THE WAVE TRACE ON THE TORUS

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Let M be a compact Riemannian manifold with metric $ds^2 = \sum_{i,j} g_{ij} dx^i dx^j$ and Laplace-Beltrami operator $\Delta : C^\infty(M) \to C^\infty(M)$.

We consider

$$Tr(\exp(it\sqrt{\Delta})) = \sum_{j=1}^{\infty} e^{i\lambda_j t} =: e(t) = \hat{\sigma}(t)$$

where the λ_j are the eigenvalues of $\sqrt{\Delta}$ and

$$\sigma(\lambda) = \sum_{j=1}^{\infty} \delta(\lambda - \lambda_j).$$

Note that $e(t)$ is a tempered distributional function of t.

The **Classical Trace Formula** gives information about the singularities of $e(t)$.

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Let H be the Hamiltonian

$$H(x,\xi) = \frac{1}{2}\sum g_x^{ij}\xi_i\xi_j$$

where $H : T^*M \to \mathbb{R}$.

Let X be the unit cosphere bundle of M, so $X \subset T^*M$.

Let Ξ denote the Hamiltonian vector field generated by H, so

$$\Xi = \sum_i \frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial \xi_i}$$

Denote by

$$\exp(t\Xi) : X \to X$$

the flow it generates, ie, geodesic flow. Denote

$$\Phi(t,v) := \exp(t\Xi)(v).$$

Fix a free homotopy class C of closed loops on M.

Let $T \in \mathbb{R}^+$.

Define

$$\Phi_T(v) := \Phi(T,v).$$

Define

$$W_T(C) := \{v \in X : \Phi(T,v) = v, \quad t \mapsto \pi \circ \Phi(t,v) \in C\} \subset X;$$

where $\pi : T^*M \to M$ is the projection map.

The set $W_T(C)$ is the set of all initial co-velocities that are periodic with period T and which, as closed loops, are contained in the homotopy class C.

Recall

$$W_T(C) := \{v \in X : \Phi(T, v) = v, \quad t \mapsto \pi \circ \Phi(t, v) \in C\} \subset X.$$

The Clean Intersection Hypothesis requires that, for v in $W_T(C)$, the fixed point set of $d(\Phi_T)_v : T_vX \to T_vX$ must equal $T_vW_T(C)$.

Let

$$F := I - d(\Phi_T) : T_vX \to T_vX.$$

The Clean Intersection Hypothesis just states that for $v \in W_T(C)$,

$$\ker(F_v) = T_vW_T(C).$$

Note that if we take a curve $v(t)$ in $W_T(C)$ then $\Phi_T(v(t)) = v(t)$, which implies the tangent space of $W_T(C)$ is automatically a subspace of the fixed point set.

Recall

$$W_T(C) := \{v \in X : \Phi(T, v) = v, \quad t \mapsto \pi \circ \Phi(t, v) \in C\} \subset X.$$

Let $d_T(C)$ denote the dimension of $W_T(C)$.

Denote

$$L(C) := \{T \in \mathbb{R}^+ : W_T(C) \neq \emptyset\}.$$

The set $L(C)$ is just the set of lengths of closed geodesics in $L(C)$. Denote

$$L_M := \cup_C L(C) (\text{disjoint union}).$$

This is the marked length spectrum.

We say $T \in L_M$ is *nondegenerate* if for all C such that $T \in L(C)$, $W_T(C)$ satisfies the CIH.

The work of Duistermaat-Guillemin showed

Trace Formula.

(1) *The singular support of* $e(t)$ *is contained in* L_M *(Corollary 1.2 of DuG).*
(2) *If* T *is nondegenerate, then* $e(t)$ *is a classical conormal distribution in a neighborhood of* T, *and*
(3) *If* T *is nondegenerate, then the leading singularity of* $e(t)$ *at* T *is a* **constant** *multiple of the distribution*

$$(t - T + i0)^{(d_T - 1)/2}.$$

These three statements together with the calculation of **constant** are what is generally called **The Trace Formula**.

Note: The paper [GU], from which I took most of the above exposition, uses power $d_T/2$. I think the power above is correct, as can be seen on the next page.

