THE TWENTY-SEVEN LINES ON THE SMOOTH CUBIC $X \subset \mathbb{P}^3_{\mathbb{C}}$

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ABSTRACT. This talk is intended to explain configuration of the 27 lines on the smooth cubic surface in $\mathbb{P}^3$ and their symmetry group, which is isomorphic to the Weyl group associated to the rational double point surface singularity E_6.

1. LINE BUNDLES ON COMPLEX PROJECTIVE SPACE

Definition 1.1. A **complex line bundle** on a topological space X can be thought of as a map $\pi : L \to X$ which locally on X looks like $X \times \mathbb{C} \to X$. By looking at the space of holomorphic (or algebraic) sections to π on open subsets $U \subset X$, one can also understand L as an invertible sheaf on X. We will use the invertible sheaf understanding here.

Example 1.2. On any holomorphic manifold M one has the **trivial bundle** $\mathcal{O}_M$ whose sections are the holomorphic functions on open sets. From the topological viewpoint is the projection $M \times \mathbb{C} \to M$.

Given a line bundle L on X and an open cover of X, the transition functions defining L define a Čech cocycle $\gamma \in H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^* \subset \mathcal{O}_X$ is the group of units, i.e. nonvanishing holomorphic functions. This association gives an isomorphism between the group $H^1(X, \mathcal{O}_X^*)$ and the group of isomorphism classes of line bundles on X, the latter group is called the **Picard group of** X, denoted $\operatorname{Pic} X$. Thus there is an isomorphism

$$\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*).$$

Remark 1.3. If L is a line bundle on X and $s \in \Gamma(X, L)$ is a non-zero holomorphic global section, then $(s)_0 = \{x \in X : s_x = 0\}$ is a Cartier divisor on X, i.e. a closed subscheme defined locally by a single equation, hence it has pure codimension one and can be written $(s)_0 = D = \sum m_i Y_i$ where Y_i are the irreducible components and m_i are the corresponding multiplicities. Conversely if D is a sum of irreducible codimension one subvarieties with multiplicities (an effective Weil divisor), its ideal sheaf $\mathcal{I}_D \subset \mathcal{O}_X$ is locally defined by a single equation, so $\mathcal{I}_D$ is a line bundle: dualizing the inclusion $\mathcal{I}_D \subset \mathcal{O}_X$ gives $\mathcal{O}_X \to \mathcal{I}_D^{-1} = L$, i.e. a global section $s \in \Gamma(X, L)$ and one locally checks that $(s)_0 = D$. This association gives an isomorphism $\operatorname{Pic} X \cong \operatorname{Cl} X$, where $\operatorname{Cl} X$ is the group of Weil divisors modulo prinicipal divisors.

Projective space $X = \mathbb{P}^n_{\mathbb{C}}$ supports many line bundles. The complex exponential function gives rise to an exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

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where $\mathbb{Z}$ refers to the constant sheaf. Taking cohomology gives a long exact sequence of which the fragment

$$\cdots \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(\mathcal{O}_X) \to \ldots$$

exhibits an isomorphism $\operatorname{Pic} X \cong H^1(\mathcal{O}_X^*) \cong H^2(X, \mathbb{Z})$, the latter isomorphism due to the vanishings $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$. Since $H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$, there is one line bundle (up to isomorphism) for each integer giving an isomorphism

$$\operatorname{Pic} \mathbb{P}^n_{\mathbb{C}} \cong \mathbb{Z}$$

where the group operation on the left is tensor product.

Example 1.4. The group $\operatorname{Pic} \mathbb{P}^n_{\mathbb{C}}$ has two generators corresponding to $\pm 1 \in \mathbb{Z}$. One of them is the **tautological line bundle** used by differential geometers. As a map $L \to \mathbb{P}^n_{\mathbb{C}}$, the fibers can be identified with the actual complex lines in $\mathbb{C}^{n+1}$ through the origin, hence the term "line bundle". As an invertible sheaf, it can be thought of as the ideal sheaf $\mathcal{I}_H \subset \mathcal{O}_{\mathbb{P}^n}$ of holomorphic functions which vanish on a fixed hyperplane $H \subset \mathbb{P}^n$.

