SYMPLECTIC MANIFOLDS

1. Two forms

Given a two-form α on a manifold M, we can evaluate it on a pair of tangent vectors to produce a number. That is, if in local coordinates

$$\begin{aligned} \alpha &= \sum_{i<j} \alpha_{ij}(x)\, dx_i \wedge dx_j \\ &= \sum_{i<j} \alpha_{ij}(x)\left(dx_i \otimes dx_j - dx_j \otimes dx_i\right), \end{aligned}$$

and if at a point p of the manifold,

$$\begin{aligned} v_p &= \sum v_j \partial_j, \\ w_p &= \sum w_j \partial_j, \end{aligned}$$

then we define

$$\alpha(v_p, w_p) = \sum_{i<j} \alpha_{ij}(p)\left(v_i w_j - v_j w_i\right).$$

This definition is independent of the choice of coordinates. Of course, this evaluation can be generalized to a k-form acting on k tangent vectors.

We say that a two-form β is *nondegenerate* at a point p if $\beta(v_p, w_p) = 0$ for all tangent vectors v_p implies $w_p = 0$. We say that a two-form on a manifold is nondegenerate if it is nondegenerate at every point of the manifold. Nondegeneracy at a point implies in particular that the dimension of the manifold is even (say dim $= 2n$). In fact, nondegeneracy is equivalent to $\beta^n = \beta \wedge ... \wedge \beta$ being nonzero (ie a nonzero multiple of the volume form).

2. Definition of symplectic manifold

A *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω is a distinguished closed, nondegenerate two-form (called the symplectic form).

The standard example is $2n$-dimensional Euclidean space with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ together with the symplectic form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

In some sense this is the only example of a symplectic structure on $\mathbb{R}^{2n}$. Given any nondegenerate, bilinear, skew symmetric linear form ω on a $2n$-dimensional vector space V, there is a way to choose a basis (called a symplectic basis) of this vector space so that ω has the same form as ω_0. In other words, there exists a vector space isomorphism $\Psi : \mathbb{R}^{2n} \to V$ such that

$$\Psi^* \omega = \omega_0.$$

In fact, even more is true:

Theorem 2.1. *(Darboux) Every symplectic manifold (M, ω) is locally diffeomorphic to $(\mathbb{R}^{2n}, \omega_0)$.*

Thus, there is not much use in studying local symplectic geometry!

The important maps in symplectic topology are the *symplectomorphisms*. A symplectomorphism between two symplectic manifolds (X, ω_X) and (Y, ω_Y) is a diffeomorphism $\psi : X \to Y$ such that

$$\psi^* \omega_Y = \omega_X.$$

Theorem 2.2. *(Moser Stability Theorem) Let (M, ω_t) be a closed manifold with a family of cohomologous symplectic forms. Then there is a family of symplectomorphisms $\psi_t : M \to M$ such that*

$$\psi_0 = \mathbf{1},\ \psi_t^* \omega_t = \omega_0.$$

Moreover, if $\omega_t(q) = \omega_0(q)$ for all points q on a compact submanifold Q of M, we may assume ψ_t is the identity on Q.

3. Examples

Besides the obvious example of $(\mathbb{R}^{2n}, \omega_0)$, there are many other typical examples of symplectic manifolds.

3.1. The cotangent bundle.

Let M be any smooth manifold, and let T^*M denote its cotangent bundle. Then (T^*M, ω) is a symplectic manifold, where the symplectic two-form ω is defined as follows. Since the sections of $\pi : T^*M \to M$ are one-forms, there is a canonical one-form λ on T^*M given by

$$\lambda(\xi_x) = \pi^*(\xi_x),$$

where since $\pi(\xi_x) = x$

$$\pi^* : T_x^*M \to T_{\xi_x}^*(T_x^*M)$$

is the pullback. In other words, given $v_{\xi_x} \in T_{\xi_x}(T_x^*M)$,

$$\lambda(v_{\xi_x}) = \xi_x(\pi_* v_{\xi_x}).$$

Then we define the symplectic form ω on T^*M by

$$\omega = -d\lambda.$$

The sign is chosen so that in $T^*(\mathbb{R}^n)$, the symplectic form becomes $\omega = \omega_0$.

Note that the phase space in classical mechanics is a cotangent bundle, where the manifold coordinates are positions, and the vertical coordinates correspond to momentum.

4. Almost Complex Manifolds

Let M be a $2n$-dimensional real, smooth manifold. An *almost complex structure* J on M is a bundle map

$$J : TM \to TM$$

such that

$$J^2 = -\mathbf{1}$$

If such a structure is integrable (compatible across coordinate systems), then the manifold is a complex manifold, but this is a much stronger condition. If M is a smooth manifold and J is an almost complex structure on M, then the pair (M, J) is called an almost complex manifold.

Each almost complex manifold (M, J) comes equipped with natural nondegenerate two-forms (canonically defined up to homotopy), and conversely, every manifold with nondegenerate 2-form comes equipped with natural almost complex structures (canonical up to homotopy). Given an almost complex manifold that also has a nondegenerate two-form β, we say that the almost complex structure J and β are *compatible* if

$$\beta(Jv, Jw) = \beta(v, w)$$

for all $v, w \in T_x M$ and every $x \in M$, and

$$\beta(v, Jv) > 0$$

for all nonzero tangent vectors v. Let

$$\mathcal{J}(M, \beta)$$

denote the space of compatible almost complex structures on (M, β).

Proposition 4.1. *Let M be a $2n$-dimensional manifold.*

1. *For each nondegenerate two-form β on M, there exists an almost complex structure compatible with β. The space $\mathcal{J}(M, \beta)$ is contractible.*
2. *For each almost complex structure J on M, there exists a compatible nondegenerate 2-form β. The space of such forms is contractible.*

Proof. 1. Given a nondegenerate 2-form β and a metric g, one may write at a given point

$$g(v, w) = v^T G w$$

for some positive definite symmetric matrix $G \in M_{2n}(\mathbb{R})$. Now we want to solve for J, where

$$\beta(v, w) = (Jv)^T w = g\left(\left(G^{-1} J\right) v, w\right).$$

And one can certainly find a matrix $G^{-1}J$ by changing the basis so that the nondegenerate two-form is the standard one.

2. Given a metric g on M and an almost complex structure J, define

$$\beta(v, w) = g(v, Jw).$$

□

Not every even-dimensional manifold admits an almost complex structure. For instance, S^2 and S^6 are the only spheres that do. On the other hand, by using the symplectic form, every symplectic manifold is an almost complex manifold.

Note that non every almost complex manifold is a symplectic manifold; the simplest example is S^6. The reason is that every compact symplectic manifold must have nonzero $H^2(M)$.

Proposition 4.2. *If M is a compact symplectic manifold, then $H^2(M, \mathbb{R})$ is nontrivial.*

Proof. Otherwise, if $\omega = d\alpha$ for some $\alpha \in \Omega^1(M)$, then the volume form

$$\omega^n = d(\alpha \wedge \omega \wedge \ldots \wedge \omega),$$

which is impossible since the volume form of an oriented manifold cannot be exact (and every symplectic manifold is oriented, using the volume form ω^n). □

In fact, the proof extends:

Proposition 4.3. *If M is a compact symplectic manifold of dimension $2n$, then $H^{2k}(M,\mathbb{R})$ is nontrivial for all k such that $0 \leq k \leq n$.*

Proof. Otherwise, if $\omega^k = d\alpha$ for some $\alpha \in \Omega^1(M)$, then the volume form

$$\omega^n = d\left(\alpha \wedge \omega^{n-k}\right),$$

which is impossible since the volume form of an oriented manifold cannot be exact (and every symplectic manifold is oriented, using the volume form ω^n). □

Even more is true for Kähler manifolds; the result is called the *Hard Lefschetz Theorem.* (See below.)

