Speculations on Mirror Symmetry and the Arithmetic of the Quintic Threefold

with

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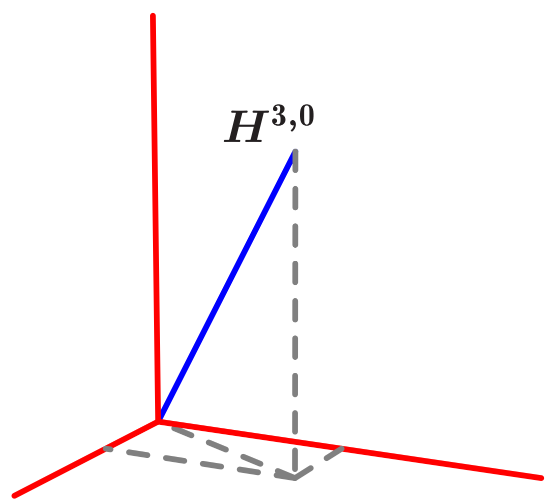
Calabi-Yau manifolds arise when we 'compactify' a $D = 10$ string theory down to $D = 4$. These manifolds have many remarkable properties owing to this relation with string theory and supersymmetry.

- **The number of $\mathbb{F}_p$-rational points may be calculated in terms of periods. Moreover this expression is computable in a practical sense.**
- **We are led to compute the corresponding ζ-function and to speculate on the possible role of 'quantum corrections' in relation to mirror symmetry and the conifold singularization.**

An important property of a Calabi–Yau manifold is that there exists a unique holomorphic three-form, Ω. One way to parametrise the complex structure is via the algebraic parameters as in the quintic threefold

$$P \;=\; x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \alpha\, x_1^2 x_2^3 + \beta\, x_4^3 x_3 x_5 + \ldots\;.$$

For the quintic there are $h^{21} = 101$ such parameters.



The (projective) coordinates for $H^{3,0}$ are the periods

$$\varpi_j(\psi) \;=\; \int_{\Upsilon^j} \Omega\;.$$

Consider, for definiteness, the one parameter family of quintics in $\mathbb{P}_4$

$$\mathcal{M} \;:\; P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1x_2x_3x_4x_5 \;.$$

$\mathcal{M}$ has $h^{11} = 1$ and $h^{21} = 101$. It also has a group $\Gamma \cong \mathbb{Z}_5^3$ of automorphisms:

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (\zeta^{n_1}x_1, \zeta^{n_2}x_2, \zeta^{n_3}x_3, \zeta^{n_4}x_4, \zeta^{n_5}x_5)$$

with $\zeta^5 = 1$ and $\sum n_i = 0$.

In this simple case there is a simple relation between $\mathcal{M}$ and its mirror

$$\mathcal{W} \;=\; \mathcal{M}/\Gamma \;.$$

$\mathcal{W}$ has $h^{11} = 101$ and $h^{21} = 1$. The 100 extra Kähler forms arise from the blow ups required to resolve the singularities associated with the fixed points of Γ.

Now, as we have said, the number of rational points is determined by the periods and there are $b^3 = 2h^{21} + 2$ of these. The Hodge number h^{21} counts the number of parameters on which the complex structure depends and, in simple cases, this corresponds to the number of ways of deforming the defining polynomial

$$P(x,c) \ = \ \sum_v c_v \, x^v \quad ; \quad x^v = x_1^{v_1} \, x_2^{v_2} \, x_3^{v_3} \, x_4^{v_4} \, x_5^{v_5} \ .$$

The directions in which $P(x,c)$ can be deformed correspond to the monomials x^v considered subject to the ideal $(\partial P/\partial x_i)$. A special role is played by fundamental monomial

$$Q \ = x_1 x_2 x_3 x_4 x_5$$

which is related by mirror symmetry to the Kähler form of the mirror.

