SMOOTHING THEOREMS IN ALGEBRAIC GEOMETRY

SCOTT NOLLET

ABSTRACT. The purpose of this series of talks is to state a few of the standard smoothing theorems in algebraic geometry, such as Hironaka's theorem, Bertini theorems, and smoothing of degeneracy loci of maps between vector bundles. In the context of these results, I'll then explain recent joint work with Prabhakar Rao.

1. INTRODUCTION

The main objects of study in algebraic geometry are smooth projective varieties $X \subset \mathbb{P}^n_k$ over an algebraically closed field k (feel free to take $k = \mathbb{C}$). In recent decades, there has been a lot of work done in trying to understand families of these varieties. Unlike in differential geometry or topology, in algebraic geometry we can often construct parameter spaces for these varieties, or moduli spaces, which turn out to be varieties (or schemes) themselves. Even if one is only interested in smooth projective varieties, one is forced to deal with singular degenerations that appear in the boundary of these moduli spaces if they are proper.

Example 1.1. Each conic $X \subset \mathbb{P}^2$ is defined by a nonzero homogeneous polynomial $f(x, y, z)$ of degree two, where x, y, z are homogeneous coordinates for $\mathbb{P}^2$. Two such polynomials f and g define the same curve X if and only if $f = \lambda g$ for some nonzero scalar $f \in k^*$, defining an equivalence relation $f \sim g$. If V is the vector space of homogeneous degree two polynomials with basis $x^2, y^2, z^2, xy, xz, yz$, then all degree two curves are parametrized by $(V - \{0\}/\sim) = \mathbb{P}^5$, the map is given by

$$f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz \mapsto (a, b, c, d, e, f)$$

This is a simple example of a Hilbert scheme.

We may express f uniquely in the form $f = (x, y, z)M(x, y, z)^T$, where M is a symmetric matrix. Orthogonally diagonalizing M gives a diagonal matrix, thus there is a change of coordinates for which $f(x, y, z) = x^2 + y^2 + z^2, x^2 + y^2$ or x^2. Each possibility can be visualized geometrically:

(A) Writing $x^2 + y^2 + z^2 = (x + iy)(x - iy) - (iz)^2 = XY - Z^2$ gives another description of a conic in family (A). Some calculation shows that $Z(XY - Z^2) \subset \mathbb{P}^2$ is exactly the image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ given by $(s, t) \mapsto (s^2, t^2, st)$, hence the conic in family (A) is isomorphic to $\mathbb{P}^1$, a smooth rational curve.

(B) In this case we have the equation $XY = 0$, which gives two lines meeting at a point. This conic is not smooth, being singular at the intersection point, it is also not a variety (see definition below), it is a union of two varieties.

(C) The equation $X^2 = 0$ is a doubling of the Y-axis. Classically this might be discounted or maybe thought of as a line, but using Grothendieck's foundations of scheme theory, it gives a closed subscheme of $\mathbb{P}^2$ supported on a line, but having more "scheme structure", i.e. having a different structure sheaf of locally rational functions.

Thus using our parameter space we can stratify our conics as $\mathbb{P}^5 = (A) \cup (B) \cup (C)$. What is the nature of this stratification? If $(\mathbb{P}^2)^\vee$ is the space of lines in $\mathbb{P}^2$, which is parametrized by $\mathbb{P}^2$ itself via $ax + by + cz = 0 \mapsto (a, b, c)$, we have a map $\Phi : (\mathbb{P}^2)^\vee \times (\mathbb{P}^2)^\vee \to \mathbb{P}^5$ given by

$$(L_1, L_2) \mapsto L_1 \cup L_2$$

This is a 2-1 map away from the diagonal, so the image of Φ has dimension 4 and consists of $(B) \cup (C)$. We can do better than this, however: $(B) \cup (C)$ is precisely the locus of $\mathbb{P}^5$ where $\det M = 0$, which gives an equation of degree three, therefore (B) is a cubic hypersurface. Furthermore, (C) is given by the vanishing of the three 2×2 minors of M, hence is the intersection of three quadric hypersurfaces and since $\dim C = 2$, it is a complete intersection, so $(C) \subset \mathbb{P}^5$ is a complete intersection of three quadrics and $\deg(C) = 8$. I suspect that (B) is a nonsingular hypersurface except along (C). It follows that (A) is a dense Zariski open subset of $\mathbb{P}^5$.

Remark 1.2. Although the proof is different, this example is also correct if char $k = 2$. Mohan-Kumar supplied a proof for Luis Aguirre to use in his proof in his 2019 PhD thesis.