Note that a complex manifold is a manifold with a complex structure that is integrable. This means that coordinate charts are chosen so that the standard complex structure in $\mathbb{R}^n$ is mapped to the almost complex structure on the manifold, and the transition functions preserve the standard complex structure. Equivalently, the transition functions may be chosen to be holomorphic.

Lemma 4.4. *Let ω be a nondegenerate 2-form on M, and let J be an almost complex structure compatible with ω and let ∇ be the covariant derivative associated to the metric g_J (note: $g_J(v,w) = \omega(v,-Jw)$.). Then the following are equivalent:*

(1) $\nabla J = 0$
(2) *J is integrable and ω is closed.*

A *Kähler manifold* is a triple (M,ω,J), where ω is a symplectic form and J is an associated integrable complex structure (so the manifold is complex). Every Riemann surface is a Kähler manifold, as is complex projective space $\mathbb{C}P^n$. In the later case, the complex structure is multiplication by i, and in homogeneous coordinates $[z_0,...,z_n]$, the form

$$\begin{aligned}\tau_0 &= \frac{i}{2\left(\sum_{j=0}^{n}\overline{z_j}z_j\right)^2}\sum_{k=0}^{n}\sum_{j\neq k}\overline{z_j}z_k dz_k\wedge d\overline{z_j} - \overline{z_j}z_k dz_j \wedge d\overline{z_k} \\ &= \sum_j \frac{i\partial\overline{\partial}f_j}{2},\ f_j(z) = \log\frac{\sum_{k=0}^{n}\overline{z_k}z_k}{\overline{z_j}z_j}\end{aligned}$$

is a symplectic form, and the associated metric is called the *Fubini-Study metric.* Further, every complex submanifold of a Kähler manifold is a Kähler manifold. For a time, people wondered whether every symplectic manifold was Kähler. Robert Gompf (UT, 1993-4) showed that there are many examples of non-Kähler symplectic manifolds. Using Gromov's J-holomorphic techniques, many results concerning Kähler manifolds can be extended to general symplectic manifolds, where the almost complex structure is not necessarily integrable.

Theorem 4.5. *(Banyaga) Let M be a compact complex manifold. Let ω_0 and ω_1 be two Kähler forms on M. If $[\omega_0] = [\omega_1] \in H^2(M,\mathbb{C})$, then (M,ω_0) and (M,ω_1) are symplectomorphic.*

Theorem 4.6. *(Hodge decomposition theorem for Kähler manifolds) For a compact Kähler manifold M, the complex cohomology satisfies*

$$\begin{aligned} H^r(M,\mathbb{C}) &\cong \bigoplus_{p+q=r} H^{p,q}_{\overline{\partial}}(M), \\ H^{p,q}_{\overline{\partial}}(M) &= \overline{H^{q,p}_{\overline{\partial}}(M)}, \end{aligned}$$

where $H^{p,q}_{\overline{\partial}}$ is the Dolbeault cohomology restricted to forms of type (p,q). Further, we could instead choose the de Rham cohomology to obtain the same result, and the cohomology classes contain unique harmonic representatives. Consequently, holomorphic forms are therefore harmonic for any Kähler metric on a compact manifold, and the odd Betti numbers are even.

Theorem 4.7. *(Hard Lefschetz Theorem) Let (M^{2n},ω) be a Kähler manifold. Let L be exterior multiplication by ω. Then*

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

is an isomorphism for $1 \leq k \leq n$. Further, if we define the primitive cohomology $P^j(M)$ by

$$P^j(M) = \ker L^{n-j+1} : H^j \to H^{2n-j+2},$$

then we have the Lefschetz decomposition

$$H^m(M) = \bigoplus_k L^k P^{m-2k}(M).$$

Thus, the Betti numbers of a compact Kähler manifold have a pyramid structure, called the Hodge pyramid.

Donaldson proved a theorem about constructing codimension two symplectic submanifolds:

Theorem 4.8. *(Donaldson, 1994) Let (M,ω) be a compact symplectic manifold with integral cohomology class $[\omega] \in H^2(M,\mathbb{Z})$. Then, for every sufficiently large integer k, there exists a codimension 2 symplectic submanifold $N_k \subset M$ which represents the Poincare dual of $k[\omega]$.*

5. Linear Symplectic Geometry

There are various kinds of vector subspaces of symplectic vector spaces that have special names.

Definition 5.1. *Let (V,ω) be a symplectic vector space. Let W be a linear subspace of V.*

(1) *The **symplectic complement** W^ω of W is*

$$W^\omega = \{v \in V : \omega(v,w) = 0 \text{ for all } w \in W\}.$$

(2) *W is called **isotropic** if $W \subset W^\omega$.*
(3) *W is called **coisotropic** if $W^\omega \subset W$.*
(4) *W is called **symplectic** if $W^\omega \cap W = \{0\}$.*
(5) *W is called **Lagrangian** if $W^\omega = W$.*

Lemma 5.2. *We have the following facts from linear symplectic geometry:*

(1) *W is isotropic iff $\omega|_W = 0$.*
(2) *W is symplectic iff $\omega|_W$ is nondegenerate.*

(3) *W is symplectic iff W^ω is symplectic.*
(4) *W is isotropic iff W^ω is coisotropic.*
(5) *If W is Lagrangian, then* $\dim W = \frac{1}{2}\dim V$.
(6) $\dim W + \dim W^\omega = \dim V$.
(7) *A linear map $\Psi : V \to V$ is a linear symplectomorphism iff the graph*

$$\Gamma_\Psi = \{(v, \Psi x) : v \in V\}$$

is a Lagrangian subspace of $(V \times V, (-\omega) \times \omega)$.
(8) *Any isotropic subspace is contained in a Lagrangian subspace.*
(9) *Any basis of a Lagrangian subspace can be extended to a symplectic basis of V.*
(10) *(Linear symplectic reduction) If W is coisotropic, then $V' = W \diagup W^\omega$ carries a natural induced symplectic structure. If $\Lambda \subset V$ is a Lagrangian subspace of V, then $\Lambda' = ((\Lambda \cap W) + W^\omega) \diagup W^\omega$ is a Lagrangian subspace of V'.*

Example 5.3. *Graphs of exact one-forms in T^*M are Lagrangian submanifolds of T^*M.*

6. Contact structures

Contact geometry is the odd-dimensional analogue of symplectic geometry.

A *contact structure* on a manifold M of dimension $2n+1$ is a field of hyperplanes $\xi \subset TM$ with the following property. Let α be a one-form such that $\xi = \ker \alpha$. We say that ξ is a contact structure if $d\alpha$ restricts to a nondegenerate form on ξ (ie symplectic form). Note that if ξ were integrable, the Frobenius condition would be equivalent to $\alpha \wedge d\alpha = 0$, and the contact condition is as far from this as possible. In fact, ξ can be thought of as a maximally nonintegrable distribution of codimension 1.

Proposition 6.1. *Let M be a manifold of dimension $2n+1$ and $\xi \subset TM$ be a transversally orientable hyperplane field.*

(1) *Let α be a one-form with $\xi = \ker \alpha$. Then $d\alpha$ is nondegenerate on ξ iff*

$$\alpha \wedge (d\alpha)^n \neq 0.$$

In this case, ξ is called a ***contact structure****, and α is called a* ***contact form*** *for ξ.*
(2) *Let α and α' be 1-forms with $\xi = \ker \alpha = \ker \alpha'$. Then α is a contact form iff α' is a contact form.*
(3) *If ξ is a contact structure, then the symplectic bilinear form on ξ induced by $d\alpha$ is independent of the choice of α, up to a nonzero scaling factor.*

Proposition 6.2. *Let (M, ξ) be a contact manifold, $\alpha \in \Omega^1(M)$ be a contact form for ξ, and let $L \subset \xi$ be an integrable submanifold. Then T_qL is an isotropic subspace of the symplectic vector space $(\xi_q, d\alpha_q)$ for every $q \in L$. In particular,*

$$\dim L \leq n.$$

If $\dim L = n$*, then L is called* Legendrian*.*

Note that specifying a specific contact form is equivalent to specifying a positive function (since α, α' contact implies $\alpha' = f\alpha$ for some function $f > 0$). Now, a function on a symplectic manifold generates a Hamiltonian flow (described later); analogously, a there is a canonically associated flow to a contact form. Given a contact form α, there exists a unique vector field $Y = Y_\alpha \in \Gamma(TM)$ such that

$$i(Y)\,d\alpha = 0,\ \ \alpha(Y) = 1.$$

This vector field is called the *Reeb vector field* determined by α. Because

$$\mathcal{L}_Y \alpha = di(Y)\,\alpha + i(Y)\,d\alpha = 0,$$

the flow of Y preserves α and thus the contact structure ξ.