Return now to our special one parameter family of polynomials

$$P(x, \psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5 \;.$$

$\mathcal{M}$ **has** $2h^{21}(\mathcal{M}) + 2 = 204 = 2 \times 100 + 4$ **periods while** $\mathcal{W}$ **has** $2h^{21}(\mathcal{W}) + 2 = 4$ **.**

$$\begin{array}{ccccccc} 1 & \longrightarrow & Q & \longrightarrow & Q^2 & \longrightarrow & Q^3 \\ & & x^v & \longrightarrow & Q\, x^v & & \end{array}$$

This leads to 1 fourth order differential operator $\mathcal{L}_0$ **and 100 second order operators** $\mathcal{L}_v$**.**

There are tenth order monomials that are not included in the above scheme and which require special attention. The generators of the ideal are

$$x_1^4 \simeq \psi\, x_2 x_3 x_4 x_5 \;\; \text{\& cyclic.}$$

Thus

$$x^{(4,3,2,1,0)} \;\simeq\; \psi\, x^{(0,4,3,2,1)} \;\simeq\; \cdots \;\simeq\; \psi^5\, x^{(4,3,2,1,0)} \;.$$

The Z-Function

We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x, \psi) = 0$. Our expressions lead us to decompose N_r into a sum of contributions $N_r = N_{r,0} + \sum_v N_{r,v}$.

The ζ-function is defined by the expression

$$\zeta(t) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r t^r}{r}\right)$$

Given the decomposition of N_r into a sum over monomials v we have

$$\zeta_{\mathcal{M}}(t, \psi) = \frac{R_0(t, \psi) \prod_v R_v(t, \psi)}{(1 - t)(1 - pt)(1 - p^2 t)(1 - p^3 t)}$$

$$\zeta_{\mathcal{W}}(t, \psi) = \frac{R_0(t, \psi)}{(1 - t)(1 - pt)^{101}(1 - p^2 t)^{101}(1 - p^3 t)} \, .$$

In all cases, apart from the conifold, R_0 is a quartic

$$R_0 = 1 + a_0\, t + b_0\, pt^2 + a_0\, p^3 t^3 + p^6 t^4 \, .$$

The Euler Curves

Classical analysis gives an expression for the hypergeometric functions in terms of Euler's integral which is of the form

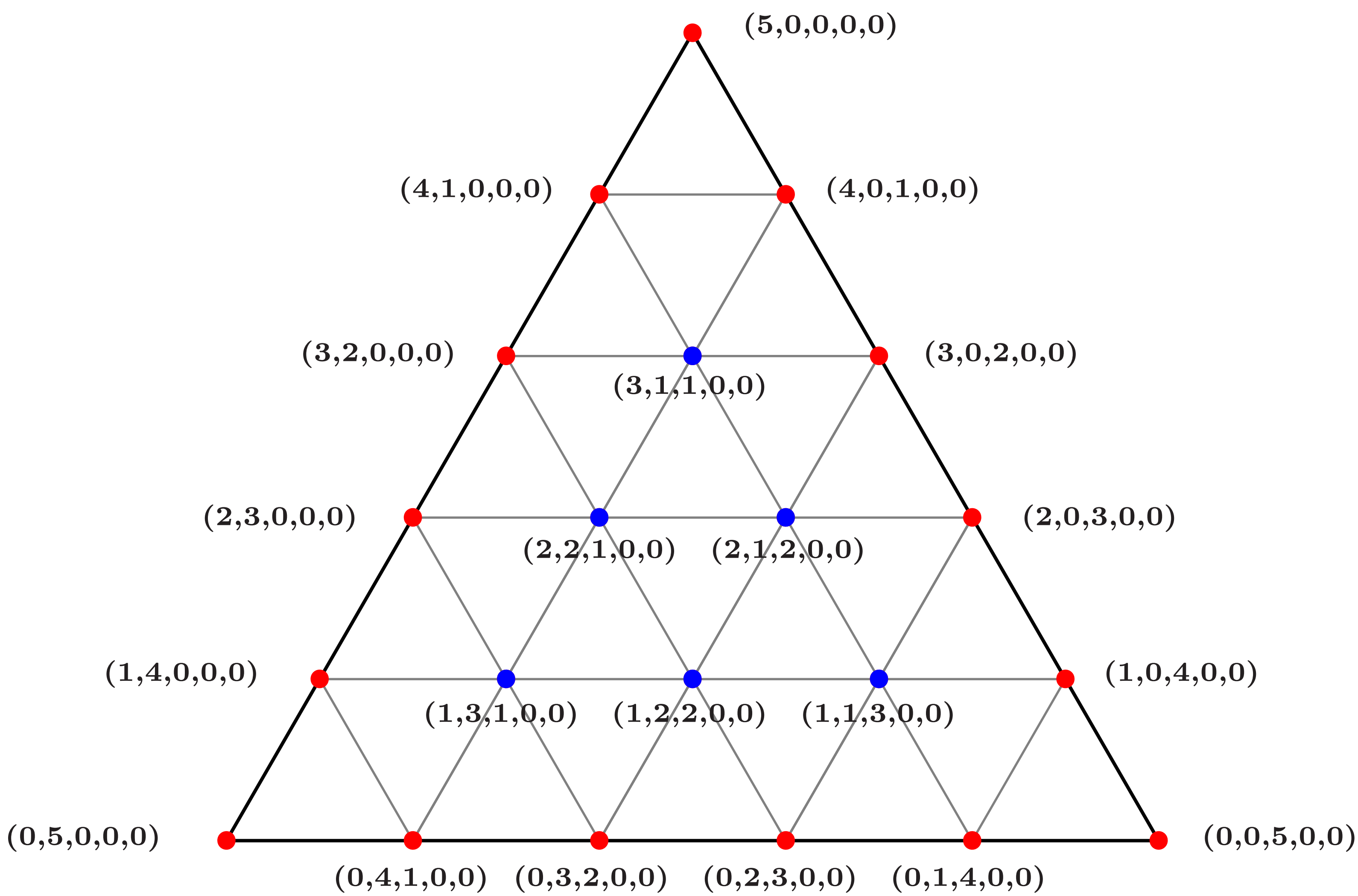
$$\int dx\, x^{-\alpha/5}(1-x)^{-\beta/5}(1-x/\psi^5)^{-(1-\beta/5)}\ .$$