We see in this example that the singular conics in families (B) and (C) can be deformed in the parameter space $\mathbb{P}^5$ to the smooth conics in family (A), so they are *smoothable*. In general, given a variety or a family of varieties, it is useful to know when the general member is smooth, as occurred in Example 1.1. If not, can one modify the general singular member to obtain a smooth variety? These are the kinds of questions the smoothing theorems attempt to answer. Here's a brief outline:

2. Singularities in algebraic geometry.
3. Theorems of Hironaka and Bertini, degeneracy loci of maps of vector bundles.
4. Recent work with Rao.

2. Singularities in algebraic geometry

In this section we define singular and smooth points on an algebraic variety, illustrating with various examples.

2.1. **Affine varieties.** For a field k, we define *affine n-space* $\mathbb{A}^n_k$ to be the set k^n with the Zariski topology, meaning that the closed sets are the common zero locus of a family of polyonomials $f_\alpha \in k[x_1, \dots, x_n]$, i.e. the closed sets $Z \subset \mathbb{A}^n$ are

$$Z(f_\alpha) = \{\overline{a} \in \mathbb{A}^n : f_\alpha(\overline{a}) = 0 \text{ for all } \alpha\}.$$

Given a closed set $Z \subset \mathbb{A}^n$, we define $I_Z = \{f \in k[x_1, \dots, x_n] : f(\overline{a}) = 0 \text{ for all } \overline{a} \in Z\}$. It is easy to show that I_Z is an ideal. If $k = \overline{k}$ is algebraically closed, Hilbert's Nullstellensatz tells us that $Z(I_Z) = Z$, so that we may take the f_α from the ideal I_Z. Furthermore Hilbert's basis theorem says that every ideal in $k[x_1, \dots, x_n]$ is finitely generated, so that we may take the f_α from a **finite** generating set for I_Z. The closed set $Z \subset \mathbb{A}^n$ is an *affine variety* if I_Z is a prime ideal. Recall that an ideal $P \subset k[x_1, \dots, x_n]$ is *prime* if $fg \in P \Rightarrow f \in P$ or $g \in P$.

Example 2.1. In $\mathbb{A}^2$ with coordinates x, y, the ideals $(y - x^2)$ and $(xy - 1)$ are prime and give affine conic varieties, while the ideal (xy) is not prime. Looking at Example 1.1, you might wonder about the non-prime ideal (x^2). It doesn't show up in this context, because $Z(x^2) = Z(x)$, so when we take the ideal defined by closed set, we get the prime ideal (x). On the other hand, (x^2) defines a *closed subscheme* $Z \subset \mathbb{A}^2$ in the language of Grothendieck's scheme theory. Topologically Z is the same as $Z(x)$, but has a different structure sheaf of regular functions.

2.2. **Singular points on affine varieties.** We begin with an example for motivation.

Example 2.2. Consider the variety $X \subset \mathbb{A}^3_{\mathbb{R}}$ defined by the equation $f = x^2+y^2-z^2 = 0$, the quadric cone. If we asked our Calculus III students about tangent planes to V, they would take a point $\overline{a} = (a_1, a_2, a_3) \in V$ and write down the equation

$$\nabla f(\overline{a}) \cdot (\overline{x} - \overline{a}) = 0.$$

This works well enough except when $\overline{a} = (0, 0, 0)$, when their "tangent plane" would turn out to be all of $\mathbb{R}^3$ because all the partials of f vanish there. A point on a variety should be smooth or nonsingular if it's tangent space has the right dimension.

The story over an arbitrary algebraically closed field k is the same. If $X \subset \mathbb{A}^n_k$ is defined by equations f_j, the tangent space to X at $\overline{a}$ is defined by

$$T_{X,\overline{a}} = \{\overline{x} \in \mathbb{A}^n : \sum_{i=1}^{n} \partial f_j/\partial x_i(\overline{a})(x_i - a_i) = 0 \text{ for all } f_j \in I_V\}$$

and $\overline{a}$ is a *nonsingular* or *smooth* point of V if $\dim T_{X,\overline{a}} = \dim X$. Over an algebraically closed field, there is a good theory of dimension for varieties (and Zariski closed subsets), namely $\dim V = \mathrm{tr.deg._k K(X)}$, where $K(X)$ is the function field of X, i.e. the fraction field of the integral domain $k[x_1, \dots, x_n]/I_X$. Since the tangent space $T_{X,\overline{a}}$ is defined by linear equations whose coefficients come from the derivative matrix $(\partial f_j/\partial x_i)$, the dimension of $T_{X,\overline{a}}$ is $n - \mathrm{rank}\ (\partial f_j/\partial x_i)(\overline{a})$. Thus $\overline{a} \in X$ is nonsingular point if

$$n - \dim X = \mathrm{rank}(\partial f_j/\partial x_i)(\overline{a}).$$

For $r \in \mathbb{Z}$, the set of points $\{\overline{a} : \mathrm{rank}(\partial f_j/\partial x_i)(\overline{a}) < r\}$ is a Zariski closed set, because it is defined by the vanishing of the $r \times r$ minors, which are polynomial equations. It follows that the locus of smooth points in X is Zariski-open in X, in other words, the set of singular points $\mathrm{Sing}\, X \subset X$ is a Zarsiki closed subset.