Example 6.3. *The basic example is the "standard" contact structure on* $\mathbb{R}^{2n+1} = \{(x_1, ..., x_n, y_1, ..., y_n, z)\}$ *given by the* 1*-form*

$$\alpha_0 = dz - \sum_j y_j dx_j.$$

One can check $Y_{\alpha_0} = \partial_z$ *is the Reeb vector field associated to this contact structure. Note that the condition* $i(Y_{\alpha_0})\,d\alpha_0 = 0$ *is equivalent to requiring that* Y_{α_0} *have no* ∂_{x_j} *or* ∂_{y_k} *components.*

Example 6.4. *On* $\mathbb{R}^{2n+1}$*, the form*

$$dz + \frac{1}{2}\sum d\theta_j = dz + \frac{1}{2}\sum_j (x_j dy_j - y_j dx_j)$$

is a contact form (contactomorphic to the previous example). Again, ∂_z *is the Reeb vector field associated to this contact structure. Note that a helix-like submanifold such as* $(t_1, t_2, ..., t_n) \mapsto \left(\sum t_j, 2\cos(t_1), 2\sin(t_1), ..., 2\cos(t_n), 2\sin(t_n)\right)$ *is* n*-dimensional and tangent to the contact structure, so it is a Legendrian submanifold.*

Example 6.5. *Similar to last example, every compact 3-manifold admits a contact structure. However, the classification of compact contact 3-manifolds is unknown as of 2000.*

Example 6.6. $\frac{1}{2}\sum d\theta_j$ *is a contact form on* $S^{2n-1} \subset \mathbb{R}^{2n}$.

Example 6.7. *Let* L *be any compact manifold. Then the* 1*-jet bundle*

$$J^1 L = T^* L \times \mathbb{R}$$

is a contact manifold with contact form

$$\alpha = dz - \lambda_{\text{can}},$$

and the Reeb vector field $\frac{\partial}{\partial z}$ *where* z *is the real parameter. For any function* $S : L \to \mathbb{R}$*, the submanifold*

$$L_S = \{(x, dS(x), S(x)) : x \in L\} \subset J^1 L$$

is Legendrian.

Example 6.8. *Let* L *be a Riemannian manifold, and let* $M = S(T^*L)$ *be the unit cotangent bundle. Then the restriction of the canonical* 1*-form*

$$\alpha = \lambda_{\text{can}}|_{S(T^*L)}$$

is a contact form. The corresponding Reeb vector field is the Hamiltonian vector field dual to the geodesic flow on TL.

Theorem 6.9. *(Darboux) Every contact structure is locally contactomorphic to the standard contact structure on* $\mathbb{R}^{2n+1}$.

Following is an analogue of Moser's stability theorem for symplectic manifolds.

Theorem 6.10. *(Gray's Stability Theorem) Every family α_t of contact forms on a closed manifold M satisfies*

$$\alpha_t = \psi_t^* \left(f_t \alpha_0\right)$$

for some family of nonvanishing functions f_t and contactomorphisms ψ_t.

The following result relates symplectic manifolds and contact manifolds.

Proposition 6.11. *Let (Q, ξ) be a transversally oriented contact manifold. Then $M = Q \times \mathbb{R}$ is a symplectic manifold with exact symplectic form*

$$\omega = e^{\theta} \left(d\alpha - \alpha \wedge d\theta\right) = d\left(e^{\theta}\alpha\right),$$

where $\theta \in \mathbb{R}$. The symplectic manifold (M, ω) is called the ***symplectization*** *of (Q, ξ).*

Proposition 6.12. *A submanifold $L \subset Q$ is Legendrian iff $L \times \mathbb{R}$ is Lagrangian.*

A **Liouville vector field** on a symplectic manifold (M, ω) is a vector field X such that

$$\mathcal{L}_X \omega = \omega.$$

Example 6.13. *The radial vector field $\partial_r = \sum \left(\frac{1}{2} x_j \partial_{x_j} + \frac{1}{2} y_j \partial_{y_j}\right)$ on $\mathbb{R}^{2n}$ is a Liouville vector field, which is transversal to the unit sphere S^{2n-1}, and $i\left(\partial_r\right)\omega_0 = \frac{1}{2}\sum d\theta_j$ is a contact form on S^{2n-1}.*

Proposition 6.14. *Let (M, ω) be a symplectic manifold, and let $Q \subset M$ be a compact hypersurface. Then the following are equivalent:*

1. *There exists a contact form α on Q such that $d\alpha = \omega|_Q$.*
2. *There exists a Liouville vector field $X : U \to TM$, defined in a neighborhood U of Q, that is transverse to Q.*

If these conditions are met, Q is said to be ***of contact type****.*

There are several open problems in contact geometry.

Problem 6.15. *(Seifert, 1948) Let V be a nowhere vanishing vector field on S^3. Does the flow of V have any periodic orbits?*

Schweitzer proved in 1974 that there exists a C^1 vector field without periodic orbits. Kristina Kuperberg proved in 1994 that there exists a smooth vector field without periodic orbits.

Problem 6.16. *What about volume-preserving vector fields?*

Greg Kuperberg proved in 1997 that there exists a C^1 counterexample to existence. There are no known smooth counterexamples, as of 2000.

Conjecture 6.17. *(Weinstein, 1978) Suppose that M is a 3-dimensional manifold with a (global) contact form α. Let v be the Reeb vector field for α. Then v has a periodic orbit.*

Theorem 6.18. *(Viterbo and Hofer, 1993) The Weinstein Conjecture is true when*

1. *$M = S^3$, or*
2. *$\pi_2(M) \neq 0$, or*
3. *the contact structure is "overtwisted". (ie not "tight")*

Remark 6.19. *A contact structure on a 3-manifold is called "overtwisted" if there exists an embedded 2-disk such that the characteristic foliation (intersection of contact structure with tangent space to disk gives a line bundle, thus a foliation) contains one closed leaf C and exactly one singular point inside C (ie C is a limit cycle). "Tight" = not "overtwisted". Note that the standard $\mathbb{R}^3$ examples of contact structures are tight. The example on $S^{2n-1} = S^3$ is also a tight contact structure, and in fact this is the only possible tight contact structure (up to isotopy). Eliashberg proved that the isotopy classes of overtwisted contact structures on closed 3-manifolds coincide with the homotopy classes of tangent plane fields.*

Problem 6.20. *Classify the tight contact structures on a given 3-manifold.*

Problem 6.21. *How many periodic orbits are there?*

Problem 6.22. *What do they look like?*

Problem 6.23. *Is there always an unknotted one?*

Problem 6.24. *What about the linking behavior?*

7. Results in symplectic topology

Given a symplectic manifold (M^{2n}, ω), there is a canonical orientation an measure given by the volume form ω^n. Thus symplectomorphisms preserve volume. But is the group of symplectomorphisms of a manifold significantly smaller than the group of volume-preserving diffeomorphisms? Nothing was known until the following fact was proven in 1985.

