Let $X$ and $Y$ be normed linear spaces, and suppose $A : X \to Y$ is a linear map. Define

$$\|A\|_{op} = \sup \left\{ \frac{\|Ax\|}{\|x\|} : x \neq 0 \right\} = \{\|Ax\| : \|x\| \leq 1\} = \{\|Ax\| : \|x\| = 1\}$$

If $\|A\| < \infty$, we call $A$ a bounded (linear) operator.

**Theorem 1.** The following are equivalent:

(a) $A$ is bounded;
(b) $A$ is continuous at 0;
(c) $A$ is uniformly continuous.

Many commonly occurring operators are bounded; here is an example that is unbounded.

**Example 2.** Let $B$ be a vector space basis (or a Hamel basis, as it is often called) of $L^2(\mathbb{R})$ and choose a sequence $\{f_n\}$ of distinct elements of $B$. By dividing each $f_n$ by its norm if necessary, we may assume that each $f_n$ has norm 1. Define a linear map $F : L^2(\mathbb{R}) \to \mathbb{C}$ by setting $Ff_n = n$ for each $n$ and $Fg = 0$ for all other elements in $B$. Then $F$ is unbounded.

From now on, we restrict our attention to linear operators from a Hilbert space $\mathcal{H}$ to itself. If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear map, its adjoint $A^* : \mathcal{H} \to \mathcal{H}$ is determined by the requirement that $\langle Af, g \rangle = \langle f, A^*g \rangle$ for all $f$ and $g$ in $\mathcal{H}$. Interestingly, the existence of an adjoint automatically implies that $A$ is linear.

**Theorem 3.** Let $A : \mathcal{H} \to \mathcal{H}$ be a function, and suppose there exists a function $B : \mathcal{H} \to \mathcal{H}$ with the property that $\langle Af, g \rangle = \langle f, Bg \rangle$ for all $f$ and $g$ in $\mathcal{H}$. Then $A$ is linear.

**Proof.** For all $f$, $g$, and $h$ in $\mathcal{H}$ and complex numbers $\alpha$ and $\beta$,

$$\langle A(\alpha f + \beta g), h \rangle = \langle \alpha f + \beta g, Bh \rangle$$

$$= \alpha \langle f, Bh \rangle + \beta \langle g, Bh \rangle$$

$$= \alpha \langle Af, h \rangle + \beta \langle Ag, h \rangle$$

$$= \langle \alpha Af + \beta Ag, h \rangle,$$

whence $A(\alpha f + \beta g) = \alpha Af + \beta Ag$. $\square$

In fact, more is true.

**Theorem 4 (Closed Graph Theorem).** Let $X$ and $Y$ be Banach spaces and suppose $A : X \to Y$ is a linear map with the following property: if $\{x_n\}$ is a sequence in $X$ converging to some $x$ in $X$ and $\{Ax_n\}$ converges to some $y$ in $Y$, then $y = Ax$. Then $A$ is bounded.

---

Date: Spring 2013.
Theorem 5. Let \( A : \mathcal{H} \to \mathcal{H} \) be a function, and suppose there exists a function \( B : \mathcal{H} \to \mathcal{H} \) with the property that \( \langle Af, g \rangle = \langle f, Bg \rangle \) for all \( f \) and \( g \) in \( \mathcal{H} \). Then \( A \) is bounded.

Proof. Let \( \{f_n\} \) be a sequence in \( \mathcal{H} \) converging to \( f \), and suppose that \( \{Af_n\} \) converges to \( g \). Then for all \( h \) in \( \mathcal{H} \),

\[
\langle g, h \rangle = \lim_{n \to \infty} \langle Af_n, h \rangle = \lim_{n \to \infty} \langle f_n, Bh \rangle = \langle f, Bh \rangle = \langle Af, h \rangle,
\]

and therefore \( Af = g \). \( \square \)

Therefore if one wants to study unbounded linear operators on a Hilbert space \( \mathcal{H} \) that have an adjoint, such operators can not be defined on all of \( \mathcal{H} \).

Definition 6. Let \( \mathcal{H} \) be a Hilbert space. An (unbounded) linear operator on \( \mathcal{H} \) consists of a dense linear subspace \( \mathcal{D}(A) \) and a linear map \( A : \mathcal{D}(A) \to \mathcal{H} \). The linear subspace \( \mathcal{D}(A) \) is called the domain of \( A \).