Theorem [DuG, Thm 4.5]**.** *Assume that the set of period Ξ solution curves of period T is a union of connected submanifolds $Z_1, Z_2, \ldots, Z_r$ in X, each Z_j being a clean fixed point set for Φ_T of dimension d_j. Then there exists an interval around T in which no other periods occur, and on such an interval we have*

$$e(t) = \hat{\sigma}(t) = \sum_{j=1}^{r} \beta_j(t - T),$$

where

$$\beta_j(t) = \int_{-\infty}^{\infty} \alpha_j(t) e^{-ist} ds$$

with

$$\alpha_j(s) \sim \left(\frac{s}{2\pi i}\right)^{(d_j-1)/2} i^{-\sigma_j} \sum_{k=0}^{\infty} \alpha_{j,k} s^{-k}$$

as $s \to \infty$ where $d_j = \dim Z_j$ and σ_j is a Morse index. Also, if d_j is even, we let $(1/i)^{(d_j-1)/2} = e^{-\pi i (d_j-1)/4}$. Finally,

$$\alpha_{j,0} = \frac{1}{2\pi} \int_{Z_j} d\mu_j,$$

where we describe μ_j below.

Remark. What I do in my paper is compute $\alpha_{j,0}$, which when multiplied by $\left(\frac{1}{2\pi i}\right)^{(d_j-1)/2} i^{-\sigma_j}$ is (usually) the leading singularity. The key to doing this is understanding the density μ_j.

Remark. Note that X is not symplectic, but T^*M is symplectic.

Citation [DuG, p. 60]**.** *Let $N = T^*M$ be a symplectic manifold, $\Phi = \Phi_T$ a symplectomorphism of N and $Z = W_T(C)$ a clean fixed point set of Φ. The tangent space to Z at each $z \in Z$ possesses an intrinsic positive density, i.e. an intrinsic smooth positive measure, denoted $d\mu_Z$. In particular, let Z be a submanifold of X (unit cosphere bundle) consisting of periodic Ξ solution curves of period T. If Z is clean for $\Phi : X \to X$ then*

$$Z' = \{(x, \lambda\xi) : \lambda \in \mathbb{R}^+, (x, \xi) \in Z\}$$

is clean for $\Phi : T^*M\backslash 0 \to T^*M\backslash 0$. So $d\mu_{Z'}$ is defined (in a moment). Dividing by $|dq|$ we get an intrinsic measure $d\mu_Z$ on Z.

What is this intrinsic measure???

Lemma [BPU, Appendix]**.** *Let Z be a set of periodic solution curves of period T. Fix $z \in Z$. Denote*

$$W = T_z Z \qquad V = T_z T^* M.$$

Let Ω be the symplectic form on T^*X. Since Z is clean,

$$W = \ker(I - d\Phi_T).$$

Denote

$$F := I - d\Phi_T.$$

Let $\mathcal{E} := \{e_1, \ldots, e_k\}$ be an arbitrary basis of W. Let $\mathcal{F} := \{f_1, \ldots, f_k\}$ be a basis of $W^\perp$ such that $\Omega(e_i, f_j) = \delta_{ij}$, where $W^\perp$ is the symplectic complement of W in V. Let $\mathcal{V} := \{v_1, \ldots, v_{2n-k}\}$ be a basis of a complement of W in V.

Let ν be an arbitrary half-density of V. Then

$$\mu(\mathcal{E}) = \frac{\nu(\mathcal{V} \wedge \mathcal{E})}{\nu(F(\mathcal{V}) \wedge \mathcal{F})}$$

We call this the Duistermaat-Guillemin or DG denisty.

Note that $F(\mathcal{V})$ is a basis of $W^\perp$ (see [DGu p. 60] or [BPU p. 523]), and the expression in the Lemma is independent of the choice of half-density ν and the choice of $\mathcal{F}$ and $\mathcal{V}$.

Our approach will be to calculate α where $F\mathcal{V} \wedge \mathcal{F} = \alpha\mathcal{V} \wedge \mathcal{E}$. We then have

$$\int_{Z_j} d\mu_j = \int_{Z_j} \frac{1}{\sqrt{|\alpha|}} ds,$$

where ds is the Riemannian volume form on Z_j coming from X. We refer to $\frac{1}{|\alpha|^{1/2}}$ as the *DG-multiplier*.

