**Lie Groups**

**Example** The orthogonal group $O(2)$ is the set of $2 \times 2$ matrices $A$ such that $A^tA = I$. The special orthogonal group $SO(2)$ is the set of orthogonal matrices with determinant 1. Let’s examine this. First, let’s think of these matrices as a pair of column vectors. So if $A \in O(2)$, then

$$A = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

with $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ as the two column vectors. So the equation $A^tA = I$ is equivalent to

$$\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix} \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\begin{pmatrix} v_1^2 + v_2^2 & v_1w_1 + v_2w_2 \\ v_1w_1 + v_2w_2 & w_1^2 + w_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\begin{pmatrix} v \cdot v & v \cdot w \\ v \cdot w & w \cdot w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Thus, $O(2)$ is the set of $2 \times 2$ matrices where the column vectors form an orthonormal frame, an orthonormal basis of $\mathbb{R}^2$. Note that $\begin{pmatrix} w_2 \\ -w_1 \end{pmatrix}$ is the result after rotating the vector $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ clockwise 90 degrees, and the determinant of the matrix is

$$\det A = v_1w_2 - v_2w_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \cdot \begin{pmatrix} w_2 \\ -w_1 \end{pmatrix},$$

which is 1 if $(v, w)$ is an oriented frame (ie $w$ is the result of a counterclockwise 90 degree rotation of $v$) and is $-1$ if $(v, w)$ has the reverse orientation (ie $w$ is the result of a clockwise 90 degree rotation of $v$). Thus $SO(2)$ corresponds to the set of oriented orthonormal frames.

**Definition** A Lie group $G$ is a manifold and group for which the multiplication map $\mu : G \times G \to G$ is smooth.

**Remark** It follows that the inverse map $i : G \to G$ defined by $i(g) = g^{-1}$ is also smooth. Proof: implicit fcn theorem + diagram
\[ \mu^{-1}(e) \subseteq G \times G \xrightarrow{\mu} G \]
\[ \pi_1 \downarrow \]

\[ G \]

Example (\( \mathbb{R}^n, + \))

Example \( S^1 \) or \( T^n = S^1 \times \ldots \times S^1 \)

Example \( Gl(n, F) \subseteq F^n \), where \( F = \mathbb{R} \) or \( \mathbb{C} \)

Example \( E_3 = \text{isometries of } \mathbb{R}^3 \) (2 connected components) Let the orthogonal group \( O_3 < E_3 \) be the subgroup that fixes the origin, and let the special orthogonal group \( SO(3) = SO_3 < O_3 \) be the orientation-preserving elements of \( O_3 \).

Visualizing \( SO(3) \): Let \( u \) be a vector of length \( l \) in \( \mathbb{R}^3 \), corresponding to a rotation of angle \( l \) around the axis \( u \). Redundancy: if \( l = |u| = \pi \), \( u \) gives the same rotation as \( -u \), so \( SO(3) \) is the ball of radius \( \pi \) with antipodal points identified = \( \mathbb{R}P^3 \).

Matrix groups

**Theorem** If \( G \) is a Lie group and \( H < G \), then \( H \) is a Lie subgroup with the subspace topology if and only if \( H \) is closed.

Example Embed \( \mathbb{R} \) as an irrational slope on \( \mathbb{R}^2 / \mathbb{Z}^2 = T^2 \); then this is a subgroup but is not a Lie subgroup.

Note that

\[ E_3 \cong \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \subseteq Gl(4, \mathbb{R}) \right\} \text{ such that } A \in O_3 \]

\( b \) is the translation vector.

Classical Lie (sub)groups: \( Sl(n, F) \) (det=1), \( O(n) \) (\( gg^t = 1 \), orthogonal group), \( SO(n) \) (\( gg^t = 1 \), det=1, special orthogonal group), \( U(n) \) (\( gg^* = 1 \), unitary group), \( SU(n) \) (\( gg^* = 1 \), det=1, special unitary group), \( Sp(n) = \{ g \in Gl(n, \mathbb{H}) : gg^* = 1 \} \) (symplectic group).

Why study general Lie groups? Well, a standard group could be embedded in a funny way.

For example, \( \mathbb{R} \) can be embedded as \((e^t)\) as matrices, or as \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) or as

\[ \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix} \]. Also, some examples are not matrix groups. For example, consider the following quotient of the Heisenberg group \( N \): Let
Let $G = N / Z$

These groups are important in quantum mechanics. Also, consider the following transformations of $L^2(\mathbb{R})$:

$$T_a(f)(x) = f(x - a)$$
$$M_b(f)(x) = e^{2\pi i b x} f(x)$$
$$U_c(f)(x) = e^{2\pi i c x} f(x)$$

The group of operators of the form $T_a M_b U_c$ corresponds exactly to

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

quantum mechanics, $T_a$ corresponds to a unitary involution of momentum, and $M$ is the momentum, $U$ is phase.

Note that every Lie group is locally a matrix group.

Low dimensional, connected examples:

1. Dim 1: $\mathbb{R}$, $S^1$
2. Dim 2: only nonabelian example is the space of affine transformations $x \mapsto mx + b$ of $\mathbb{R}$.
3. Dim 3: $SO_3$, $SL_2(\mathbb{R})$, $E_2$, $N$ (only new ones up to local isomorphism: $G_1$ and $G_2$ are locally isomorphic if there exist open neighborhoods around the identities that are homeomorphic through multiplication-preserving homeo).

### Relationships between Lie groups

Observe that $U_2 = \{ g : gg^* = 1 \}$, $SU_2 = \{ g \in U_2 : \det g = 1 \}$.

For every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_2$, then $g^* = g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$.

so
\[ SU_2 = \{ g \in U_2 : \det g = 1 \} \]
\[ = \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} = S^3 \]
\[ = \left\{ \begin{pmatrix} t + ix & y + zi \\ -y + zi & t - ix \end{pmatrix} : (t, x, y, z) \in S^3 \right\} \]
\[ = \left\{ q = \mathbf{1} + x\hat{i} + y\hat{j} + z\hat{k} \in H : (t, x, y, z) \in S^3 \right\} = Sp(1), \]

where

\[ \hat{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \]
\[ \hat{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]
\[ \hat{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \]

satisfy the relations

\[ \hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -\mathbf{1} \]
\[ \hat{i}\hat{j} = -\hat{j}\hat{i} = \hat{k}; \quad \hat{j}\hat{k} = -\hat{k}\hat{j} = \hat{i}; \quad \hat{k}\hat{i} = -i\hat{k} = \hat{j}. \]