Definition 1.5. On projective space $X = \mathbb{P}^n$, the line bundle $\mathcal{O}(1)$ is the inverse to the tautological line bundle from Example 1.4. It can also be constructed algebraically by sheafifying the graded module $S(1)$, where $S = \mathbb{C}[x_0, x_1, \ldots, x_n]$ is the polynomial ring.

Remarks 1.6. The line bundle $\mathcal{O}(1)$ plays an important role in algebraic geometry (see next section). Note the following:

(a) Because $\mathcal{O}(1)$ generates $\operatorname{Pic} X \cong \mathbb{Z}$, we can obtain any line bundle from $\mathcal{O}(1)$ by taking tensor powers and duals. Thus for $n > 0$ we use the notation $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$ and for $m < 0$, $\mathcal{O}(m)$ is the dual to $\mathcal{O}(-m)$. In particular, the tautological line bundle of Example 1.4 is written $\mathcal{O}(-1)$.

(b) The global holomorphic sections $\Gamma(\mathbb{P}^n, \mathcal{O}(1))$ can be identified with the $(n+1)$-dimensional vector space of linear forms in the homogeneous coordinates on $\mathbb{P}^n$, i.e. with the vector space $\langle x_0, x_1, \ldots, x_n \rangle$. In general $\Gamma(\mathbb{P}^n, \mathcal{O}(m))$ appears as the space of homogeneous n-forms in the variables x_i, so that $\Gamma(\mathbb{P}^n \mathcal{O}) = \mathbb{C}$ are the constants and $\Gamma(\mathbb{P}^n \mathcal{O}(m)) = 0$ for $m < 0$.

2. Maps to projective space

The importance of $\mathcal{O}(1)$ on $\mathbb{P}^n$ springs from the following.

Construction 2.1. Let $f : X \to \mathbb{P}^n$ be a holomorphic (algebraic) map. The pull-back $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ is then a line bundle on X and $f^*(x_0), \ldots f^*(x_n) \in \Gamma(X, L)$ generate an $(n+1)$-dimensional subspace $V \subset \Gamma(X, L)$.

This motivates the following definition.

Definition 2.2. A **linear system** on an algebraic variety X consists of a line bundle L on X and a vector subspace $V \subset \Gamma(X, L)$. The **dimension** of the linear system is $\dim V - 1$.

Remark 2.3. Geometrically we think of the elements of the linear system not as the sections $s \in V$, but as the effective divisor $(s)_0 \subset X$ where s vanishes. For example when

$X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ and $s = x_0 \in \Gamma(X, L)$, we would visualize s as the hyperplane $H = \{x_0 = 0\}$. This explains why the dimension is $\dim V - 1$, because if two sections agree up to scalar, then they give rise to the same zero section.

Not every linear system comes from an embedding $j : X \hookrightarrow \mathbb{P}^n$ or even from a map to projective space, but these conditions have well-known characterizations.

Definition 2.4. Let $V \subset \Gamma(X, L)$ be a linear system on X.

(1) V is **base point free** if for each $x \in X$, there is some $s \in V$ such that $s_x \neq 0$ in the stalk L_x.
(2) V **separates points** if for each pair of points $x, y \in X$, there is some $s \in V$ with $s_x = 0$ and $s_y \neq 0$.
(3) V **separates tangent vectors** if for each point $x \in X$, the stalks s_x of the elements $s \in V$ span the vector space $L_x \otimes \mathcal{O}_{x,X}/\mathfrak{m}_x$.