Let $B^{2n}(r)$ denote the closed Euclidean ball of radius r centered at the origin of $\mathbb{R}^{2n}$. Let

$$Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2}$$

be the solid cylinder. We say that a map

$$\phi : U \to V$$

between subsets U and V in $\mathbb{R}^{2n}$ is a *symplectic embedding* if ϕ is a smooth embedding that preserves the symplectic form (i.e. $\phi^*\omega_0 = \omega_0$).

Theorem 7.1. *(Gromov nonsqueezing theorem) If there is a symplectic embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$, then $r \leq R$.*

(Proof to come later.)

Note that this theorem implies that the group of symplectomorphisms of the ball is significantly smaller than the group of volume-preserving diffeomorphisms.

Another idea is that of the *symplectic camel*. Let

$$j : B^{2n} \to \mathbb{H}^+ = \left\{x \in \mathbb{R}^{2n} : x_{2n} > 0\right\}$$

denote the inclusion of the standard unit ball in the upper half space, which is clearly a symplectomorphism with respect to ω_0. Similarly, let $k : B^{2n} \to \mathbb{H}^-$ be the inclusion of a ball in the lower half space. One may think of B^{2n} as the "camel". Let $B_r \subset \partial\mathbb{H}^+ \cong \mathbb{R}^{2n-1}$ denote an open $(2n-1)$-ball of radius r in $\partial\mathbb{H}^+$. We call B_r "the eye of the needle". We say that *the camel passes through the eye of the needle* if there exists a homotopy

$$h : [0,1] \times B^{2n} \to \mathbb{R}^{2n}$$

such that

$$h(0,x) = j(x), \;\; h(1,x) = k(x),$$

and

$$(\mathrm{Im} h) \cap \partial \mathbb{H}^{+} \subset B_r.$$

Theorem 7.2. *(Gromov symplectic camel theorem) If $r < 1$, there does not exist such a homotopy for which each $h(t,\cdot) : B^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism for $0 \le t \le 1$. Thus, in this case the camel cannot pass through the eye of the needle.*

8. Hamiltonian flows

8.1. **Hamiltonian vector fields and flows.** Let (M,ω) be a symplectic manifold. Given a differentiable function $H : M \to \mathbb{R}$ (called the *Hamiltonian*), there exists a unique vector field X_H such that

$$\begin{aligned} -dH(Y) &= \omega(X_H, Y), \text{ or} \\ -dH &= i(X_H)\,\omega \end{aligned}$$

for every vector field Y on M. (Note: some authors use a negative sign in the equation above.) Such a vector field X_H exists and is unique because the nondegeneracy of ω implies that the linear map

$$i(\cdot)\,\omega : T_x M \to T_x^* M$$

has a trivial kernel. In canonical coordinates (ie $\omega = \sum dx_j \wedge dy_j$), we have

$$X_H = \sum -\frac{\partial H}{\partial y_j}\partial_{x_j} + \frac{\partial H}{\partial x_j}\partial_{y_j}.$$

Check:

$$\begin{aligned} i(X_H)\,\omega &= \sum i\left(-\frac{\partial H}{\partial y_j}\partial_{x_j} + \frac{\partial H}{\partial x_j}\partial_{y_j}\right) dx_j \wedge dy_j \\ &= -\sum\left(\frac{\partial H}{\partial y_j}dy_j + \frac{\partial H}{\partial x_j}dx_j\right) \\ &= -dH. \end{aligned}$$

The flow generated by the vector field X_H is called the Hamiltonian flow. That is, for each $t \in (-\varepsilon, \varepsilon)$, let the Hamiltonian flow $\phi_H^t : M \to M$ be the diffeomorphism defined by

$$\frac{\partial}{\partial t}\phi_H^t(x) = X_H\left(\phi_H^t(x)\right);\ \phi_H^0(x) = x.$$

For fixed $x \in M$, $\phi_H^t(x)$ is just an integral curve of X_H with initial point x. Note that the map $f \mapsto X_f$ is a linear map from smooth functions to smooth vector fields. By construction, observe that the function H is constant along the integral curves of X_H (ie H is an *integral* of ϕ_H^t), because

$$\frac{\partial}{\partial t}H\left(\phi_H^t(x)\right) = dH\left(\frac{\partial}{\partial t}\phi_H^t(x)\right) = dH\left(X_H\left(\phi_H^t(x)\right)\right),$$

and

$$dH(X_H) = -\omega(X_H, X_H) = 0$$

since ω is a 2-form.

Observe that each X_H is a symplectic vector field, meaning that

$$\begin{aligned}\mathcal{L}_{X_H}\omega &= (d\circ i(X_H)+i(X_H)d)\,\omega\\ &= d(i(X_H)\omega)=0.\end{aligned}$$

(True since $i(X_H)\omega = -dH$, which is closed.) A consequence of this is that for each $t\in(-\varepsilon,\varepsilon)$, ϕ_H^t is a symplectomorphism, because

$$\frac{\partial}{\partial t}\left(\left(\phi_H^t\right)^*\omega\right)=\left(\phi_H^t\right)^*\left(\mathcal{L}_{X_H}\omega\right)=0.$$

Another interesting fact: the set of symplectic vector fields forms a Lie subalgebra of the Lie algebra of vector fields, and in fact it is the Lie algebra corresponding to the Lie group of symplectomorphisms. This follows from the fact that if X and Y are symplectic vector fields, then

$$\begin{aligned}i([X,Y])\,\omega &= i(\mathcal{L}_XY)\,\omega = i\left(\left.\frac{d}{dt}\right|_{t=0}\left(\phi_X^t\right)_*Y\right)\omega\\ &= \left.\frac{d}{dt}\right|_{t=0} i\left(\left(\phi_X^t\right)_*Y\right)\omega\\ &= \left.\frac{d}{dt}\right|_{t=0}\left(\phi_X^t\right)^*(i(Y)\,\omega)=\mathcal{L}_X(i(Y)\,\omega)\\ &= d\circ i(X)(i(Y)\,\omega)=d(\omega(Y,X)),\end{aligned}$$

so that $i([X,Y])\,\omega$ is also closed.

Note that not every symplectic vector field V is Hamiltonian; V is symplectic iff $i(V)\,\omega$ is closed. If in fact $i(V)\,\omega$ is exact, then V is Hamiltonian. Thus, the obstruction to a symplectic vector field being Hamiltonian lies in $H^1(M)$.

Example 8.1. *This is the typical example of a Hamiltonian vector field and flow. Consider the symplectic manifold* $(\mathbb{R}^2,\omega=dx\wedge dy)$*, with Hamiltonian function* $H(x,y)=\frac{1}{2}(x^2+y^2)$*. Then* $dH=xdx+ydy$*, and*

$$i(x\partial_y-y\partial_x)\,\omega=-xdx-ydy=-dH,$$

so that the Hamiltonian vector field is

$$X_H=\partial_\theta=(x\partial_y-y\partial_x).$$

Note that H *is constant on the integral curves of* X_H*, which are circles.*

8.2. **Poisson bracket.** A skew-symmetric, bilinear operation on the differentiable functions on a symplectic manifold M, the *Poisson bracket* $\{f,g\}$, is defined by the formula

$$\{f,g\}=\omega(X_f,X_g)=-df(X_g)=-X_g(f)=X_f(g),$$

which in canonical coordinates (ie $\omega=\sum dx_j\wedge dy_j$ is

$$\{f,g\}=\sum_{j=1}^n\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j}-\frac{\partial f}{\partial y_j}\frac{\partial g}{\partial x_j}.$$

Note that

$$\{g,f\}=-\{f,g\}$$

Proposition 8.2. *We have*

$$X_{\{f,g\}} = [X_f, X_g],$$

where the right hand side represents the Lie bracket of vector fields.