If we think of Euler's integral as $\int \frac{dx}{y}$ then we are led to curves

$$\mathcal{E}_{\alpha\beta}(\psi):\quad y^5 = x^\alpha(1-x)^\beta(1-x/\psi^5)^{5-\beta}\ .$$

v	α	β
$(4,1,0,0,0)$	2	3
$(3,2,0,0,0)$	1	4
$(3,1,1,0,0)$	2	4
$(2,2,1,0,0)$	4	3

$$\mathcal{E}_{\alpha\beta} = \begin{cases} \mathcal{A} & \alpha+\beta=5 \\ \mathcal{B} & \alpha+\beta\neq 5 \text{ and } \alpha\neq\beta\ . \end{cases}$$


(5,0,0,0,0)
(4,1,0,0,0)
(4,0,1,0,0)
(3,2,0,0,0)
(3,0,2,0,0)
(3,1,1,0,0)
(2,3,0,0,0)
(2,0,3,0,0)
(2,2,1,0,0)
(2,1,2,0,0)
(1,4,0,0,0)
(1,0,4,0,0)
(1,3,1,0,0)
(1,2,2,0,0)
(1,1,3,0,0)
(0,5,0,0,0)
(0,0,5,0,0)
(0,4,1,0,0)
(0,3,2,0,0)
(0,2,3,0,0)
(0,1,4,0,0)

For the curve $\mathcal{A}$ there is a corresponding ζ-function

$$\zeta_{\mathcal{A}}(t) \;=\; \frac{R_{\mathcal{A}}(t)^2}{(1-t)(1-pt)}\;.$$

Now the existence of nontrivial fifth roots of unity is important for the mirror construction. Such roots of unity exist in $\mathbb{F}_{p^r}$ precisely when $5|p^r-1$. For given p let $\rho = 1$, 2 or 4 be the smallest r for which $5|p^r-1$.

The R_v pair up in the following way:

$$\begin{aligned} R_{(4,1,0,0,0)}(t)\, R_{(3,2,0,0,0)}(t) \;&=\; R_{\mathcal{A}}(p^\rho t^\rho)^{1/\rho} \\ R_{(3,1,1,0,0)}(t)\, R_{(2,2,1,0,0)}(t) \;&=\; R_{\mathcal{B}}(p^\rho t^\rho)^{1/\rho}\;. \end{aligned}$$

So the ζ-function for $\mathcal{M}$ takes the form

$$\boxed{\zeta_{\mathcal{M}}(t,\psi) \;=\; \frac{R_0(t,\psi)\, R_{\mathcal{A}}(p^\rho t^\rho,\psi)^{\frac{30}{\rho}}\, R_{\mathcal{B}}(p^\rho t^\rho,\psi)^{\frac{20}{\rho}}}{(1-t)(1-pt)(1-p^2t)(1-p^3t)}\;.}$$

The Z-Function and Mirror Symmetry

As defined the ζ-function does not respect mirror symmetry

$$\zeta(t) \;=\; \frac{\text{Numerator of deg. } 2h^{21}+2 \text{ depending on the cpx. structure of } \mathcal{M}}{\text{Denominator of deg. } 2h^{11}+2}\;.$$