With these definitions, we have the following result, whose content is that maps to projective space (and in particular projective embeddings) are intrinsically linked with linear systems via Construction 2.1:

Theorem 2.5. *Let $V \subset \Gamma(X, L)$ be a linear system on an algebraic variety*[1] *X. Then*

(a) *There's a morphism $f : X \to \mathbb{P}^n$ with $f^*(\mathcal{O}_{\mathbb{P}^n}(1)) = L$ and $V = \langle f^*(x_0), \dots f^*(x_n)\rangle$ if and only if V is base point free.*
(b) *There is a closed immersion $f : X \hookrightarrow \mathbb{P}^n$ as above if and only if V separates points and tangent vectors.*

Remark 2.6. Part (b) was an element of Kodaira's embedding theorem. He proved that if L is *positive in the sense of Kodaira*[2], then some tensor power $L^{\otimes m}$ with $m > 0$ has global holomorphic sections in $\Gamma(X, L)$ satisfying condition (b), from which he proved that there is a holomorphic embedding of X into a projective space [6]. Nowadays we say that L is **ample** if some tensor power $L^{\otimes m}$ comes from an embedding into projective space, and that L is **very ample** if we may take $m = 1$.

Example 2.7. We give a few examples.

(a) Taking $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$ and $V = \Gamma(X, L)$ the corresponding map is simply an isomorphism $\mathbb{P}^n \cong \mathbb{P}^n$. Thus the automorphisms of $\mathbb{P}^n$ come from linear automorphisms of the vector space $V = \Gamma(X, L)$ modulo scalar, i.e. $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n)$.
(b) Take $X = \mathbb{P}^1$, $L = \mathcal{O}(d)$ and $V = \Gamma(X, L)$. Then V is a vector space of dimension $d + 1$ and we obtain a closed immersion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ whose image is the *rational normal curve of degree d*. When $d = 2$ we obtain the plane conic and when $d = 3$ we obtain the twisted cubic curve.
(c) Taking $X = \mathbb{P}^2$, $L = \mathcal{O}(2)$ and $V = \Gamma(X, L)$ to be the 6-dimensional vector space generated quadratic forms in x_0, x_1, x_2 gives a closed immersion $X \hookrightarrow \mathbb{P}^5$. The image is the *Veronese surface*.

[1] This theorem holds for any scheme of finite type over $\mathbb{C}$.
[2] This was described in my GAGA talk last semester.

(d) Continuing the previous example, the vector space $V \subset \Gamma(\mathbb{P}^2, \mathcal{O}(2))$ generated by the quadrics $x_0^2, x_1^2, x_2^2, x_1(x_0 - x_2), x_2(x_0 - x_1)$ also can be shown to separate points and tangent vectors, giving an embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^4$, also called the Veronese surface. It is obtained from the Veronese in $\mathbb{P}^5$ by projecting from a point.

(e) The space of all cubics in $\mathcal{O}(3)$ on $\mathbb{P}^2$ gives an embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$.

3. Lines on the smooth cubic in $\mathbb{P}^3$

Understanding the lines on cubic surfaces was trendy mathematics in the mid to late 1800s. The result that the smooth cubic contains 27 lines was proved through letters mailed between George Salmon and Arthur Cayley in 1849 [1]: basically Cayley proved that the number of lines must be bounded and Salmon proved that the expected number of such lines is 27. Two whole books have been devoted to the subject [4, 10]. Here's a quote from Yu Manin [7, p.112]:

"Their elegant symmetry both enthrals and at the same time irritates: what use is it to know, for instance, the number of coplanar triples of such lines (forty five) or the number of Schläfli double sixes (thirty six)? The answer to this rhetorical question is one of the two recurring themes in this chapter. In just a few words: the classes of lines on $X \otimes \overline{k}$ generate the group $N(X) = \mathrm{Pic}(X \otimes \overline{k})$ and the action of the Galois group $G = \mathrm{Gal}(\overline{k}/k)$ on $N(X)$ preserves symmetry and it implicitly contains an extremely large amount of information on the arithmetic and geometry of X"

3.1. **Construction of the smooth cubic X.** We continue with example 2.7 (e). For $\mathbb{P}^2$, instead of taking $V \subset \Gamma(\mathbb{P}^2, \mathcal{O}(3))$ to the space of all cubic forms, we take the cubic forms which vanish at six points $p_1, p_2, \dots p_6$ chosen generally enough that

(a) No three of the p_i are collinear.
(b) The p_i are contained in no conic.