Proof.

$$\begin{aligned} \omega\left(X_{\{f,g\}}, Y\right) &= -Y(\{f,g\}) \\ &= Y(X_g(f)) \\ &= Y(df(X_g)) \end{aligned}$$

and since X_f and X_g are symplectic, by the previous section we have

$$\begin{aligned} (i([X_f, X_g])\,\omega)(Y) &= d(\omega(X_g, X_f))(Y) \\ &= Y(\omega(X_g, X_f)) \\ &= -Y(\omega(X_f, X_g)) \\ &= Y(df(X_g)). \end{aligned}$$

□

As a consequence, the Poisson bracket satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0,$$

which means that the vector space of differential functions on M, endowed with the Poisson bracket, has the structure of a Lie algebra over $\mathbb{R}$, and the assignment $f \mapsto X_f$ is a Lie algebra homomorphism, whose kernel consists of the locally constant functions (constant functions if M is connected). Note that many books (such as McDuff-Salamon) on symplectic geometry define $[X,Y] = \mathcal{L}_Y X = -\mathcal{L}_X Y$ and $i(X_H)\,\omega = dH$ so that this still becomes a Lie algebra homomorphism.

A function $g : M \to \mathbb{R}$ is called an *integral* of the flow ϕ_H^t if g is constant on the integral curves of X_H, ie $X_H(g) = 0$. Clearly, H is an integral.

Proposition 8.3. *A function $g : M \to \mathbb{R}$ is an integral of the Hamiltonian flow associated to $H : M \to \mathbb{R}$ if and only if $\{g, H\} = 0$.*

Proof. $X_H(g) = \{H, g\} = -\{g, H\}$. □

8.3. **Relationship to Physics: Hamiltonian mechanics.** The canonical coordinates for the phase space of a mechanical system are $(\mathbf{p}, \mathbf{q}) = (p_1, ..., p_n, q_1, ..., q_n)$, where $\mathbf{q}$ corresponds to position and $\mathbf{p}$ corresponds to momentum. The symplectic form is

$$\omega = \sum dp_j \wedge dq_j.$$

Given a Hamiltonian function H (the total energy!), the Hamiltonian flow is the solution to the system of differential equations

$$\frac{\partial}{\partial t}(\mathbf{q}, \mathbf{p}) = X_H(\mathbf{q}, \mathbf{p}) = \sum -\frac{\partial H}{\partial q_j}\partial_{p_j} + \frac{\partial H}{\partial p_j}\partial_{q_j},$$

which implies

$$\sum \frac{\partial p_j}{\partial t}\partial_{p_j} + \frac{\partial q_j}{\partial t}\partial_{q_j} = X_H = \sum -\frac{\partial H}{\partial q_j}\partial_{p_j} + \frac{\partial H}{\partial p_j}\partial_{q_j},$$

or

$$\begin{aligned} \frac{\partial q_j}{\partial t} &= \frac{\partial H}{\partial p_j} \\ \frac{\partial p_j}{\partial t} &= -\frac{\partial H}{\partial q_j}. \end{aligned}$$

These are know as Hamilton's equations of motion. For example, if

$$\begin{aligned} H &= KE + PE \\ &= \frac{\|p\|^2}{2m} + V(q), \end{aligned}$$

then the equations of the integral curves become

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial t} &= \frac{\mathbf{p}}{m} \\ \frac{\partial \mathbf{p}}{\partial t} &= -\nabla V, \end{aligned}$$

coinciding with the physical laws that

$$\begin{aligned} momentum &= mass * velocity, \\ Force &= -\nabla V = \frac{\partial \mathbf{p}}{\partial t}. \end{aligned}$$

Note that the fact that H is constant along the integral curves is the principle of conservation of energy.

8.4. **Lagrangian mechanics.** Let $L = L(t, \mathbf{q}, \mathbf{v})$ be a function that is intuitively understood to be kinetic minus potential energy, with $\mathbf{q}$ the position varables and $\mathbf{v}$ the velocity variables. To minimize the action functional for paths $\mathbf{q} : [t_0, t_1] \to \mathbb{R}^n$

$$I(\mathbf{q}) = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}}) \; dt,$$

the Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial v_j} = \frac{\partial L}{\partial q_j}.$$

Setting

$$H(t, \mathbf{q}, \mathbf{p}) = \sum v_j p_j - L(t, p, v(t, p, q)),$$

and setting

$$p_j = \frac{\partial L}{\partial v_j}(t, q, v(t, p, q)), v_j(t, p, q) = \dot{q}_j,$$

we obtain Hamilton's equations, by

$$\begin{aligned} \frac{\partial H}{\partial q_k} &= -\frac{\partial L}{\partial q_k}, \frac{\partial H}{\partial p_k} = v_k(t, p, q), \text{ or} \\ \dot{q}_j &= \frac{\partial H}{\partial p_j}, \; \dot{p}_j = -\frac{\partial H}{\partial q_j} \end{aligned}$$

This transformation to (p_j, q_j) is called the Legendre transformation.

The Legendre transformation is its own inverse, so if we assume Hamilton's equations and set

$$\begin{aligned} L(t,\mathbf{q},\mathbf{v}) &= \sum p_j \frac{\partial H}{\partial p_j}(t,) - H, \\ v_j &= \frac{\partial H}{\partial p_j}, p_j = \dot{q}_j, \end{aligned}$$

Then we get the Euler-Lagrange equations from Hamilton's equations.

8.5. **Geodesic flow on the cotangent bundle.** We repeat the notation given previously. Let M be any smooth manifold, and let T^*M denote its cotangent bundle. Then (T^*M,ω) is a symplectic manifold, where the symplectic two-form ω is defined as follows. Since the sections of $\pi : T^*M \to M$ are one-forms, there is a canonical one-form λ on T^*M given by

$$\lambda(\xi_x) = \pi^*(\xi_x),$$

where since $\pi(\xi_x) = x$

$$\pi^* : T_x^*M \to T_{\xi_x}^*(T_x^*M)$$

is the pullback. The symplectic form ω on T^*M is

$$\omega = d\lambda.$$

Suppose that we have chosen a Riemannian metric on M. Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian function defined by

$$H(\xi_x) = \|\xi_x\|^2.$$

The Hamiltonian differential equations obtained are precisely the differential equations of geodesics.

8.6. **Completely integrable Hamiltonian systems.** If (M,ω) is a symplectic manifold of dimension $2n$, then it is called *completely integrable* (or simply *integrable*) if there exist n independent Poisson commuting integrals $F_1,...,F_n$. The word *independent* means that the vectors $\nabla F_1,...,\nabla F_n$ are linearly independent at each point and that $\{F_j,F_k\} = 0$ for each j, k. Given such a system, the level sets of the $\{F_j\}$ are invariant under the Hamiltonian flows of all n of the functions. Since the functions commute, the compact, connected components of level sets of the form $T_c = \{x \in M : F_j(x) = c_j \text{ for all } j\}$ are n-dimensional tori and are in fact Lagrangian submanifolds (ω vanishes on them). Near such an invariant torus, the symplectic manifold is foliated by such tori, as explained in the next paragraph.

These completely integrable Hamiltonian systems have an extremely simple structure. There are special coordinates called *action-angle variables* that are chosen as follows. In a neighborhood of each invariant torus T_c, the variables $(\xi,\eta) \in \mathbb{T}^n \times \mathbb{R}^n$ are defined by

$$\begin{aligned} \eta &= (c_1,...,c_n), \\ &\xi \text{ is a coordinate on each } T_c \end{aligned}$$

such that the Hamiltonian flow of $H(\xi,\eta)$ (thought of as a function on M) is given by the straight-line solutions of the differential system

$$\dot{\xi} = \frac{\partial H}{\partial \eta},\ \dot{\eta} = 0.$$

Thus the physical trajectories can be solved completely by doing integrals!

This dynamical behavior of integrable systems is very exceptional, and an arbitrarily small perturbation of the dynamical system will destroy many of the invariant tori. On the other hand, if the frequency vector $\dot{\xi} = \frac{\partial H}{\partial \eta}$ has rationally independent coordinates which satisfy certain Diophantine inequalities and if $\frac{\partial^2 H}{\partial \eta^2}$ is nonsingular, then the corresponding invariant torus survives under sufficiently small perturbations which are sufficiently smooth. This is roughly the content of the Kolmogorov-Arnold-Moser Theorem, from which KAM theory evolved.