Example 7. Let \( \phi \) be a continuous unbounded function on \( \mathbb{R} \), and define \( M_\phi' \) on \( L^2(\mathbb{R}) \) by \( (M_\phi' f)(x) = \phi(x) f(x) \) with domain \( \mathcal{D}_1 \) consisting of compactly supported continuous functions on \( \mathbb{R} \).

Example 8. Let \( \phi \) be a continuous unbounded function on \( \mathbb{R} \), and define \( M_\phi \) on \( L^2(\mathbb{R}) \) by \( (M_\phi f)(x) = \phi(x) f(x) \) with domain

\[
\mathcal{D}_2 = \{ f \in L^2(\mathbb{R}) : \phi f \in L^2(\mathbb{R}) \}.
\]

Example 9. \( D \) on \( L^2(0,1) \), defined by \( Df = f' \) with domain \( C_0^\infty(0,1) \).

We shall see that different choices of domain can give a linear operator very different properties.

Definition 10. Let \( A, B \) be linear operators on \( \mathcal{H} \). We say that \( A \) and \( B \) are equal if \( \mathcal{D}(A) = \mathcal{D}(B) \) and \( Af = Bf \) for all \( f \) in \( \mathcal{D}(A) \). We will say that \( B \) is an extension of \( A \) if \( \mathcal{D}(A) \subseteq \mathcal{D}(B) \) and \( Af = Bf \) for all \( f \) in \( \mathcal{D}(A) \); if \( B \) is an extension of \( A \), we write \( A \subseteq B \).

Note that in the examples above, the operator \( B \) is an extension of \( A \).

Definition 11. The graph of a linear operator \( A \) is the set

\[
\mathcal{G}(A) = \{ (f, Tf) : f \in \mathcal{D}(A) \}.
\]

Note that if \( A \subseteq B \), then \( \mathcal{G}(A) \subseteq \mathcal{G}(B) \) as sets.

Definition 12. A linear operator \( A \) is closed if \( \mathcal{G}(A) \) is a closed subset of \( \mathcal{H} \oplus \mathcal{H} \).

Theorem 13. Let \( A \) be a linear operator on \( \mathcal{H} \). The following are equivalent:

(a) \( A \) is closed;
(b) If \( \{f_n\} \) is a sequence in \( \mathcal{D}(A) \) converging to some \( f \) in \( \mathcal{H} \) and if \( \{Af_n\} \) converges to some \( g \) in \( \mathcal{H} \), then \( f \) is in \( \mathcal{D}(A) \) and \( Af = g \);
(c) The linear space \( \mathcal{D}(A) \) is a Hilbert space for the inner product

\[
\langle f, g \rangle_A := \langle f, g \rangle + \langle Af, Ag \rangle.
\]

Definition 14. A linear operator is closable if it has an extension that is closed.

Theorem 15. Let \( A \) be a linear operator on \( \mathcal{H} \). The following are equivalent:

(a) \( A \) is closable;
(b) If \( \{f_n\} \) is a sequence in \( D(A) \) converging to 0 and if \( \{Af_n\} \) converges to some \( g \) in \( H \), then \( g = 0 \);

(c) The (set-theoretic) closure of \( \mathcal{G}(A) \) is the graph of a linear operator \( \bar{A} \).

If \( A \) is closable, the operator \( \bar{A} \) in the previous theorem is unique and is called the closure of \( A \). The operator \( \bar{A} \) is minimal in the following sense: if \( B \) is an extension of \( A \) that is closed, then \( B \) is also an extension of \( \bar{A} \).

Our operator \( M_\phi \) is closed. The other examples are not closed, but they are closable. In fact, one has to do some amount of work to construct an unbounded operator that is not closable.

**Example 16.** Choose a linear subspace \( D \) of \( H \), an unbounded linear functional \( F \) on \( D \), and a vector \( h \neq 0 \) in \( H \). Define a linear operator \( A \) with domain \( D \) by the formula
\[
Tf = F(f)h.
\]
Because \( F \) is not continuous at 0, there exists a sequence \( \{f_n\} \) in \( D \) with the property that \( \{f_n\} \) converges to 0, but \( \{F(f_n)\} \) does not converge to 0. By passing to a subsequence if necessary, we may assume that there exists a constant \( c > 0 \) such that
\[
|F(f_n)| \geq c
\]
for all natural numbers \( n \). Define
\[
g_n = \frac{x_n}{F(x_n)}.
\]
Then \( \{g_n\} \) converges to 0, while
\[
T(g_n) = T\left( \frac{f_n}{F(f_n)} \right) = \frac{1}{F(f_n)}T(f_n) = \frac{1}{F(f_n)}F(f_n)e = e \neq 0.
\]

We next discuss adjoints of unbounded operators.