Theorem 2.3. *If $X \subset \mathbb{A}^n$ is an algebraic variety over $k = \overline{k}$, then $\mathrm{Sing}\, X$ is a proper closed subset of X, so X has a dense open subset of nonsingular points.*

Proof. The idea is to show that X is birational to a hypersurface in $\mathbb{A}^{\dim X+1}$ to reduce to the case when X is defined by a single equation $f = 0$. Then if $\mathrm{Sing}\, X = X$, the partials $\partial f/\partial x_i$ lie in (f), contradicting the degrees. $\square$

Example 2.4. A few plane curve singularities at the origin for $X \subset \mathbb{A}^2$. See figure 1.

(a) $xy = x^6 + y^6$ defines a node.
(b) $x^3 = y^2 + x^4 + y^4$ defines a cusp.
(c) $x^2 = x^4 + y^4$ defines a tacnode.
(d) $x^2y = xy^2 + y^4$ defines a triple point.

Example 2.5. We look examine the cusp $Z = Z(x^2 - y^3) \subset \mathbb{C}^2$, following Milnor [5] and Mumford [6, p. 13]. Let $B = \{(x, y) : |x|^2 + |y|^2 \leq 1$ be the closed unit ball about the origin in $\mathbb{C}^2 \cong \mathbb{R}^4$. Then $\partial B \cong S^3$ meets Z transversely in a one dimensional manifold, i.e. $Z \cap \partial B$ is a real curve on S^3, a possibly interesting knot or link. For $t \in \mathbb{C}$, the complex line $x = ty$ intersects the curve $x^2 = y^3$ in three points: substitution gives $t^2y^2 = y^3$ which has a double root $y = 0$ and simple root $y = t^2$, so that $x = t^3$. Thus each point on Z is uniquely written $x = t^3, y = t^2$ for $t \in \mathbb{C}$. Now $(x, y) = (t^3, t^2) \in \partial B$ iff $|t|^6 + |t|^4 = 1$. If λ is the unique positive solution to $\lambda^6 + \lambda^4 = 1$, then

$$Z \cap \partial B = \{(\lambda^3 e^{3i\theta}, \lambda^2 e^{2i\theta}) : 0 \leq \theta \leq 2\pi\}.$$

Observe that Z lies on the torus $T \subset \partial B$ given by

$$T = \{(x, y) : |x| = \lambda^3, |y| = \lambda^2\} \subset \partial B \cong S^3$$

and we see that $Z \cap \partial B$ is the torus knot corresponding to the rational slope $3/2$, so it is the trefoil knot. See figure 2. It appears that the singularity $x^r - y^s$ could be treated similarly for $(r, s) = 1$. It might be interesting to determine exactly which knots and links arise from singularities of complex plane curves. The node $xy = 0$ gives not a knot, but a link.

2.3. **Singular points on Projective varieties.** We define complex projective space by

$$\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^{n+1} - (0, 0, \dots, 0)/\sim$$

where $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if there is $\lambda \in \mathbb{C}^*$ for which $(a_0, \dots, a_n) = \lambda(b_1, \dots, b_n)$. The vanishing of a point $(a_0, \dots, a_n) \in \mathbb{P}^n$ at a polynomial $f(x_1, \dots, x_n)$ is not well-defined unless the polynomial f is *homogeneous*, meaning that is a linear combination of monomials of total degree d, or equivalently if $f(\lambda\overline{x}) = \lambda^d f(\overline{x})$ for all $\overline{x} \in \mathbb{A}^{n+1}$. A *homogeneous ideal* $I \subset S$ is one that can be generated by homogeneous polynomials. Now we can define zero sets $X = Z(f_\alpha) \subset \mathbb{P}^n$ of a family of homogeneous polynomials f_α and take these as the closed sets for the Zariski topology on $\mathbb{P}^n$. As before, we can always take the f_α to be a finite set. Hilbert's Nullstellensatz gives a bijection between zero sets and radical homogeneous ideals $I \subset S$, **except** for the *irrelevant maximal ideal* $(x_0, x_1, \dots, x_n)$ whose zero set does not correspond to any point in $\mathbb{P}^n$. The homogeneous coordinate ring for $\mathbb{P}^n$ is $S = \mathbb{C}[x_0, \dots, x_n]$, now considered as a graded ring

$$S = \bigoplus_{d \geq 0} S_d$$

where S_d is the vector space of homogeneous forms of degree d.