9. Symplectic reduction

9.1. **The moment map and symplectic reduction for circle actions.** A Hamiltonian circle action on a symplectic manifold (M^{2n}, ω) is simply a one-parameter family of symplectomorphisms that is periodic (we force it to have period 1) and that is the integral of a Hamiltonian vector field X_H. In this case, the moment map (or momentum map) is the Hamiltonian function $H : M \to \mathbb{R}$. Note that H is well-defined up to an additive constant. If M is compact, then the function could be normalized so that $\int_M H\ \omega^n = 0$. For example, if S^1 acts on S^2 by rotations around the z-axis, then the normalized moment map is the function $(x, y, z) \to 2\pi z$. For another example, consider multiplication by $e^{2\pi i t}$ on $\mathbb{C}^{n+1}$, with Hamiltonian function $H(z) = -\pi |z|^2$. Note that the quotient of the level set $H^{-1}(\pi)$ by this action is $\mathbb{C}P^n$.

Lemma 9.1. *Suppose that S^1 acts freely on the regular level set $H^{-1}(\lambda) \subset M$ of H. Then the quotient manifold*

$$B_\lambda = H^{-1}(\lambda) \diagup S^1$$

*is symplectic, and the symplectic form may be defined by the condition that its pullback to $H^{-1}(\lambda)$ is the restriction of ω to $H^{-1}(\lambda)$. The manifold B_λ is called the **symplectic quotient** of (M, ω) at $\lambda \in \mathbb{R}$.*

Proof. Because it is codimension 1, the tangent space H_x to the hypersurface $H^{-1}(\lambda)$ at each point is coisotropic, meaning that $H_x^\omega := \{v \in T_x M : \omega(v, w) = 0 \text{ for all } w \in H_x\}$ is a subspace of H_x. In this case, each H_x^ω is also one-dimensional. To see this, just look at standard symplectic form on $\mathbb{R}^n$; fixing v, the set of vectors w such that $\omega(v, w) = 0$ is a subspace that contains v and is codimension 1. Next, the dimension of H_x^ω is $\dim T_x M - \dim H_x = 1$. The resulting foliation is called the **characteristic foliation**. In fact, the leaves of the characteristic foliation are the S^1-orbits. Thus, ω induces a nondegenerate form $\bar{\omega}$ on $H_x \diagup H_x^\omega$, and this form is invariant under the Hamiltonian flow. Thus it descends to a well-defined form on B_λ. □

The spaces B_λ are also called **reduced spaces**. Note that not all symplectic circle actions are Hamiltonian. An obvious obstruction is that a Hamiltonian action must have fixed points (corresponding to the critical points of H).

Theorem 9.2. *(McDuff, 1988) A symplectic circle action on a closed 4-manifold is Hamiltonian iff it has fixed points.*

Remark 9.3. *In the same paper, it was shown that the assertion is false in 6 dimensions (and above).*

Now, how does the symplectic quotient depend on the choice of $\lambda \in \operatorname{Im}(H)$, say if we allow λ to move in an interval of regular values of H? We assume that the S^1 action is

semi-free, meaning that it is free away from the fixed points, ie that the principal isotropy subgroup is trivial. In this case, the reduced spaces B_λ have no singularities. Note that in general B_λ is an orbifold. Anyway, we our assumptions above, fixing an interval I, all B_λ with $\lambda \in I$ may be identified with a single manifold B.

Proposition 9.4. *We have*

(1) *Let $I \in \mathbb{R}$ be an interval and $\{\tau_\lambda\}_{\lambda \in I}$ be a family of symplectic forms on B such that*
$$[\tau_\lambda] = [\tau_\mu] + (\lambda - \mu)\, c_I,$$
for $\lambda, \mu \in I$ and where $c_I \in H^2(B, \mathbb{Z})$. Moreover, let $\pi : P \to B$ be a circle bundle with first Chern class c_I. Then there is an S^1-equivariant symplectic form ω on the manifold $P \times I$ with Hamiltonian function H equal to the projection $P \times I \to I$ and with reduced spaces (B, τ_λ).

(2) *Conversely, any compact, connected symplectic manifold (M, ω) with a free Hamiltonian action of S^1 is equivariantly symplectomorphic to a manifold of the above form. Moreover, (M, ω) is determined up to an equivariant symplectomorphism by the family (B, τ_λ).*

Example 9.5. *Consider the action of S^1 on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ which corresponds to multiplication by $e^{2\pi i t}$. If $H(z) = \pi |z|^2$, then*
$$\begin{aligned} dH &= 2\pi \sum (x_k dx_k + y_k dy_k) \\ &= -i\left(2\pi(-y_0, x_0, -y_1, x_1, ..., -y_n, x_n)\right) \sum dx_k \wedge dy_k \\ &= -\omega_0(X_H, \cdot). \end{aligned}$$
Thus the infinitessimal generator $X_H = 2\pi\partial_\theta$ of the S^1 action is the Hamiltonian vector field corresponding to the Hamiltonian function H. Consider the set $H^{-1}(\pi)$: this is the unit sphere. If we take the symplectic quotient, we get $H^{-1}(\pi) \diagup S^1 = \mathbb{C}P^n$, and $H^{-1}(\pi) = S^{2n+1} \xrightarrow{p} \mathbb{C}P^n$ is the Hopf fibration. One may check that the induced $U(n+1)$-invariant symplectic form τ_0 on $\mathbb{C}P^n$ satisfies
$$\begin{aligned} p^*\tau_0 &= \frac{i}{2}\partial\overline{\partial}\left(\log \sum z_k \overline{z_k}\right), \\ \tau_0 &= \frac{i}{2}\partial\overline{\partial}\left(\log \frac{\sum z_k \overline{z_k}}{z_j \overline{z_j}}\right) \text{ in the chart } z_j \neq 0. \end{aligned}$$
the same as the restriction of ω_0 to S^{2n+1}. Note that the form τ_0 is the standard symplectic structure on $\mathbb{C}P^n$, and the induced metric is the Fubini-Study metric.

9.2. **Moment maps in general.** Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ that acts on a symplectic manifold (M, ω) by symplectomorphisms. Let $R_g : M \to M$ denote right multiplication by $g \in G$, so that $R_g \circ R_h = R_{hg}$. This induces a Lie algebra homomorphism $\mathfrak{g} \to \chi(M, \omega)$ (say $\xi \mapsto X_\xi$) to the space of symplectic vector fields. The action of G is called **weakly Hamiltonian** if each X_ξ is Hamiltonian, ie that each $i(X_\xi)\,\omega$ is exact. Thus, for each ξ, there exists a Hamiltonian function H_ξ that is determined only up to a constant, ie so that $X_\xi = X_{H_\xi}$. Note we could choose an orthonormal basis for $\mathfrak{g}$, find the corresponding Hamiltonian for each basis element and then extend linearly to get a linear map
$$\mathfrak{g} \to C^\infty(M) : \ \xi \mapsto H_\xi.$$

The G action is called **Hamiltonian** if this linear map can be chosen to be a Lie algebra homomorphism with respect to the Poisson structure on $C^{\infty}(M)$. In general weakly Hamiltonian actions need not be Hamiltonian. The standard example of a Hamiltonian action comes from a G-action on a manifold X, and then the induced action on $T^{*}X$ is automatically Hamiltonian with respect to the standard symplectic structure on the cotangent bundle.