The vector space V has dimension 4, but clearly is not base point free, since by construction the points $p_1, \dots p_6$ are base points for the linear system. Ignoring the base points, we would obtain an embedding

$$\mathbb{P}^2 - \{p_1, p_2, p_3, p_4, p_5, p_6\} \hookrightarrow \mathbb{P}^3.$$

The closure of the image is a smooth cubic in $\mathbb{P}^3$, but we can understand what this closure looks like. There is a general theorem in algebraic geometry that says that if $V \subset \Gamma(X, L)$ is a linear system with base locus B, then V gives a map $X - B \to \mathbb{P}^n$ as above and that if $\tilde{X} \to X$ is the blow-up of the base locus, then the rational map $X \to \mathbb{P}^n$ extends to $\tilde{X}$. In the situation above, the base locus is $B = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ and we can complete the map to

$$\begin{array}{ccccc} & & \widetilde{\mathbb{P}^2} & & \\ & \pi \swarrow & & \searrow \varphi & \\ \mathbb{P}^2 & \supset & \mathbb{P}^2 - B & \hookrightarrow & \mathbb{P}^3 \end{array}$$

The map φ is a closed immersion whose image X is the closure we seek, a smooth cubic surface in $\mathbb{P}^3$.

Theorem 3.1. *Every smooth cubic surface $S \subset \mathbb{P}^3$ arises by the construction above.*

Proof. If $S \subset \mathbb{P}^3$ is a smooth surface, the adjunction formula gives $K_S = (K_{\mathbb{P}^3} + L)|_S$, where K_X denotes the canonical sheaf on X (analytically speaking, this is the sheaf of differential d-forms, if $d = \dim X$) and $L \in \operatorname{Pic} \mathbb{P}^3$ is the line bundle corresponding to S (see Remark 1.3). Now $K_{\mathbb{P}^3} \cong \mathcal{O}(-4)$ and $L = \mathcal{O}(3)$, so we find that $K_S = \mathcal{O}_S(-1)$, meaning that S is a *Del Pezzo surface.* These have been classified [7, §24] and in this case S is obtained from $\mathbb{P}^2$ by blowing up six points in general position as above. $\square$

3.2. **Divisors on the smooth cubic** X**.** We showed above that $\operatorname{Pic} \mathbb{P}^2 \cong \mathbb{Z}$ generated by $\mathcal{O}(1)$. If we use the isomorphism $\operatorname{Pic} \mathbb{P}^2 \cong \operatorname{Cl} \mathbb{P}^2$, we would write this generator as a line $l \subset \mathbb{P}^2$, thought of as an effective Cartier divisor. The blow up $\pi : \tilde{X} \to X$ construction replaces the six points p_i with exceptional divisors E_i which are isomorphic to $\mathbb{P}^1$, essentially each tangent direction to p_i on $\mathbb{P}^2$ has been replaced by an actual corresponding point in E_i. From the theory of blow ups, one knows that $\operatorname{Cl} \mathbb{P}^2 \cong \mathbb{Z}^7$ generated by $L = \pi^*(l)$ and the E_i.

Intersection theory works on any smooth variety, so on X there is a bilinear pairing

$$\operatorname{Cl} X \times \operatorname{Cl} X \to \mathbb{Z}$$

obtained by extending the rule $D.E = \#\{D \cap E\}$ when D and E meet transversely. For blow-ups the intersection theory is well-known: the intersection pairing on X is given by

$$L^2 = 1, \quad L.E_i = 0, \quad E_i.E_j = -\delta_{i,j}$$

Intersection theory can be used to compute many integer invariants of divisors. If $C \subset X$ is an effective divisor (i.e. a curve), given by the class $C \sim aL - \sum b_i E_i$, then the basic invariants of C are given by

$$\begin{array}{rcl} \deg C &=& 3a - \sum b_i \\ C^2 &=& a^2 - \sum b_i^2 \\ p_a(C) &=& \frac{1}{2}(C^2 - \deg C) + 1 \end{array} \tag{1}$$

3.3. **The lines on** X**.** Using the formulas (1) above, we are in position to hunt for the straight lines on X, which are precisely the curves of degree 1 and genus 0.

Example 3.2. We apply the formulas above to some curves on X to compute their invariants.