Finally, note that calculating $T\mathcal{V}$ requires calculating the derivative $d\Phi_T$ restricted to a complement of the fixed points of $d\Phi_T$. That is, we compute the Poincaré or first return map on a complement of $T_z W_T(C)$ in T^*M.

Remark. Note that V above is symplectic and W is a fixed point set for $\Phi : V \to V$. So to calculate μ, we must set $W = Z'$, where Z' is described above. We then calculate the DG-multiplier $1/sqrt|\alpha|$ using $W = Z'$, but then we then "divide out" by the added direction, i.e., we take our volume on Z, not Z'.

Example: Flat Torus. We calculate the wave invariant associated to an arbitrary element of the length spectrum of a flat torus.

The manifold $L\backslash\mathbb{R}^n$ with the metric induced from the Euclidean metric on $\mathbb{R}^n$ is a closed Riemannian manifold. Note that

$$X \cong L\backslash\mathbb{R}^n \times S^{n-1} = \{(p, v) : p \in L\backslash\mathbb{R}^n, v \in \mathbb{R}^n, |v| = 1\}.$$

Geodesics in $\mathbb{R}^n$ are just straight lines, so geodesics in $L\backslash\mathbb{R}^n$ are projections of straight lines. For $(p, v) \in L\backslash\mathbb{R}^n \times S^{n-1}$,

$$\Phi(t, (p, v)) = (p + tv, v).$$

Free homotopy classes of $L\backslash\mathbb{R}^n$ correspond to conjugacy classes in L. As $\mathbb{R}^n$ is abelian, the free homotopy classes of $L\backslash\mathbb{R}^n$ are in one-to-one correspondence with the elements of L.

Now, $\Phi(T, (p, v)) = (p, v)$ iff $(p + Tv, v) = (p, v)$ iff $Tv = l \in L$. In this case, $T = |l|$ and $v = l/|l|$. The free homotopy class of the curve $p+tl/|l|$ corresponds to the element $l \in L$. From this computation, we conclude that the length spectrum of $L\backslash\mathbb{R}^n$ is

$$\{|l| : l \in L\}.$$

One easily concludes that for $T = |l|$ and $C = l$,

$$Z = W_T(C) = L\backslash\mathbb{R}^n \times \{l/|l|\}.$$

Clearly, Z is diffeomorphic to $L\backslash\mathbb{R}^n$, and $\dim Z = n$.

Let $\alpha(s) = (p(s), v(s))$ be a curve in X with $\alpha(0) = v_p = (p, v) \in Z$. Then $\Phi_T(p(s), v(s)) = (p(s) + Tv(s), v(s))$. Thus

$$d(\Phi_T)_{v_p}(p'(0), v'(0)) = (p'(0) + Tv'(0), v'(0)),$$

and

$$F(p'(0), v'(0)) = (-Tv'(0), 0),$$

where $p'(0) + Tv'(0) \in T_pL\backslash\mathbb{R}^n$ and $v'(0) \in T_v\mathbb{R}^n = \mathbb{R}^n$. From this we conclude that $(p'(0), v'(0))$ is a fixed point of $d\Phi_T$ iff it is tangent to Z, ie, T is clean for all T in the length spectrum.

We now compute the wave invariants. Recall that the symplectic form on $T\mathbb{R}^n$, obtained from $T^*\mathbb{R}^n$ via the musical isomorphisms, is just

$$\Omega\left((A,B),(A',B')\right) = \langle A, B'\rangle - \langle B, A'\rangle\,,$$

for $A, B \in \mathbb{R}^n$.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ such that $e_1 = l/|l|$. We must amplify Z to Z', ie,

$$Z' = L\backslash\mathbb{R}^n \times l\mathbb{R}^+.$$

We set

$$\mathcal{E} = \{(e_1, 0), \cdots, (e_n, 0), (0, e_1)\}.$$

Then using Ω,

$$\mathcal{F} = \{(0, e_1), \cdots, (0, e_n), (-e_1, 0)\}.$$

We set

$$\mathcal{V} = \{(0, e_2), \cdots, (0, e_n)\}.$$

We must calculate $F\mathcal{V} \wedge \mathcal{F}$ as a multiple of $\mathcal{V} \wedge \mathcal{E}$, up to sign. Now $F\mathcal{V} \wedge \mathcal{F}$ equals