Now, suppose that the action of G on (M,ω) is Hamiltonian with homomorphism $\xi \mapsto H_{\xi}$. Then the **moment map** for the action is a map

$$\mu : M \to \mathfrak{g}^{*}$$

defined by

$$H_{\xi}(p) = \mu(p)(\xi),$$

ie such that at each point $\mu(p)$ is the linear functional $\xi \mapsto H_{\xi}(p)$. Note that any other map $\widetilde{\mu} : M \to \mathfrak{g}^{*}$ is a moment map for the Hamiltonian G-action if and only if there exists a constant element $\nu \in [\mathfrak{g},\mathfrak{g}]^{0}$ such that $\widetilde{\mu} = \mu + \nu$. We have

$$\begin{aligned} [\mathfrak{g},\mathfrak{g}]^{0} \quad : &= \{\xi \in \mathfrak{g}^{*} : \xi([X,Y]) = 0 \text{ for all } X,Y \in \mathfrak{g}\} \\ &= \{\xi \in \mathfrak{g}^{*} : Ad^{*}(g)(\xi) = \xi \text{ for all } g \in G\}, \end{aligned}$$

where

$$Ad^{*} : G \to GL(\mathfrak{g})$$

is the coadjoint representation defined by

$$Ad^{*}(h) := Ad\left(h^{-1}\right)^{*} = \left((Ad(h))^{*}\right)^{-1}$$

Note that

$$X_{\xi}(p) = \left.\frac{d}{dt}\right|_{t=0} R_{\exp(t\xi)}(p)$$

for each $p \in M$.

Lemma 9.6. *We have*

$$X_{Ad(g^{-1})\xi} = X_{\text{"}g^{-1}\xi g\text{"}} = (R_g)_{*} X_{\xi}.$$

Proof. Note that if $\xi = \left.\frac{d}{dt}\right|_{t=0} h(t)$ for some path $h : (-\varepsilon,\varepsilon) \to G$ with $h(0) = \mathbf{1}$, then for each $p \in M$, the curve $\beta_p(t) := R_{h(t)}(p)$ satisfies $\beta_p'(0) = X_{\xi}(p)$. Then

$$\begin{aligned} (R_g)_{*}(X_{\xi}(p)) &= (R_g)_{*}\left(\beta_p'(0)\right) = (R_g \circ \beta_p)'(0) \\ &= \left(R_g \circ R_{h(t)}(p)\right)'(0) \\ &= \left(R_{g^{-1}h(t)g}(R_g(p))\right)'(0) \\ &= X_{Ad(g^{-1})\xi}(R_g(p)). \end{aligned}$$

∎

Lemma 9.7. *Also,*

$$X_{[\xi,\eta]} = [X_{\xi}, X_{\eta}].$$

Proof. If x and h are two curves on G such that $x(0) = h(0) = \mathbf{1}$, $x'(0) = \xi$, and $h'(0) = \eta$, then

$$\begin{aligned} X_{[\xi,\eta]}(p) &= X_{ad(\xi)\eta}(p) \\ &= X_{d(Ad)(\xi)\eta}(p) = \left.\frac{d}{dt}\right|_{t=0} X_{Ad(x(t))\eta}(p) \\ &= \left.\frac{d}{dt}\right|_{t=0} X_{Ad(x(t))\eta}(p) = \left.\frac{d}{dt}\right|_{t=0} \left(R_{x(t)}\right)_*^{-1} X_\eta \left(R_{x(t)}p\right) \\ &= [X_\xi, X_\eta](p), \end{aligned}$$

using the formula

$$\mathcal{L}_X(Y)_p = [X,Y]_p = \left.\frac{d}{dt}\right|_{t=0} (\phi_{-t})_* Y_{\phi_t(p)},$$

where ϕ_t is the flow of the vector field X. □

Lemma 9.8. *With notation as above, if $g \in G_0$ (identity component of G) and $\xi \in \mathfrak{g}$,*

$$H_{Ad(g^{-1})\xi} = H_\xi \circ R_g.$$

Proof. By the equation $X_{Ad(g^{-1})\xi} = (R_g)_* X_\xi$, the functions $H_{Ad(g^{-1})\xi}$ and $H_\xi \circ R_g$ generate the same Hamiltonian vector field, since

$$\begin{aligned} dH_{Ad(g^{-1})\xi}\left((R_g)_* Y\right) &= -\omega\left(X_{Ad(g^{-1})\xi}, (R_g)_* Y\right) \\ &= -\omega\left((R_g)_* X_\xi, (R_g)_* Y\right) = R_g^*(\omega)(X_\xi, Y) \\ &= R_g^* dH_\xi(Y) = d(H_\xi \circ R_g)(Y) \end{aligned}$$

So their difference is constant, and

$$\begin{aligned} H_{Ad(g^{-1})[\xi,\eta]} &= \left\{H_{Ad(g^{-1})\xi}, H_{Ad(g^{-1})\eta}\right\} \\ &= \{H_\xi, H_\eta\} \circ R_g = H_{[\xi,\eta]} \circ R_g. \end{aligned}$$

Now let $g(t)$ be a path in G_0 connecting the identity to an arbitrary g_1, and let $\eta(t) = \left(R_{g(t)^{-1}}\right)_* g'(t) \in \mathfrak{g}$. Then

$$\frac{d}{dt} R_g = X_\eta \circ R_g \text{ and } \frac{d}{dt} Ad\left(g^{-1}\right)\xi = Ad\left(g^{-1}\right)[\xi,\eta].$$

Thus,

$$\begin{aligned} \frac{d}{dt}\left(H_\xi \circ R_g - H_{Ad(g^{-1})\xi}\right) &= d(H_\xi \circ R_g)(R_g)_* X_\eta - H_{Ad(g^{-1})[\xi,\eta]} \\ &= -\omega\left(X_{H_\xi \circ R_g}, (R_g)_* X_\eta\right) - H_{[\xi,\eta]} \circ R_g \\ &= -\omega\left(X_{H_\xi \circ R_g}, X_{H_\eta \circ R_g}\right) - \{H_\xi, H_\eta\} \circ R_g \\ &= -\{H_\xi, H_\eta\} \circ R_g - \{H_\xi, H_\eta\} \circ R_g = 0. \end{aligned}$$

Integrate. □

The next lemma shows that the moment map is equivariant with respect the coadjoint action on $\mathfrak{g}^*$.

Lemma 9.9. *For all $g \in G_0$ (connected component of the identity),*

$$\left(\left(R_{g^{-1}}\right)^* \mu\right)(p) = \mu\left(R_g(p)\right) = "g\mu(p)g^{-1}" = Ad\left(g^{-1}\right)^*(\mu(p)),$$

where $R_g : M \to M$ is the symplectomorphism induced from g.

Proof. We have

$$\begin{aligned} \mu\left(R_g(p)\right)(\xi) &= H_\xi\left(R_g(p)\right) \\ &= H_\xi \circ R_g(p) \\ &= H_{Ad(g^{-1})\xi}(p) \\ &= \mu(p)\left(Ad\left(g^{-1}\right)\xi\right). \end{aligned}$$

□

Note that from the last lemma, if G is connected and abelian, then μ is constant on the orbits.

The significance of the moment map is as follows. If $H : M \to \mathbb{R}$ is a Hamiltonian function that is G-invariant, then H is an integral for the Hamiltonian flow of H_ξ, for all $\xi \in \mathfrak{g}$, so that $\{H, H_\xi\} = 0$. Thus the moment map is constant along the integral curves of the Hamiltonian flow of H. Angular momentum in $\mathbb{R}^3$ is a particular case of this. One can think of this as an instance of Noether's Theorem: each infinitessimal symmetry (ie dimension of $\mathfrak{g}$) yields a conserved quantity. Thus, if you choose a basis ξ_1,...,ξ_r of $\mathfrak{g}$, then the functions $H_{\xi_1}, ..., H_{\xi_r}$ are independent and are conserved by the Hamiltonian flow.