Explicitly for the quintic we have

$$\zeta_{\mathcal{M}}(t,\psi) \;=\; \frac{R_0(t,\psi)\,R_{\mathcal{A}}(pt,\psi)^{30}\,R_{\mathcal{B}}(pt,\psi)^{20}}{(1-t)(1-pt)(1-p^2t)(1-p^3t)}$$

$$\zeta_{\mathcal{W}}(t,\psi) \;=\; \frac{R_0(t,\psi)}{(1-t)(1-pt)^{101}(1-p^2t)^{101}(1-p^3t)}$$

Clearly what we would like to see is

$$\zeta_{\mathcal{W}} \;=\; \frac{1}{\zeta_{\mathcal{M}}}$$

however we cannot have this relation and the classical definition in terms of the N_r with positive N_r.

The Conifold

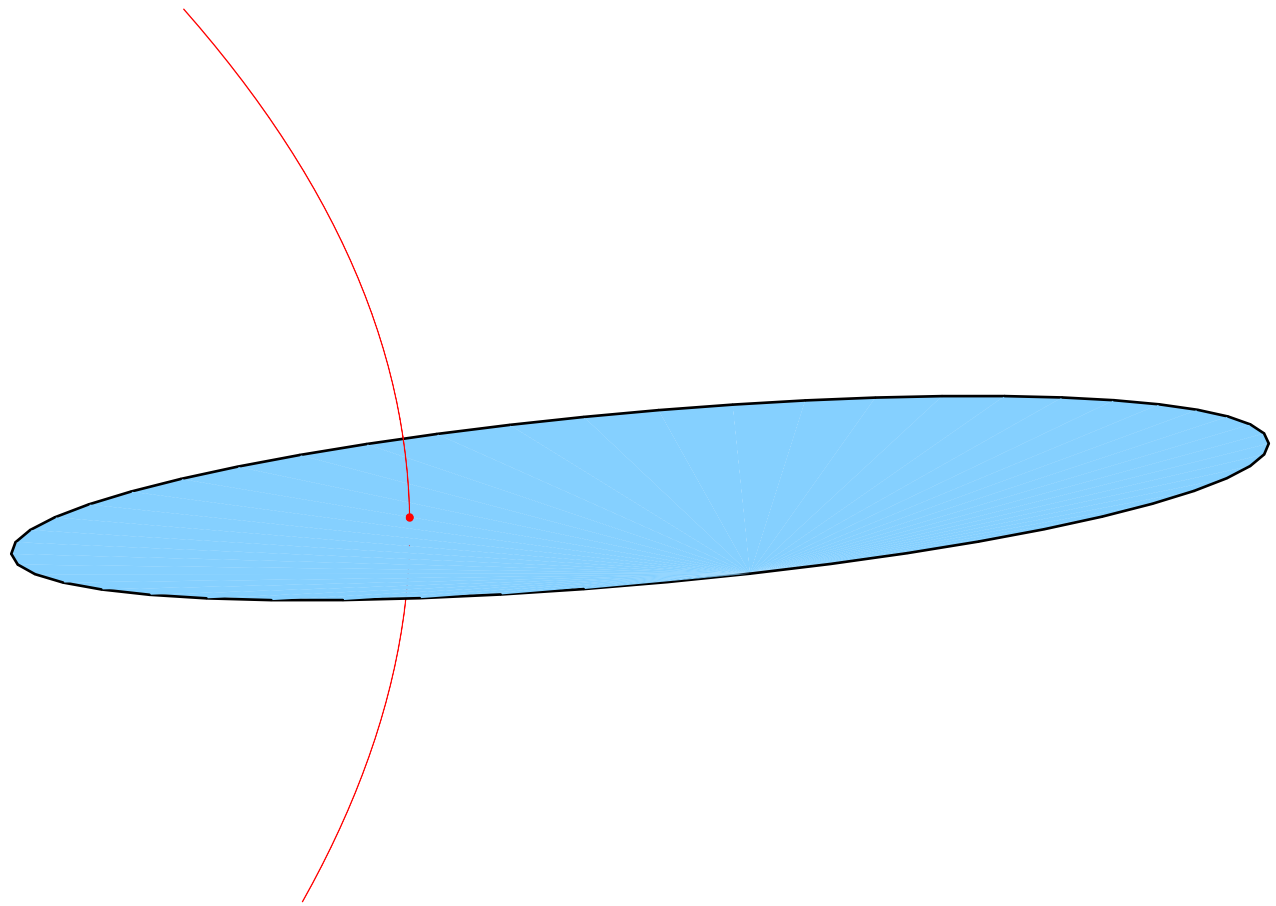
For the conifold $\psi^5 = 1$ the ζ-function seems to be especially simple