$$F(0, e_2) \wedge \cdots \wedge F(0, e_n) \wedge (0, e_1) \wedge \cdots \wedge (0, e_n) \wedge (-e_1, 0)$$

$$= (-\tau e_2, 0) \wedge \cdots \wedge (-\tau e_n, 0) \wedge (0, e_1) \wedge \cdots \wedge (0, e_n) \wedge (-e_1, 0)$$

$$= \pm \tau^{(n-1)} \mathcal{V} \wedge \mathcal{E}.$$

Thus, the DG-multiplier is $\frac{1}{\tau^{(n-1)/2}}$, and the wave invariant associated with $\tau = |l|$ and $l \in L$ is

$$\mathrm{Wave}(|l|, l) = \frac{\mathrm{Vol}(L\backslash\mathbb{R}^n)}{(2\pi i\tau)^{(n-1)/2}}.$$

If we sum up over all free homotopy classes that contain a closed geodesic of length τ, we obtain the wave invariant

$$\mathrm{Wave}(\tau) = \frac{\mathrm{Vol}(L\backslash\mathbb{R}^n)}{(2\pi i\tau)^{(n-1)/2}} \sum_{l\in L, |l|=\tau} 1 = \frac{\mathrm{mult}(\tau)}{(2\pi i\tau)^{(n-1)/2}} \mathrm{Vol}(L\backslash\mathbb{R}^n).$$

Note that we may add up over the free homotopy classes since the dimension of $W_T(C)$ is independent of the length τ. Also, the Morse index for periodic geodesics is zero in this case. $\square$

Introduction of my Paper.

The *spectrum* of a closed Riemannian manifold (M, g), denoted $\mathrm{spec}(M, g)$, is the collection of eigenvalues of the Laplace–Beltrami operator Δ acting on smooth functions. Two manifolds (M, g) and (M', g') are *isospectral* if $\mathrm{spec}(M, g) = \mathrm{spec}(M', g')$. The *length spectrum* of (M, g), denoted by $\mathrm{spec}_{[L]}(M, g)$, is the collection of lengths of smoothly closed geodesics of (M, g), counted with multiplicity. The *multiplicity* of a length is defined as the number of free homotopy classes of loops containing a closed geodesic of that length. The *absolute length spectrum* of a Riemannian manifold (M, g), denoted $\mathrm{spec}_L(M, g)$, is the set of lengths of smoothly closed geodesics with no multiplicity assigned. (The absolute length spectrum is also referred to in the literature as the *weak length spectrum.*)

A major open question in spectral geometry is the precise relation between the Laplace spectrum on functions and the (absolute) length spectrum. Using the heat equation, Colin de Verdière [CdV] has shown that *generically* (in the family of all Riemannian manifolds), the Laplace spectrum determines the absolute length spectrum. This result can also be obtained from the classical (wave) trace formula [DGu] (described below). In the case of compact, hyperbolic manifolds, this arises from the Selberg Trace Formula [Sel]. (See also [Ch, Chapter XI].)

In sharp contrast, Miatello and Rossetti [MR] have constructed pairs of compact flat manifolds that are isospectral on one-forms but which do not have the same absolute length spectrum. (See also [CR1,2].) There is no known example of a pair of manifolds that are isospectral on functions but with unequal absolute length spectra.

Also in contrast, C. S. Gordon [G1] has constructed pairs of isospectral Heisenberg manifolds that have unequal multiplicities in the length spectrum, and the author [Gt2] has constructed other higher-step nilmanifolds with this property. These examples are of great interest, as it has been shown that all known methods for producing examples of isospectral nilmanifolds necessarily yield examples with the same absolute length spectrum [G1], [GtM3]. A *Riemannian two-step nilmanifold* is a closed manifold of the form $(\Gamma\backslash G, g)$, where G is a simply connected two-step nilpotent Lie group, Γ is a uniform (i.e., $\Gamma\backslash G$ compact) discrete subgroup of G, and g is a left invariant metric on G, which descends to a Riemannian metric on $\Gamma\backslash G$ that we also denote by g. A *Heisenberg manifold* is a two-step Riemannian nilmanifold whose covering Lie group G is one of the $(2n+1)$-dimensional Heisenberg Lie groups.

