April 3

Discrete Mathematics

Announcements:

• Quiz 12 is graded, it’s in your file in TCU Box "Discrete Spring 2020".
• Today is Quiz 13, a take home quiz on strong induction, due Monday.

Quiz 12

Quiz 12 asks about another summation formula, namely

\[ P_n : 3^1 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 3}{2} \quad \text{for } n \geq 1 \]

1. Checking the statement for \( n = 1 \) and \( n = 2 \) is easy:

For \( n = 1 \), the left side is 3 and the right side is \( \frac{3^2 - 3}{2} = \frac{6}{2} = 3 \checkmark \)

For \( n = 2 \) the left side is 3 + 3^2 = 12 and the right side is \( \frac{3^3 - 3}{2} = \frac{24}{2} = 12 \checkmark \).

2. We prove \( P_n \) for \( n \geq 1 \) by induction on \( n \). The induction base \( n = 1 \) we checked above, so we proceed immediately to the main point:

\text{Induction step: Assume } P_n \text{ is true, i.e. that } 3^1 + 3^2 + \cdots + 3^n = \frac{3^{n+1} - 3}{2}, \text{ we need to show that } P_{n+1} \text{ is true, namely that } 3^1 + 3^2 + \cdots + 3^n + 3^{n+1} = \frac{3^{n+2} - 3}{2}.

The left side of this last expression is

\[ 3^1 + 3^2 + \cdots + 3^n + 3^{n+1} = \frac{3^{n+1} - 3}{2} + 3^{n+1} \]

with equality due to the formula for the underlined part from \( P_n \). We need to see that the last expression is \( \frac{3^{n+2} - 3}{2} \), but this is algebra:

\[ \frac{3^{n+1} - 3}{2} + 3^{n+1} = \frac{1 \cdot 3^{n+1} - 3 + 2 \cdot 3^{n+1}}{2} = \frac{3 \cdot 3^{n+1} - 3}{2} = \frac{3^{n+1} - 3}{2}. \]

The middle equality is distributing the \( 3^{n+1} \). This proves \( P_n \) for all \( n \geq 1 \) by induction. \( \square \)
The next two sections deal with functions, one of the most important notions in mathematics. If you’ve taken any calculus classes, you’ve spent plenty of hours studying functions, but they are also essential in computer science (for example one uses hash functions for a hash table when working on data structures). The inventors of calculus (Newton and Leibniz) thought about functions in terms of power series, but it wasn’t until the late 1800s that mathematicians came to a unified definition, which you’ve probably seen before:

**Definition 1.** Let $A, B$ be two sets. A function $f : A \to B$ is an assignment of each $a \in A$ to exactly one $b \in B$. The set $A$ is the **domain** of $f$ and $B$ is the **codomain** or **target** of $f$.

**Notation:** When $a$ is assigned to $b$, we write $f(a) = b$.

If you’ve taken calculus, you are certainly aware of two more words associated with functions:

**Definition 2.** The **range** of $f : A \to B$ is the set $\{ f(a) : a \in A \} \subseteq B$. The range is also called the **image** of $f$, written $\text{Im } f$. The **graph** of $f$ is the set $\{(a,b) \in A \times B : f(a) = b\}$.

It’s the graph of $f$ that we spend so much time analyzing in calculus. Here are a few calculus examples.

**Example 1.** Familiar examples might be (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and (b) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3 - x$. See the graphs below.

![Graphs of $f(x) = x^2$ and $g(x) = x^3 - x$.](image)
You can see that in (a) the range of $f$ is $[0, \infty) = \text{Im} f = \{x \in \mathbb{R} : x \geq 0\}$ as the set of $y$-coordinates in the graph and in (b) the range of $g$ is $\text{Im} g = \mathbb{R}$.

Example 1 does not look discrete! The graphs and real numbers are continuously varying quantities. In the more discrete setting one can visualize functions by arrows which explain the assignment.

**Example 2.** For $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$, Figure (a) indicates a function $f : A \to B$ but Figure (b) does not because $f(a)$ has two values.

**Example 3.** In the last example you saw a function described by a picture. It is also possible to define a function by a verbal description. For example, define

$$f : \mathbb{N} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

by the rule

$$f(n) = \text{the nth decimal digit of } \pi$$

Since the decimal expansion is given by $\pi = 3.14159\ldots$, we have the values $f(0) = 3, f(1) = 1, f(2) = 4, f(3) = 1, f(4) = 5, \ldots$.

Here are definitions that you will encounter in math and computer science:

**Definition 3.** Let $f : A \to B$ be a function.

1. $f$ is 1-1 or injective if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. 
2. $f$ is **onto** of surjective if $\text{Im } f = B$.

3. $f$ is a **one to one correspondence** or **bijective** if $f$ is 1-1 and onto.

**Example 4.** Let’s check these definitions on the previous three examples.

1. The function $f$ in Example 1 (a) is neither 1-1 ($f(1) = f(-1)$ but $1 \neq -1$) nor onto (since $\text{Im } f \neq \mathbb{R}$). The function $g$ in part (b) is not 1-1 but it is onto.

2. The function of Example 2 (a) is onto but not 1-1. Example 2 (b) doesn’t even represent a function.

3. The function of Example 3 is onto (all digits 0-9 appear in $\pi$), but not 1-1 ($f(1) = f(3) = 1$ but $1 \neq 3$).

In precalculus or calculus we often use inverse functions:

**Definition 4.** A function $g : B \rightarrow A$ is the **inverse** function to $f : A \rightarrow B$ if $f(g(b)) = b \ \forall b \in B$ and $g(f(a)) = a \ \forall a \in A$.

What is easy to show is that functions with inverses are bijective:

**Example 5.** Prove that if $f : A \rightarrow B$ has an inverse $g$, then $f$ is bijective.

Proof: Assume that $g$ is the inverse of $f$. We must show $f$ is 1-1 and onto.

$f$ is 1-1: $f(a_1) = f(a_2) \Rightarrow a_1 = g(f(a_1)) = g(f(a_2)) = a_2 \Rightarrow a_1 = a_2$, so $f$ is 1-1 by definition 3.

$f$ is onto: We must show that $\text{Im } f = B$. Let $b \in B$. Then $f(g(b)) = b$, so $\exists a \in A$ with $f(a) = b$, namely $a = g(b)$.

**Example 6.** Bijective functions in calculus often appear with inverses.

1. The function $f : (0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = \ln x$ is bijective. It’s inverse function is $g(x) = e^x$.

2. The function $f : [-\pi/2, \pi/2] \rightarrow [-1, 1]$ given by $f(x) = \sin x$ is bijective. It’s inverse function is $g(x) = \arcsin x$.

3. The function $f : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ by $f(x) = \tan x$ is bijective, it’s inverse is $g(x) = \arctan x$. 