**Definition 17.** Let \( A \) be a linear operator on a Hilbert space \( H \). Set
\[
D(A^*) = \{ g \in H : \text{there exists } h \in H \text{ such that } \langle Af, g \rangle = \langle f, h \rangle \text{ for all } f \in D(A) \}.
\]
Because \( D(A) \) is dense in \( H \), the element \( h \), if it exists, is unique. The operator \( A^* \) with domain \( D(A^*) \) defined by \( A^*h = g \) is called the adjoint of \( A \).

By definition, we have the equation
\[
\langle Af, g \rangle = \langle f, A^*g \rangle
\]
for all \( f \) in \( D(A) \) and \( g \) in \( D(A^*) \).

It is possible that \( D(A^*) \) is not dense in \( H \). In fact, for our example \( T \) above, \( D(A^*) = 0 \).

In general, it is very difficult to explicitly determine the elements of \( D(A^*) \), but in certain cases we can do this.

**Theorem 18.** \((M_\phi)^* = M_{\phi^*}\).

**Proof.** Integration by parts shows that \((M_\phi)^*\) is an extension of \( M_{\phi^*}\):
\[
\langle M_\phi f, g \rangle = \int_R \phi f \overline{g} \, dx = \int_R f \overline{\phi g} \, dx = \langle f, M_{\phi^*}g \rangle,
\]
and an easy argument with characteristic functions shows that these two operators are equal. \qed
Theorem 19. Let $A$ be an operator on $\mathcal{H}$.

(a) $A$ is closable if and only if $\mathcal{D}(A^*)$ is dense in $\mathcal{H}$.
(b) If $A$ is closable, then $(\overline{A})^* = A^*$, and $\overline{A} = A^{**}$.
(c) $A$ is closed if and only if $A = A^{**}$.

We can use this part (a) of this theorem to prove that $\frac{d}{dx}$ on $(0,1)$ is closable.

To come up with a domain for which $\frac{d}{dx}$ is closed, we begin with a definition. Let us consider another example.

Definition 20. A function $f$ on $[0,1]$ is absolutely continuous if there exists a function $h$ in $L^1(0,1)$ such that

$$f(x) = f(0) + \int_0^x h(t) \, dt$$

for every $x$ in $[0,1]$.

Absolutely continuous functions are continuous, and are differentiable almost everywhere; in fact, $f'(x) = h(x)$ almost everywhere. For this reason, we typically write $f'$ for $h$. Let $AC[0,1]$ denote the set of absolutely continuous functions on $[0,1]$, and define $H^1(0,1) = \{ f \in AC[0,1] : f' \in L^2(0,1) \}$.

Theorem 21. Let $D$ be the operator $\frac{d}{dx}$ on $L^2(0,1)$ with domain $H^1(0,1)$. Then $D$ is closed.

Before considering more examples, we need the notion of a multi-index.

Definition 22. A $d$-dimensional multi-index is a $d$-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ of nonnegative integers. We set $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

Definition 23. Let $\Omega$ be an open subset of $\mathbb{R}^d$. For each $d$-dimensional multi-index $\alpha$, set

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$$

and

$$D^\alpha = (-i)^{|\alpha|} \partial^\alpha.$$ 

A $n$-th order linear partial differential operator on $\Omega$ is an operator with domain $C_0^\infty(\Omega)$ and formula

$$L = \sum_{|\alpha| \leq n} a_\alpha D^\alpha$$

for smooth functions $a_\alpha$ in $C^\infty(\Omega)$. The formal adjoint of $L$ is the operator

$$L^+ = \sum_{|\alpha| \leq n} D^\alpha a_\alpha = \sum_{|\alpha| \leq n} a_\alpha^+ D^\alpha.$$ 

We say $L$ is formally self-adjoint if $L^+ = L$.

By integrating by parts, we can show that

$$\langle Lf, g \rangle = \langle f, L^+ g \rangle$$

for all $f$ and $g$ in $C_0^\infty(\Omega)$.

We let $L_0$ and $L_0^+$ respectively denote $L$ and $L^+$ with $C_0^\infty(\Omega)$ as its domain. The closure of the graph of $L_0$ and $L_0^+$ determine the minimal extensions $L_{\text{min}}$ and
There also exist maximal extensions \( L_{\text{max}} \) and \( L_{\text{max}}^+ \); these are defined using distribution theory. The following equalities hold:

\[
(L_0^+)^* = L_{\text{max}}, \quad (L_{\text{max}})^* = L_{\text{min}}.
\]

**Definition 24.** Let \( A \) be a linear operator on \( \mathcal{H} \).

(a) \( A \) is symmetric if \( A \subseteq A^* \).