Definition 2.6. *A closed set $X \subset \mathbb{P}^n$ is a* variety *if I_X is a homogeneous prime ideal.*

Example 2.7. Looking at Example 1.1, we see that only conics of type A are varieties, because the ideals (xy) and (x^2) are not prime, whereas $(xy - z^2)$ is a prime ideal because $xy - z^2$ is irreducible.

To define singular and nonsingular points, we use the standard affine cover of $\mathbb{P}^n$. The Zariski closed set $Z(x_0)$ is the same as the space $\mathbb{P}^{n-1}$ obtained by ignoring one variable and the open compliment $U_{x_0} = \mathbb{P}^n - Z(x_0)$ is homeomorphic to $\mathbb{C}^n$ with its Zariski topology: we define a map $\phi : U_{x_0} \to \mathbb{A}^n$ and its inverse by

$$\phi(x_0, \ldots x_n) = (x_1/x_0, \ldots, x_n/x_0)$$

$$\phi^{-1}(y_1, \ldots, y_n) = (1, y_1, \ldots, y_n)$$

These are two-sided inverses, giving a bijection of sets. For continuity, notice that for $f \in \mathbb{C}[y_1, \ldots, , y_n]$ of degree d, the polynomial $F(y_0, \ldots, y_n) = y_0^d f(y_1/y_0, \ldots, y_n/y_0)$ is homogeneous of degree d and $\phi^{-1}(Z(f)) = Z(F)$, showing continuity of ϕ. Similarly if F is homogeneous of degree d, we can put $f(y_1, \ldots, y_n) = F(1, y_1, \ldots, y_n)$ and observe that $\phi(Z(F) \cap \mathbb{A}^n) = Z(f)$, establishing bicontinuity. If we do the analogous construction to build U_{x_i}, then $\mathbb{P}^n = \cup_i U_{x_i}$ is an open affine cover of projective space. If $X \subset \mathbb{P}^n$ is a projective variety, then X has the open cover $X \cap U_{x_i}$ so we may locally consider X as a subset of $\mathbb{A}^n$ and use the coordinates there to determine smoothness of points.

Example 2.8. Take $X \subset \mathbb{P}^2$ to be the conic $xy - z^2 = 0$. When we restrict X to the open affine $U_x \cong \mathbb{C}^2$, the equation becomes $y - z^2 = 0$ and we see a standard parabola. Similarly the restriction of X to U_y has equation $x - z^2$. The restriction to U_z has equation $xy - 1 = 0$ and we get a hyperbola.

Remark 2.9. One can also give a smoothness criterion directly for a projective variety. If $X \subset \mathbb{P}^n$ is a projective variety defined by homogeneously generated ideal $I_X = (f_1, \ldots, f_r)$, then $p \in X$ is a smooth point if and only if

$$\mathrm{codim}(X, \mathbb{P}^n) = \mathrm{rank}(\mathrm{\partial f_j/\partial x_i(p)}).$$

The proof uses Euler's formula, which says that if $f(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree d, then $\sum_{i=0}^{n} x_i \partial f/\partial x_i = df$.

2.4. **Zariski's theorem and regular local rings.** Grothendieck's scheme theory foundations for algebraic geometry dramatically changed the field in the early 1960s. His theory is based on commutative algebra. The local model for a scheme is the spectrum of a commutative ring A, that is $X = \mathrm{SpecA} = \{\mathrm{p} : \mathrm{p} \subset \mathrm{A} \text{ is a prime ideal}\}$ as a topological space equipped with a structure sheaf $\mathcal{O}_X$ of regular functions. The dimension of X is the dimension of the ring A, which is the longest length of a chain of prime ideals contained in A. This allows interaction with number theory, because one can consider schemes like $\mathrm{Spec}\,\mathbb{Z}$. The definition of nonsingular point is not applicable to general schemes not embedded in $\mathbb{A}^n_k$. In this section we mention Zariski's theorem and how to define a nonsingular scheme.