The purpose of this paper is to understand the behavior of the length spectrum on the isospectral Heisenberg manifolds of Gordon in particular, and on two-step nilmanifolds in general. Our approach is to compute certain wave invariants, which are defined below. En route to calculating the wave invariants on two-step nilmanifolds, we must calculate the Duistermaat-Guillemin density. This in turn requires us to calculate the first return or Poincaré map. We apply these calculations to the Heisenberg manifolds of Gordon and show how a pair of manifolds can have the same Laplace spectrum, i.e., the same wave invariants, and yet have unequal multiplicities in the length spectrum. See (3.?) for conclusions about the definition of the multiplicity of a length in the length spectrum.

For (M, g) a Riemannian manifold, define

$$e_M(t) = \text{trace}(\exp(it\sqrt{\Delta})) = \sum_{\lambda \in \text{spec}(M,g)} e^{it\sqrt{\lambda}}.$$

This is a tempered distribution that is determined by the Laplace spectrum. That is, if M and M' are isospectral, then $e_M(t) = e_{M'}(t)$. The *classical trace formula*, arising from the study of the wave equation, provides information about the singularities of $e_M(t)$. In particular [DGu],

(1) the singular support of $e_M(t)$ is contained in $\text{spec}_L(M, g)$, and
(2) if τ is in $\text{spec}_L(M, g)$ and τ satisfies a *Clean Intersection Hypothesis*, then $e_M(t)$ is a classical conormal distribution in a neighborhood of τ, and the singularities of $e_M(t)$ at τ provide geometric information about (M, g), the *wave invariants.*

See Section 1.1 below for more details about $e_M(t)$. Good references for distributions and singular support are [Si] and [St].

The advantage of working with nilmanifolds is that they are "getatable" in the sense that the spectrum [P3],[GW3] and the length spectrum [E1], [GtM1] are explicitly computable. Moreover, the generic results of Colin de Verdière and Duistermaat-Guillemin require that all closed geodesics be isolated and that lengths be multiplicity free. Nilmanifolds possess a great deal of symmetry so that closed geodesics always come in large dimensional families, thus failing these generic hypotheses. Consequently, generic results relating the Laplace and length spectra say nothing about the many known examples of isospectral nilmanifolds and other isospectral families of manifolds [DG1,2], [O], [P1,2,3,4], [GW1,2,3], [G1,2,3], [E1].

This paper, while self-contained, is a continuation of [Gt6], in which the author computes a necessary and sufficient condition for a two-step nilmanifold to satisfy the Clean Intersection Hypothesis. The computations of [Gt6] are closely related to those here, and we use the necessary and sufficient condition to avoid two-step nilmanifolds failing the Clean Intersection Hypothesis in this paper. Of interest is the fact that the Heisenberg manifolds of Gordon that exhibit unequal multiplicities in the length spectrum are not related to the Clean Intersection Hypothesis. In particular, lengths of a Heisenberg manifold that are "unclean" are associated to free homotopy classes coming from central elements of the fundamental group. Whereas lengths of a Heisenberg manifold that exhibit unequal multiplicities in the length spectrum come from noncentral elements of the fundamental group.

This paper is organized as follows. In Section 1 we review necessary background information on the wave invariants and nilmanifolds. Large parts of Section 1 summarize material included in more detail in [Gt6]. We include in Section 1 a warm up calculation of the wave invariants on the n-dimensional flat torus. We end Section 1 with the computation of the wave invariants of interest on the Heisenberg manifolds of Gordon. Because of broad interest in Heisenberg manifolds, the author attempts to present this example so that a reader with a background in differential geometry may find it readable without getting bogged down with the many technical details of two-step nilmanifolds. In Section 2 we calculate the first return or Poincaré map and the Duisermaat-Guillemin density for all two-step nilmanifolds. The Poincaré map is quite explicit, but we are not able to get a closed form for the Duistermaat-Guillemin density in all cases. In Section 3 we examine other special cases, including the wave invariants on two-step nilmanifolds with a one-dimensional center, and the Duisermaat-Guillemin density for a specific 5-dimensional example with a two-dimensional center. The 5-dimensional example exhibits many of the subtleties possible in these calculations.

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