha}(2s - 1) = \alpha(2 - 2s)$, we see that $\overline{\gamma} = \overline{\beta} \ast \overline{\alpha}$.

For a loop $f$ at $x_0$ we compute $\hat{\gamma}([f])$ to be

$$[\overline{\gamma} \ast f \ast \gamma] = [\overline{\beta} \ast \overline{\alpha}] \ast f \ast \alpha \ast \beta = [\hat{\beta}(\hat{\alpha}(f))] = [(\hat{\beta} \ast \hat{\alpha})(f)]$$

so that $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

3. Suppose that $\pi_1(X, x_0)$ is abelian. Then for paths $\alpha, \beta$ from $x_0$ to $x_1$ we have $[(\beta \ast \overline{\alpha}) \ast f] = [f \ast (\beta \ast \overline{\alpha})]$ and since $\overline{\beta} \ast \beta$ and $\overline{\alpha} \ast \alpha$ are homotopic to the identity we can multiply on the left by $\overline{\beta}$ and on the right by $\overline{\alpha}$ to obtain $[\overline{\alpha} \ast f \ast \alpha] = [\overline{\beta} \ast \beta \ast \alpha]$ so that $\hat{\alpha} = \hat{\beta}$.

Conversely let $f, g \in \pi_1(X, x_0)$. Taking $x_1 = x_0$, $\alpha = f$ and $\beta = e$ the identity, the equality $[\overline{f} \ast g \ast f] = [\overline{e} \ast g \ast e]$ says that $[f]^{-1} \ast [g] \ast [f] = [g]$. Multiplying on the left by $[f]$ gives $[g] \ast [f] = [f] \ast [g]$.

4. Let $i : A \hookrightarrow X$ be the inclusion. The composition $r \circ i$ yields the composite map $\pi_1(A, a_0) \xrightarrow{i_*} \pi_1(X, a_0) \xrightarrow{r_*} \pi(A, a_0)$ of groups, which is the identity since $i \circ r = \text{Id}_A$. $i_* \circ r_*$ is the identity on $\pi_1(A, a_0)$, hence surjective, so too is $r_*$.  

5. Let $i : A \hookrightarrow \mathbb{R}^n$ be the inclusion and $e : (\mathbb{R}^n, a_0) \rightarrow (Y, y_0)$ an extension of $h$ so that $h = e \circ i$. This gives $\pi_1(A, a_0) \xrightarrow{i_*} \pi_1(\mathbb{R}^n, a_0) \xrightarrow{e_*} \pi_1(Y, y_0)$ and $e_* \circ i_* = h_*$ by functoriality. Since $\pi_1(\mathbb{R}^n, a_0) = 0$, the image of the composite map is the identity, thus $h_*$ is the trivial homomorphism.

6. We need that $\hat{\beta} \circ (h_{x_0})_* = (h_{x_1})_* \circ \hat{\alpha}$. Unwinding the maps, we must show that $[(h \circ \alpha) \ast (h \circ f) \ast (h \circ \alpha)] = [h \circ (\overline{\alpha} \ast f \ast \alpha)]$ for any loop $f$ at $x_0$. But $(h \circ \alpha) = h \circ \overline{\alpha}$ and the order of grouping modulo homotopy doesn’t matter, so it suffices that $((h \circ \overline{\alpha}) \ast (h \circ f)) \ast (h \circ \alpha) = h \circ ((\overline{\alpha} \ast f) \ast \alpha)$ modulo homotopy, but both sides are equal to $(h \circ \overline{\alpha})(4s)$ for $0 \leq s \leq 1/4$, $(h \circ f)(4s - 1)$ for $1/4 \leq s \leq 1/2$ and $(h \circ \alpha)(2s - 1)$ for $1/2 \leq s \leq 1$.

A. If $f \in \text{Mor}(A, A)$ is another such, then $f = f \circ \text{Id}_A = \text{Id}_A$.

B. If $h$ is another two sided inverse, then $h = h \circ (f \circ g) = (h \circ f) \circ g = g$.  

1