Let $p = \overline{a}$ be a point on a variety $X \subset \mathbb{A}^n$. An important object in algebraic geometry is the local ring of germs of regular functions at $\overline{a}$, it is defined as follows. The point $\overline{a}$

corresponds to the maximal ideal $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n) \subset k[x_1, \dots, x_n]$, which corresponds to the maximal ideal $\overline{\mathfrak{m}} \subset A_X = k[x_1, \dots, x_n]/I_X$. If we invert the polynomials in A/I_X which do not vanish at $\overline{a}$, or equivalently the polynomials not in $\overline{\mathfrak{m}}$, we obtain a the ring

$$\mathcal{O}_{X,p} = \{\frac{f}{g} : f, g \in A/I_X, g \notin \overline{\mathfrak{m}}\}$$

In this ring, $\overline{\mathfrak{m}}\mathcal{O}_{X,p}$ is the unique maximal ideal, thus $\mathcal{O}_{X,p}$ is a local ring. Geometrically, the local ring sees the variety X from the perspective of the point p. Zariski characterized whether p is a nonsingular point on X in terms of the structure of the ring $\mathcal{O}_{X,p}$.

Theorem 2.10. *Let $X \subset \mathbb{A}^n$ be a variety of dimension d and $p \in X$ with local ring $\mathcal{O}_{X,p}$ having maximal ideal $\mathfrak{m}$. Then p is a nonsingular point of X if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$.*

A local ring $(A, \mathfrak{m}, k)$ is a *regular local ring* if $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2$, where $k = A/\mathfrak{m}$ is the residue field. Thus the smooth points $p \in X$ are precisely those for which $\mathcal{O}_{X,p}$ is a regular local ring. This definition extends to arbitrary schemes.

Example 2.11. Let's compare the algebraic version by going back to the previous example. Take $p = (0, 0)$ on the conic $y - z^2 = 0$. The corresponding maximal ideal is (x, y), so the local ring at p is

$$\mathcal{O}_{X,p} = \{\frac{f}{g} : f, g \in k[y, z]/(y - z^2), g \notin (x, y)\} = ((k[y, z]/(y - z^2))_{(y,z)}$$

but $k[y, z]/(y - z^2) \cong k[z]$ with corresponding maximal ideal $\mathfrak{m} = (z)$. The dimension of the ring is 1 because of the chain $(0) \subset (z)$. Also the vector space $(z)/(z)^2$ is clearly of dimension 1 because it is generated by one element z.

Example 2.12. The scheme $X = \operatorname{Spec} \mathbb{Z}$ is nonsingular, because if $x \in X$ is a (closed) point, then x corresponds to a prime ideal $P \subset \mathbb{Z}$ and hence $P = (p)$ for some prime number p. The local ring is $\mathbb{Z}_{(p)}$ is a discrete valuation ring, it has dimension one (because (p) is a height one prime) and the maximal ideal (p) is generated by one element, so $(p)/(p^2)$ is a one dimensional vector space over the field $\mathbb{Z}/(p)$.

This is an important example in algebraic geometry, because sometimes characteristic p methods yield useful results in characteristic zero. If $X \subset \mathbb{P}^n_{\mathbb{Z}}$ is a variety, we could look at the fibers over each prime $p \in \mathbb{Z}$ to get a family of varieties $X_p \subset \mathbb{P}^n_{\mathbb{Z}/(p)}$, one for each prime p.

2.5. **Analytic isomorphism.** It might seem like one should use isomorphism of local rings to define isomorphism of singularities. This doesn't work out well, because the local rings $\mathcal{O}_{X,p}$ remember the birational equivalence class of X through the function field $K(X)$, which is the same as the fraction field for $\mathcal{O}_{X,p}$. Thus $\mathcal{O}_{X,p}$ carries too much information about X, in particular, it sees an open dense set of points. To get closer, we use analytic isomorphism. For this, we define the *completion* of a local ring $(A, \mathfrak{m})$ to be the inverse limit $\hat{A} = \varprojlim A/\mathfrak{m}^n$. Then we define two points $p \in X$ and $q \in Y$ to be *analytically isomorphic* if $\hat{\mathcal{O}}_{X,p} \cong \hat{\mathcal{O}}_{Y,p}$ as k-algebras. Note that if $(A, \mathfrak{m})$ is a local ring, then so is $\hat{A}$, moreover

(1) $\dim \hat{A} = \dim A$.
(2) A is a regular local ring $\iff$ $\hat{A}$ is a regular local ring.

Example 2.13. Let $X = \mathbb{A}^1$ and $p = 0$ the origin. Then the local ring of X at p is $k[x]_{(x)}$. If $A = k[x]$ and $\mathfrak{m} = (x)$, then we have the inverse system

$$A \to A/\mathfrak{m} \to A/\mathfrak{m}^2 \to \cdots \to A/\mathfrak{m}^n \to \dots$$

and the direct limit is the power series ring $k[[x]]$. Similarly if $p \in \mathbb{X} = A^n$ is a point, then $\hat{\mathcal{O}}_{X,p} \cong k[[x_1, \dots, x_n]]$.