(a) The exceptional divisors $E_i \subset X$ are effective. As a divisor written $aL - \sum b_i E_i$, this is the case where $a = 0, b_j = -\delta_{i,j}$. The formulas give $\deg E_i = 1, E_i^2 = -1$ and $p_a(E_i) = 0$, so the exceptional divisors E_i give **SIX** straight lines on X.

(b) The class of the line $l \subset \mathbb{P}^2$ is represented by $L \in \operatorname{Cl} X$. We have $\deg L = 3, L^2 = 1, p_a(L) = 0$, so these are rational cubic curves on X, not lines.

(c) Let $C \subset X$ correspond to a line $l \subset \mathbb{P}^2$ passing through p_1. Then $\pi^{-1}(l) = L + E_1$, so the strict transform would be written $C \sim L - E_i$, reflecting the fact that l passes through p_1. We have $\deg C = 2, C^2 = 0, p_a(C) = 0$, so C appears as a conic.

(d) Let C correspond to a line passing through p_1 and p_2. Arguing as above, $C \sim L - E_1 - E_2$ and we find that $\deg C = 1, C^2 = -1, p_a(C) = 0$, so we have uncovered another line. Choosing different pairs of points p_i, p_j, we have discovered **FIFTEEN** lines $F_{i,j} \sim L - E_i - E_j$.

(e) Similar to (b) above, a general conic in $\mathbb{P}^2$ appears on X as a rational curve of degree 4, but looking at conics passing through some of the p_i is more interesting.

(f) Cutting to the chase, there is a unique conic $q \subset \mathbb{P}^2$ passing through any 5 points, say $p_1, \ldots p_5$. The corresponding curve on X is $C \sim 2L - E_1 - E_2 - E_3 - E_4 - E_5$ and we find that $\deg C = 1, C^2 = -1, p_a(C) = 0$. This produces **SIX** more lines $G_j \sim 2L - \sum_{i \neq j} E_i$.

Theorem 3.3. ([3, V,Theorem 4.9]) *The 27 lines on $X \subset \mathbb{P}^3$ are those listed above.*

3.4. **Intersections of the lines.** The intersection theory on X tells us exactly which lines intersect. The answer is this:

(1) The E_i are skew lines, as are the G_j. $F_{i,j} \cap F_{k,l} \neq \emptyset \iff i \neq k, l$ and $j \neq k, l$.
(2) $E_i \cap G_j \neq \emptyset \iff i \neq j$.
(3) $F_{i,j} \cap E_k \neq \emptyset \iff k = i, j$.
(4) $F_{i,j} \cap G_k \neq \emptyset \iff k = i, j$.

Remark 3.4. It is a bit difficult to draw (visualize) this configuration of lines and their intersections. If you want to see them, you should get a ticket to the 1894 World Expo in Chicago, where Felix Klein displayed plaster models of such cubics over $\mathbb{R}$ exhibiting the intersections above.

Remark 3.5. Any twelve lines like E_i, G_j such that the E_i and G_k are skew families and $\#\{E_i \cap G_j\} = 1 - \delta_{i,j}$ are called a *Schläfli's double six.* Hilbert and Cohn-Vossen proved that any Schläfli double six in $\mathbb{P}^3$ are contained in a unique nonsingular cubic surface [5]. This is an analog of the much easier fact that any skew three lines are contained in a unique nonsingular quadric surface.

3.5. **Symmetries of the 27 lines.** Using the intersection computations above, you will find that each of the 27 lines has self-intersection -1 and meets exactly 10 of the other lines, so you might expect some degree of symmetry among them. Indeed, the symmetry group acting on the configuration is rather large, as we will show in a moment. The key to realizing the symmetries is geometric in nature:

Theorem 3.6. *Let $L_1, \ldots L_6$ be any skew six lines on X. Then there is a blow up of $\mathbb{P}^2$ as contructed above for which the six exceptional divisors E_i become the L_i.*

Remark 3.7. This means that any six mutually skew lines on X look like any other six mutually skew lines. Note that the choice of E_i determines the roles of the other 21 lines, because $F_{i,j}$ is the unique line which meets both E_i and E_j and G_k is the unique line which misses E_k and intersects the remaining E_i.