(b) \( A \) is self-adjoint if \( A = A^* \).

(c) \( A \) is essentially self-adjoint if its closure \( \overline{A} \) is self-adjoint.

The operator \( M_\phi \) is self-adjoint if \( \phi \) is real-valued, and \( i \frac{d}{dx} \) is self-adjoint with the domain \( H^1(0, 1) \). The relationship between these three notions is complicated in general. For example, as we shall see, there are symmetric operators that are not self-adjoint. To explore this further, we need some terminology.

**Definition 25.** Let \( A \) be a linear operator on \( \mathcal{H} \). A complex number \( \lambda \) is a regular point for \( A \) if there exists a positive number \( c_\lambda \) with the property that

\[
\| (A - \lambda I)f \| \geq c_\lambda \| f \|
\]

for all \( f \) in \( D(A) \). We let \( \pi(A) \) denote set of regular points of \( A \). For \( \lambda \) in \( \pi(A) \), we define the deficiency number \( d_\lambda(A) \) of \( T \) to be the dimension of the orthogonal complement of the range of \( T - \lambda I \).

**Theorem 26.** Let \( A \) be a linear operator on \( \mathcal{H} \).

(a) \( \pi(A) \) is a nonempty open set in \( \mathbb{C} \);

(b) A complex number \( \lambda \) is in \( \pi(A) \) if and only if \( (A - \lambda I) \) has a bounded inverse on the range of \( A - \lambda I \).

(c) If \( A \) is closable, then \( d_\lambda(A) \) is locally constant.

The set of regular points of \( A \) is related to another quantity:

**Definition 27.** A complex number \( \lambda \) belongs to the resolvent set \( \rho(A) \) of \( A \) if \( A - \lambda I \) has a bounded inverse defined on all of \( \mathcal{H} \). The complement of the resolvent is called the spectrum of \( A \) and is denoted \( \sigma(A) \).

**Proposition 28.** \( \rho(A) = \{ \lambda \in \pi(A) : d_\lambda(A) = 0 \} \).

**Proposition 29.** If \( A \) is symmetric, then \( \mathbb{C} \setminus \mathbb{R} \) is contained in \( \pi(A) \).

The number \( d_\lambda(A) \) is constant for \( \exists \lambda > 0 \), and also constant for \( \exists \lambda < 0 \). We shall call these numbers the deficiency indices of \( A \) and denote them \( d_+(A) \) and \( d_-(A) \), respectively.

Observe that if \( \pi(A) \) contains any real number, then \( d_+(A) = d_-(A) \).

Because we are requiring our operators to be densely defined, when \( A \) is symmetric we can make alternate definitions

\[
d_+(A) = \dim \ker(A^* - iI) \\
d_-(A) = \dim \ker(A^* + iI)
\]

**Theorem 30.** Let \( A \) be a symmetric closable operator on \( \mathcal{H} \).

(a) \( A \) has a self-adjoint extension if and only if \( d_+(A) = d_-(A) \).

(b) \( A \) is essentially self-adjoint if and only if \( d_+(A) = d_-(A) = 0 \).
Example 31. Let $I$ denote one of these three intervals:
- $(0, 1)$ (or $(a, b)$ for any $a < b$ in $\mathbb{R}$);
- $(0, \infty)$;
- $\mathbb{R}$.

Define $A$ by $Af = -i f'$ and take the domain of $A$ to be $H^1(I)$. Then $A \subseteq A^*$, so $A$ is symmetric. Furthermore, $g$ is in the kernel of $A^* \mp i I$ if and only if $g$ is in $\mathcal{D}(A^*)$ and $g' = \pm g$ on $I$.

If $I = (0, 1)$, then the kernel of $A^* \mp i I$ is spanned by $e^{\pm x}$. Both of these functions are in $H^1(I)$, so $d_+(A) = d_-(A) = 1$.

If $I = (0, \infty)$, then $e^{-x}$ is in $\mathcal{D}(A^*)$, but $e^x$ is not, because its derivative is not in $L^2(0, \infty)$. Therefore $d_+(A) = 1$ and $d_-(A) = 0$.

If $I = \mathbb{R}$, then neither $e^x$ nor $e^{-x}$ is in $\mathcal{D}(A^*)$ so. Therefore $d_+(A) = d_-(A) = 0$. 