The following theorem (Cohen structure theorem) says that the previous example captures the analytic isomorphism class of any smooth point of a variety over the field k:

Theorem 2.14. *Let $(A, \mathfrak{m})$ be a complete local ring containing a field with residue field $k = A/\mathfrak{m}$. If A is regular of dimension d, then $A \cong k[[x_1, \dots, x_d]]$.*

Example 2.15. A few surface singularities $X \subset \mathbb{A}^3$.

(1) $x^2 + y^2 = z^2$ gives a conical double point at the origin.
(2) $xy = x^3 + y^3$ defines a surface singular long the entire z-axis.
(3) $xy^2 = z^2$ defines a pinch point at the origin.
(4) Rational double points (a certain kind of isolated surface singularity) have been classified: There are three basic types called the ADE classification.
    (a) $A_n : xy + z^{n+1} = 0$.
    (b) $D_n : x^2 + y^2 z + z^{n-1} = 0$.
    (c) $E_6 : x^2 + y^3 + z^4 = 0$.
    (d) $E_7 : x^2 + y^3 + yz^3 = 0$.
    (e) $E_8 : x^2 + y^3 + z^5 = 0$.

3. Hironaka's theorem

Here we explain the statement of Hironaka's theorem [3, 4]. To set up the vocabulary, we need to define blow ups and simple normal crossing singularities.

3.1. **Geometric blow up of the origin in $\mathbb{A}^n_k$.** Let $O = (0, 0, \dots, 0) \in \mathbb{A}^n$ be the origin. Take $x_1, \dots x_n$ as coordinates on $\mathbb{A}^n$ and let $y_1, \dots, y_n$ be projective coordinates on $\mathbb{P}^{n-1}$. We define an algebraic set $X \subset \mathbb{A}^n \times \mathbb{P}^n$ by the equations

$$x_i y_j - x_j y_i = 0, 1 \leq i, j \leq n$$

and compose with the first projection:

$$\begin{array}{ccc} X & \subset & \mathbb{A}^n \times \mathbb{P}^{n-1} \\ & \varphi \searrow & \downarrow \\ & & \mathbb{A}^n \end{array}$$

Some observations about the map $\phi : X \to \mathbb{A}^n$.

(1) The restriction $\varphi : X - \varphi^{-1}(O) \to \mathbb{A}^n - O$ is an isomorphism. If $(a_1, \dots, a_n) \neq O$, say $a_1 \neq 0$, we can use the equations to write $y_i = (a_i/a_1)y_1$, so that up to scalar, $(y_1, \dots y_n) = (a_1, \dots a_n)$, thus there is a unique point in X lying over

each point $p \neq O$. Moreover, we can write down the inverse map, it is given by $(a_1, \dots a_n) \mapsto (a_1, \dots a_n) \times (a_1, \dots, a_n)$.

(2) $\varphi^{-1}(O) = \{O\} \times \mathbb{P}^{n-1}$, because when all the $x_i = 0$, the equations place no restriction on the y_i.

(3) X is irreducible. I could write down some calculations, but the reason is roughly this. $\mathbb{P}^{n-1}$ is literally the space of lines through the origin (by definition!). Now if $L_v \subset \mathbb{A}^n$ is the line through the origin corresponding to the tangent direction $v \in \mathbb{P}^{n-1}$, then one can calculate that $\varphi^{-1}(L)$ is the union of $\overline{\varphi^{-1}(L - O)}$ and $\varphi^{-1}(O)$ and that these two intersect at the point v. In particular, every point $q \in \varphi^{-1}(O) \cong \mathbb{P}^{n-1}$ is in the closure of the irreducible space $X - \varphi^{-1}(O)$, so X is an irreducible variety and $\phi : X \to \mathbb{A}^n$ is a birational map, i.e. is an isomorphism on a dense open set.

Example 3.1. Take $n = 2$. Here we can visualize the construction over $k = \mathbb{R}$. The lines through the origin are parametrized by $\mathbb{RP}^1$ and one can imagine the picture of a spiral staircase over $\mathbb{R}^2$ with the lines winding up the staircase about the central pole. A picture of this can be found in just about every algebraic geometry text, for example [2, p. 29] or [7, p. 100]. Looking at this picture, if $Y \subset \mathbb{R}^2$ is the nodal curves $xy = x^4 + y^4$ we drew before, then $\tilde{Y} = \overline{\phi^{-1}(Y - O)}$ is a nonsingular curve in which the node at the origin has been replaced by two distinct points. Similarly the cuspical cubic curve Y given by $y^2 = x^3$ yields a nonsingular curve $\tilde{Y} = \overline{\phi^{-1}(Y - O)}$ which is homeomorphic to Y via the map ϕ, but now $\tilde{Y}$ is tangent to the central $\mathbb{P}^1$ and the kink at the origin is ironed out.