Proposition 3.8. *The automorphism group G of the configuration of the 27 lines has order* $51,840$.

Proof. Given the theorem above, it suffices to count the number of ordered 6-tuples of mutually skew lines, a relatively easy count:

- There are 27 choices for E_1.
- For E_1 fixed, there are 16 choices for E_2, namely $E_2, \ldots E_6, G_1, F_{i,j}$ with $i, j > 1$.
- For E_1, E_2 fixed, there are 10 choices for E_3, namely $E_3, \ldots E_6, F_{i,j}$ with $i, j > 2$.
- For E_1, E_2, E_3 fixed, there are 6 choices for E_4, namely $E_4, E_5, E_6, F_{4,5}, F_{4,6}, F_{5,6}$.

For E_1, E_2, E_3, E_4 fixed there remain only the lines $E_5, E_6, F_{5,6}$ left, but E_5 and E_6 both meet $F_{5,6}$, so $F_{5,6}$ is not available, giving only two choices between E_5 and E_6, when the last line must be the one not chosen. Therefore $|G| = 27 \cdot 16 \cdot 10 \cdot 6 \cdot 2 = 51,840$. $\square$

Example 3.9. We use linear systems to describe a symmetry of the 27 lines.
(a) Three non-collinear points $p_1, p_2, p_3 \in \mathbb{P}^2$ define the linear system $V \subset \Gamma(\mathbb{P}^2, \mathcal{O}(2))$ of conics which contain the p_i. The linear system V has base locus $B = \{p_1, p_2, p_3\}$ and $\dim V = 3$, for example if $p_1 = (1,0,0), p_2 = (0,1,0), p_3 = (0,0,1)$, then an explicit basis for V consists of the degenerate conics x_0x_1, x_0x_2, x_1x_2. This linear system gives a map $\mathbb{P}^2 - \{p_1, p_2, p_3\} \hookrightarrow \mathbb{P}^2$ which extends to the blow-up $X = \widetilde{\mathbb{P}^2}$ at B as in the construction of the smooth cubic earlier:

$$\begin{array}{ccccc} & & \widetilde{\mathbb{P}^2} & & \\ & \pi \swarrow & & \searrow \varphi & \\ \mathbb{P}^2 & \supset & \mathbb{P}^2 - B & \hookrightarrow & \mathbb{P}^2 \end{array}$$

The birational map from $\mathbb{P}^2$ to itself is called a *quadratic transformation*. The map φ can be realized as the blow-up of $\mathbb{P}^2$ at points $\{q_1, q_2, q_3\}$, the exceptional divisors being the strict transforms of the lines $l_{i,j}$ passing through p_i and p_j [3, V, Example 4.2.3]. Thus the exceptional divisors of π become the three lines determined by the q_i under φ while the exceptional divisors of φ become the three lines $l_{i,j}$ under the projection π.
(b) Associated to the quadratic transformation above one obtains an automorphism of the configuration of the 27 lines by visualizing the blow-up at the points $p_1, \ldots, p_6$ versus the points $q_1, q_2, q_3, p_4, p_5, p_6$. Since the lines through the first 3 points become exceptional divisors for the q_i and similarly when the roles of p_i and q_i are reversed for $i \leq 3$, we obtain an automorphism ψ of the 27 lines given by $\psi(E_1) = F_{2,3}, \psi(E_2) = F_{1,3}, \psi(E_3) = F_{1,2}$, $\psi(E_j) = E_j$ for $j = 4, 5, 6$ and $\psi(F_{1,2} = E_3, \psi(F_{1,3}) = E_2, \psi(F_{2,3}) = E_1$. It's easy to check that ψ fixes $F_{i,j}$ for $i < 3.5 < j$ and fixes G_1, G_2, G_3 as well. The rest of the automorphism is more interesting, for consider $\psi(G_4)$. The line G_4 corresponds to a conic in $\mathbb{P}^2$ which passes through all the p_i except p_1. Because the quadratic transformation is defined by a family of *conics*, G_4 must map to a straight line via φ. The blow-up π separates G_4 from the lines $F_{1,2}, F_{1,3}, F_{2,3}$ because of the different tangent directions, hence $\varphi(G_4)$ misses q_1, q_2, q_3. Since G_4 misses p_4 by definition, it maps to the line $F_{5,6}$ passing through p_5, p_6. Similarly $\psi(G_5) = F_{4,6}, \psi(G_6) = F_{4,5}$. Reasoning from the other side we see that $\psi(F_{4,5}) = G_6, \psi(F_{4,6}) = G_5, \psi(F_{5,6}) = G_4$. Notice that ψ is an automorphism of order two, which corresponds to the fact that the quadratic transformation has order two as a birational map.