$$\zeta(t, \psi^5 = 1) \;=\; \frac{(1 - \epsilon\, pt)\,(1 - a_p\, t + p^3 t^2)\,(1 - p^\rho t^\rho)^{\frac{200}{\rho}}}{(1-t)(1-pt)(1-p^2 t)(1-p^3 t)\,(1 - p^{2\rho} t^\rho)^{\frac{24}{\rho}}}$$

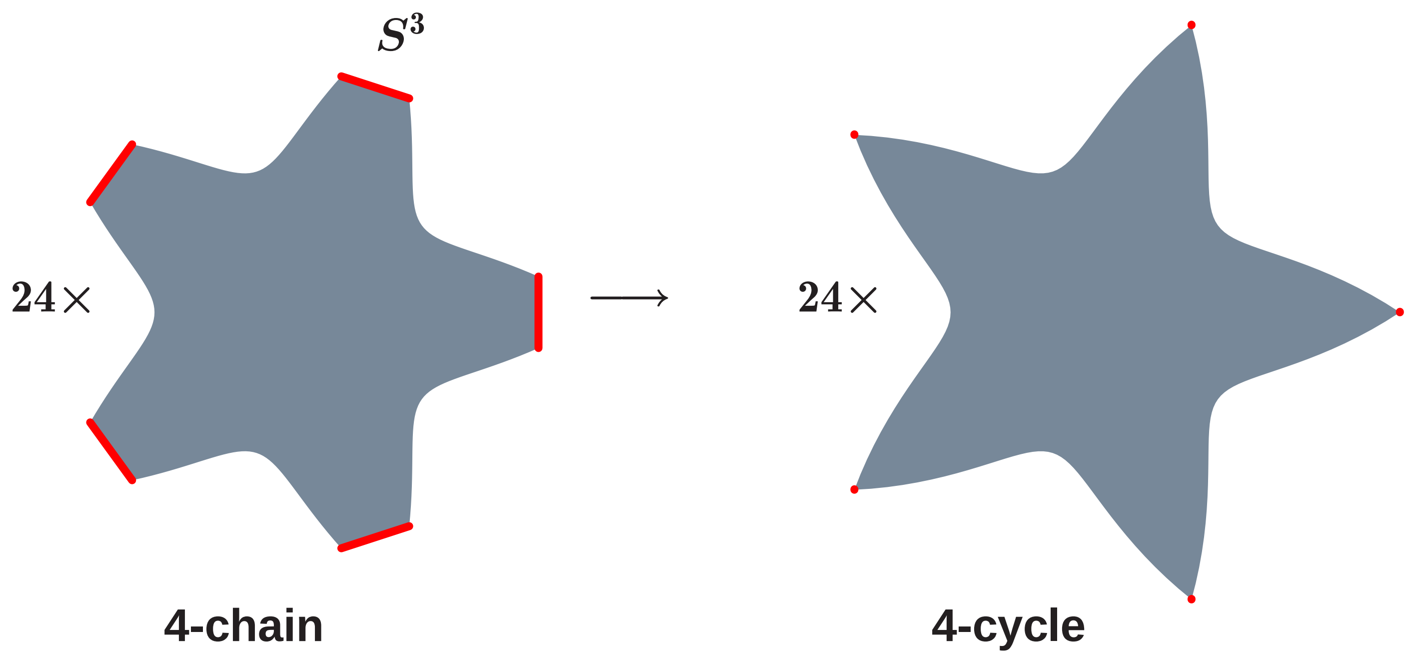
where $\epsilon = \left(\frac{5}{p}\right) = \pm 1$ and a_p is the p-th coefficient in the q-expansion of the eigenform, g, found by Schoen; it is the unique cusp form of weight 4 for the group $\Gamma_0(25)$.