With this example in mind, we define the blow up of $\mathbb{A}^n$ at O to be X, usually denoted $\tilde{\mathbb{A}^n}(0)$ to indicate the origin was blown up. For $O \in Y \subset \mathbb{A}^n$, the blow up of Y at O is $\overline{\phi^{-1}(Y - O)}$.

3.2. **Algebraic blow up of the origin in $\mathbb{A}^n$.** There is an algebraic way to think about blow up that leads to a more general and flexible notion. Looking back at Example 3.1, the equations for the blow up live in the ring $R = k[x, y][X, Y]$, where I'm now using x, y instead of x_1, x_2 and X, Y for y_1, y_2. The ring R corresponds to the space $\mathbb{A}^2 \times \mathbb{P}^1$ in the sense that (closed) points of the space correspond to height 3 prime ideals which are homogeneous in the variables X, Y: the point $(a, b, c, d) \in \mathbb{A}^2 \times \mathbb{P}^1$ corresponds to the ideal $(x - a, y - b, cY - dX) \subset R$. The analogous construction of a the space $\mathbb{A}^2 \times \mathbb{P}^1$ from the ring R is called the Proj construction, i.e. $\mathbb{A}^2 \times \mathbb{P}^1 = \text{Proj R}$. This space is a scheme, defined locally as a space of prime ideals with the Zariski topology. Similarly, if A is any ring and $R = A[X_1, \dots X_n]/I$ is a graded algebra over A defined by homogeneous ideal I, one can construct the space Proj R which comes with a natural projection to Proj R $\to$ Spec A.

Looking back at Example 3.1, $X = \tilde{\mathbb{A}^n} = \text{Proj k[x, y][X, Y]/(xY - yX)}$, but let's look more closely at this ring and it's graded pieces. Thinking of $A = k[x, y]$ as the scalars and X, Y linear forms over A, the graded pieces are $k[x, y]$ in degree zero, $Xk[x, y] \oplus Yk[x, y]/(xY - yX)$ in degree one, $X^2k[x, y] \oplus XYk[x, y] \oplus Y^2k[x, y]$ in degree two and so on. The degree one part is isomorphic as $k[x, y]$-module to the ideal (x, y), which is generated by x, y subject to the cross relation $xy = yx$. The degree two part is isomorphic as $k[x,y]$-module to the ideal $(x,y)^2$. In general, putting together these graded pieces we find that as graded algebras over $k[x,y]$ there is an isomorphism

$$k[x,y][X,Y]/(xY-yX) = k[x,y] \oplus (x,y) \oplus (x,y)^2 \oplus \cdots = \oplus_{d\geq 0}(x,y)^d$$

This motivates a general definition of blow up. If X is a variety covered by open affine sets $U_i = \mathrm{Spec\ A_i}$ and $Z \subset X$ is a closed subset given by local ideal $J_i \subset A_i$, then we construct the blow up $\tilde{X} \to X$ locally by $\mathrm{Proj\ \oplus_{d\geq 0} J_i^d \to Spec\ A_i}$. These spaces are independent of the choice of generators for the ideals J_i and as such glue together to give the space $\tilde{X} \to X$. The preimage $E = \phi^{-1}(Z)$ is callled the exceptional divisor for the blow up, it always has codimension one in $\tilde{X}$ and is locally defined by a single equation.

Example 3.2. We could blow up the linear subspace $Z \subset \mathbb{A}^n$ defined by the ideal $I_Z = (x_1, \ldots x_r)$ so that Z is a smooth subvariety of dimension $n-r$. The result is $\phi : \tilde{\mathbb{A}^n} \to \mathbb{A}^n$ which is an isomorphism away from Z and Z has been replaced by $Z \times \mathbb{P}^{r-1}$ and is defined by those same quadratic equations $x_i y_j - x_j y_i = 0$. The exceptional divisor is $E = \mathbb{Z} \times \mathbb{P}^{r-1} \subset \tilde{\mathbb{A}^n}$, an irreducible subvariety of dimension $n-1$. In general, if $Z \subset X$ is an inclusion of nonsingular varieties and Z has codimension r, then the exceptional divisor of the blow up $\tilde{X} \to X$ along Z is a $\mathbb{P}^{r-1}$-bundle over Z.