3.6. **Relation to the Weyl group for E_6 surface singularity.** There are various ways to describe the Weyl group $\mathbf{E}_6$. An intrinsic definition is that $\mathbf{E}_6$ is the automorphism group of the unique simple group of order 25920, which can be described as any of $\mathrm{PSU}_4(2), \mathrm{PSp}_4(3)$ or $\mathrm{PS}\Omega_5(3)$. There are various descriptions in terms of generators and relations (wikipedia gives at least two of them).

There are also such groups for any surface singularity $\mathbf{A}_n, \mathbf{B}_n, \dots$, so I'll use the general description coming from the **Dynkin diagram** for the singularity, which is a graph representing of the configuration of exceptional divisors in a resolution of the surface singularity. Given a graph, we can define a group by generators and relations as follows: the generators x_i are given by the vertices of the graph, and the relations are $x_i^2 = 1$ for each i; $(x_i x_j)^2 = 1$ if x_i, x_j are not adjacent, and $(x_i x_j)^3 = 1$ if x_i is adjacent to x_j.

The Dynkin diagram for $\mathbf{E}_6$ consists of 6 points x_1, x_2, x_3, x_4, x_5 and y. The x_i form a linear chain (i.e. x_i is adjacent to x_{i+1}) and y is only adjacent to x_3. According to [3, IV, Ex. 4.11], we can define an isomorphism from $\mathbf{E}_6 \to G$ (G is the automorphism group of the 27 lines) as follows: $x_i \mapsto (E_i E_{i+1})$ (the automorphism induced by permuting E_i, E_j, which will also exchange G_i, G_{i+1} and the lines $F_{i,k}, F_{i+1,k}$) for each $1 \leq i \leq 5$.

The image of y is the automorphism explained in Example 3.9 above. It is quite easy to check by the action on the E_i that the images of the x_i and y have order two, that the image of $x_i y$ has order two if $i \neq 3$ and that $x_3 y$ has order three, so that the map $\mathbf{E}_6 \to G$ is well-defined. It takes some calculation to confirm that it is bijective (apparently injectivity is the hard part, as it is a starred exercise [3, V, Ex. 4.11]).

References

[1] A. Cayley, *The Collected Mathematical Papers of Arthur Cayley* I, no. 76 (1849) 445–446.

[2] A. Cayley, A memoir on cubic surfaces, Philosophical Transactions of the Royal Society of London **159** (1869) 231–326.

[3] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer-Verlag, 1978.

[4] A. Henderson, *The twenty-seven lines upon the cubic surface*, Cambridge Univ. Press, Cambridge, 1911.

[5] D. Hilbert and S. Cohn-Vossen, *Geometry and the imagination*, Chelsea Pub. Co., New York, 1952.

[6] K. Kodaira, On Kähler varieties of restricted type (An intrinsic characterization of algebraic varieties), Annals of Math. **60** (1954) 28–48.

[7] Y. Manin, *Cubic forms: Algebra, Geometry, Arithmetic*, North-Holland, Amsterdam, 1974.

[8] G. Salmon, *The Geometry of Three Dimensions*, Dublin, 1865.

[9] D. Schläfli, On the distribution of surfaces of the third order into species, in reference to the absence or presence of singular points, and the reality of their lines, Philosophical Transactions of the Royal Society of London **153** (1863) 194–241.

[10] B. Segre, The non-singular cubic surfaces, Clarendon, Oxford, 1942.