$$g \;= \eta(q^5)^4 \left[\eta(q)^4 + 5\,\eta(q)^3\eta(q^{25}) + 20\,\eta(q)^2\eta(q^{25})^2 + 25\,\eta(q)\eta(q^{25})^3 + 25\,\eta(q^{25})^4\right]$$

$$\begin{aligned} &= q + q^2 + 7\,q^3 - 7\,q^4 + 7\,q^6 + 6\,q^7 - 15\,q^8 + 22\,q^9 - 43\,q^{11} - 49\,q^{12} - 28\,q^{13} \\ &\quad + 6\,q^{14} + 41\,q^{16} + 91\,q^{17} + 22\,q^{18} - 35\,q^{19} + 42\,q^{21} - 43\,q^{22} + 162\,q^{23} \\ &\quad - 105\,q^{24} - 28\,q^{26} - 35\,q^{27} - 42\,q^{28} + 160\,q^{29} + 42\,q^{31} + \cdots \end{aligned}$$

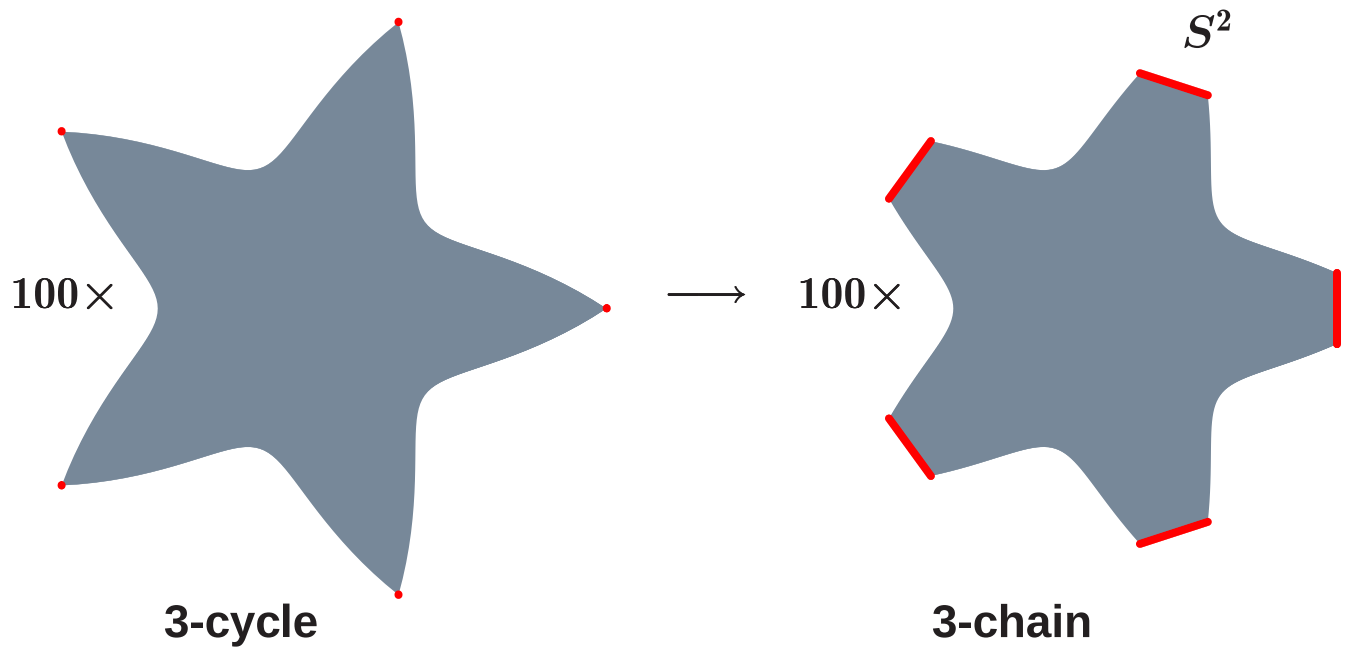


125 S^3's are blown down but only 101 are independent so 24 4-cycles are created.



$$\zeta(t, \psi^5 = 1) \;=\; \frac{(1 - a_p\, t + p^3 t^2)\,(1 - pt)^{200}}{(1 - t)(1 - p^2 t)^{25}(1 - p^3 t)}$$

Now we resolve 125 nodes with $\mathbb{P}^1$'s, but there are 100 relations so we destroy 100 3-cycles.



$$\zeta(t, \psi^5 = 1) = \frac{(1 - a_p\, t + p^3 t^2)\,(1 - pt)^{200-100}}{(1-t)(1-pt)^{125}(1-p^2 t)^{25}(1-p^3 t)}$$

$$= \frac{(1 - a_p\, t + p^3 t^2)}{(1-t)(1-pt)^{25}(1-p^2 t)^{25}(1-p^3 t)}\ .$$