The algebraic blow up has the following properties. If we blow up $Z \subset X$, say $X \subset \mathbb{A}^n$, $I_Z \subset A_X = k[x_1, \ldots, x_n]/I_X$ so that the blow up is locally defined by $\tilde{X} = \mathrm{Proj} \oplus_{\mathrm{d}\geq 0} \mathrm{I_Z^d}$, we say that Z is the *center* of the blow up and $E = \varphi^{-1}(Z)$ is the *exceptional divisor* giving a diagram

$$\begin{array}{ccc} E & \subset & \tilde{X} \\ \downarrow & & \varphi \downarrow \\ Z & \subset & X \end{array}$$

(1) $\varphi : \tilde{X} - E \to X - Z$ is an isomorphism.
(2) The exceptional divisor $E \subset \tilde{X}$ is of codimension one and locally defined by a single equation.
(3) In the best case that $Z \subset X$ are both nonsingular and Z has codimension r, the map $\varphi : E \to Z$ is a $\mathbb{P}^{r-1}$-bundle, in particular both E and $\tilde{X}$ are nonsingular.
(4) If $Z \neq X$, the $\varphi : \tilde{X} \to X$ is a birational projective morphism, meaning that φ is an isomorphism on a dense open set (see the first property) such that there is a closed embedding $g : \tilde{X} \to X \times \mathbb{P}^N$ with $\varphi = \pi_1 \circ g$.

The last property has a surprising converse:

Theorem 3.3. *If $f : Y \to X$ is any birational projective morphism, then there is a closed subscheme $Z \subset X$ and an isomorphism $g : \tilde{Y} \to \tilde{X}$, where $\varphi : \tilde{X} \to X$ is the blow up of X at Z so that $f = \varphi \circ g$.*

In other words, every birational projective morphism is a blow up!

Here is Hironaka's theorem (Ann. of Math. 79 (1964), 109–203, 205–326), it got split into two parts because of the change in issue.

Theorem 3.4. *Assume k is an algebraically closed field of characteristic zero. Let X be a nonsingular variety which contains the possibly singular variety Y. Then there is a* *sequence of blow ups with smooth centers $Z_i \subset X_{i-1}$*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \dots X_m = \tilde{X}$$

such that the strict with transforms $Y_i = \overline{\sigma_i^{-1}(Y_{i-1} - Z_{i-1})}$ we have $\tilde{Y} = Y_m$ is nonsingular. Furthermore, the union of exceptional divisors in $\tilde{Y}$ has at worst simple normal crossing singularies.

Remarks 3.5. Some quick comments:

(1) Zariski proved resolution of singularities for threefolds over a field of characteristic zero in 1944 [8]
(2) Abhyankar proved this result if Y is a surface and char $k \neq 2, 3, 5$ [1].
(3) Nowadays there are proofs of Hironaka's theorem less than 40 pages long, but nobody has improved on his result.
(4) There's a nice expository article in the AMS Bulletin on this topic [**?**].

Example 3.6. To see Hironaka's theorem in a simple setting, let $X = \mathbb{A}^3$ and $Y \subset X$ be the surface with equation $xy - z^3 = 0$. Then Y is singular only at the origin. If $X_1 = \tilde{X} \to X$ is the blow up of X at the origin and the strict transform $Y_1 = \tilde{Y} \to Y$ is the strict transform, one can check that $E_1 \cap \tilde{Y} \cong \mathbb{P}^1$ and that $\tilde{Y}$ has exactly one singular point p in $E_1 \cap \tilde{Y}$. Moreover, the local equation of the singular point p has the form $xy - z^2 = 0$. If $X_2 \to X_1$ is the blow up at p, the strict transform $Y_2 \to Y_1$ is a nonsingular surface. The exceptional divisor is $E_2 \subset Y_2$ and $E_2 \cong \mathbb{P}^1$. The two exceptional divisors meet transversely, forming what look like intersecting x and y axes in $\mathbb{P}^2$.

References

[1] S. Abhyankar, Resolution of singularities of embedded algebraic surfaces, Springer Monogr. Math., Springer-Verlag, Berlin, 1998, xii+312 pp.

[2] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer-Verlag, 1978.

[3] H. Hawser, The Hironaka theorem on resolution of singularities (or: A proof we always wanted to understand), Bull. Amer. Math. Soc. (N.S.) 40 (2003) 323–403.

[4] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, Ann. Math. 79 (1964) 109–203.

[5] ———, Resolution of singularities of an algebraic variety over a field of characteristic zero, II, Ann. Math. 79 (1964) 205–326.

[6] J. Milnor, *SIngular points of complex hypersurfaces*, Annals of Math. Studies 61, Princeton Univ. Press, 1968.

[7] D. Mumford, *Algebraic Geometry I, Complex projective varieties* Grundlehren der mathematischen Wissenschaften 221, Berlin-Heidelberg-New York: Springer-Verlag, 1976.

[8] I. Shafarevich, Basic algebraic geometry, Springer Study Ed., Springer-Verlag, Berlin-New York, 1977.

[9] O. Zariski, Reduction of the singularities of algebraic three dimensional varieties, Ann. of Math. 45 (1944) 472–542.