Example 9.10. *Angular momentum in $\mathbb{R}^3$. Note that $G = SO(3)$ acts on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ (with the standard symplectic structure) by*

$$R_A(x, y) = (Ax, Ay)$$

for $A \in SO(3)$. This action is Hamiltonian with $H_B(x, y) = -y \cdot (Bx) = -\sum_{m,k} y_m B_{mk} x_k$ with $B = -B^T \in \mathfrak{so}(3)$. We check that the Hamiltonian vector field associated to H_B is

$$\begin{aligned} \sum \frac{\partial H_B}{\partial y_j}\partial_{x_j} - \frac{\partial H_B}{\partial x_j}\partial_{y_j} &= \sum_k \left(B_{jk}x_k\right)\partial_{x_j} - \left(y_k B_{kj}\right)\partial_{y_j} \\ &= \sum_k \left(B_{jk}x_k\right)\partial_{x_j} + \left(B_{jk}y_k\right)\partial_{y_j}, \end{aligned}$$

and the vector field associated to $B \in \mathfrak{so}(3)$ is

$$(Bx, By) = \sum_k \left(\left(B_{jk}x_k\right)\partial_{x_j} + \left(B_{jk}y_k\right)\partial_{y_j}\right).$$

Anyway, the moment map is, say for $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$,

$$\begin{aligned} (x, y) &\mapsto -y \cdot (Bx) \\ &= -\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \cdot \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} = -y_1 x_3 + y_3 x_1, \end{aligned}$$

which is the $\hat{\jmath}$-component of the cross-product of the vectors y and x, and thus the $\hat{\jmath}$-component of the angular momentum, if x = momentum and y = position. Thus the angular momentum *determines three independent integrals of motion, if the Hamiltonian function depends only on $|x|$ and $|y|$, such as motions in a central force field.*

9.3. **Symplectic Quotients.** We have already seen the symplectic quotient in the case of circle actions. We now consider symplectic reduction in general.

Let (M, ω) be a symplectic manifold, and let $N \subset M$ be a coisotropic submanifold, meaning that

$$\begin{aligned} T_pN^\omega &= \{v \in T_pM : \omega(v, w) = 0 \text{ for all } w \in T_pN\} \\ &\subset T_pN, \text{ for all } p \in N. \end{aligned}$$

The distribution $TN^\omega \subset TN$ is isotropic, meaning that $TN^\omega \subset (TN^\omega)^\omega$; this is automatically true from a lemma in the linear symplectic geometry section.

Lemma 9.11. *If $N \subset M$ is a coisotropic submanifold, then TN^ω is integrable.*

Proof. Let $X, Y \in \Gamma(N, TN^\omega)$. For any vector field $Z \in \Gamma(N, TN)$, we have

$$\begin{aligned} 0 &= d\omega(X, Y, Z) \\ &= \mathcal{L}_X(\omega(Z, Y)) + \mathcal{L}_Z(\omega(Y, X)) + \mathcal{L}_Y(\omega(X, Z)) \\ &\quad +\omega([Y, Z], X) + \omega([Z, X], Y) + \omega([X, Y], Z) \\ &= 0 + 0 + 0 + 0 + 0 + \omega([X, Y], Z) \end{aligned}$$

Thus, $[X, Y] \in \Gamma(N, TN^\omega)$. □

Remark 9.12. *A more general fact: if N is any submanifold, the distribution $TN^\omega \cap TN$ is integrable as long as this distribution has constant dimension. If N is symplectic, you get a foliation by points, and if N is Lagrangian, then the whole submanifold is a single leaf.*

We denote the foliation corresponding to the distribution TN^ω by N^ω.

Definition 9.13. *We say that a coisotropic submanifold $N \subset M$ is **regular** if the foliation N^ω is regular, meaning that each leaf is a compact (connected) submanifold.*

Lemma 9.14. *Let (V, ω) be a symplectic vector space and $W \subset V$ is a coisotropic subspace. Then $V' = W \diagup W^\omega$ carries a natural symplectic structure ω' induced from ω.*

Proof. Let $[w] = w + W^\omega$ for each $w \in W$. Then for all $w_1, w_2 \in W$,

$$\omega(w_1, w_2) = \omega(w_1 + u, w_2 + v)$$

for every $u, v \in W^\omega$, so in fact $\omega(w_1, w_2)$ only depends on $[w_1]$ and $[w_2]$. Hence we may define the two-form ω' on V' by

$$\omega'([w_1], [w_2]) := \omega(w_1, w_2).$$

If $\omega'([w_1], [w_2]) = 0$ for all $w_2 \in W$, then $\omega(w_1, w_2) = 0$ for all $w_2 \in W$, so $w_1 \in W^\omega$, or $[\omega_1] = 0$. Thus ω' is nondegenerate. □

Lemma 9.15. *If $N \subset M$ is a regular coisotropic submanifold, then the quotient $B = N \diagup N^\omega$ is a symplectic manifold.*

Proof. The tangent space to the quotient of $N \xrightarrow{\pi} B = N \diagup N^\omega$ is

$$T_{\pi(x)}B \cong \frac{T_xN}{T_xN^\omega} = Q_x.$$

By the previous lemma, the two-form induced by the symplectic form on Q_x is nondegenerate; we may think of this as a family of such two-forms parametrized by points of B via a local section. The restriction $\omega|_N$ is closed since it is the pullback of ω to N by the inclusion map, and the two-form on the base is closed because it is locally the pullback of $\omega|_N$ by a local section. Thus Q_x is symplectic. It remains to show that if x and y are on the same leaf of N^ω, then Q_x and Q_y are naturally symplectomorphic (independent of choice of local section. The holonomy map gives the isomorphism; given a leafwise path $\gamma : [0,1] \to L$ in a leaf $L \subset N$, there is an induced isomorphism from $Q_{\gamma(0)}$ to $Q_{\gamma(1)}$. We must still show that the symplectic structure is preserved by this isomorphism and is thus given by a basic two-form on N with respect to the foliation N^ω. This is equivalent to showing that for all leafwise vector fields $X \in \Gamma(N, TN^\omega)$, $i(X)(\omega|_N) = 0$ and $i(X)\,d(\omega|_N) = 0$. The second equation is clear since $\omega|_N$ is closed. The first equation follows from the definition of TN^ω. □

Now we apply the results above to the moment map. Suppose we are given a Hamiltonian G-action on a symplectic manifold (M, ω), with moment map $\mu : M \to \mathfrak{g}^*$. This map is G-equivariant via the coadjoint action. Since $0 \in \mathfrak{g}$ is a fixed point of the coadjoint action, $\mu^{-1}(0)$ is G-invariant.

Proposition 9.16. *Suppose that 0 is a regular value of μ, so that $\mu^{-1}(0)$ is a submanifold of M. Suppose that G acts freely and properly on $\mu^{-1}(0)$, so that $\mu^{-1}(0)\diagup G$ is a manifold. Then $\mu^{-1}(0)$ is coisotropic and the quotient*

$$M\diagup\diagup G = \mu^{-1}(0)\diagup G$$

*is a symplectic manifold called the **Marsden-Weinstein quotient**, and its dimension is*

$$\dim(M\diagup\diagup G) = \dim M - 2\dim G.$$

Proof. Let $\mathcal{O}_p$ denote the G-orbit of p, so that

$$T_p\mathcal{O}_p = \{X_\xi(p) : \xi \in \mathfrak{g}\}.$$

We will show that if $p \in \mu^{-1}(0)$, then $T_p\mathcal{O}_p = T_p(\mu^{-1}(0))^\omega$. To see this, observe that if $\mu(p) = 0$, then $H_\xi(p) = \mu(p)(\xi) = 0$ for all $\xi \in \mathfrak{g}$. Since H_ξ is constant on $\mu^{-1}(0)$, $\omega(X_\xi, v) = -dH_\xi(v) = 0$ for all $v \in T_p(\mu^{-1}(0))$, so we have $T_p\mathcal{O}_p \subset T_p(\mu^{-1}(0))^\omega$. Recall that in general $\dim W^\omega + \dim W = \dim V$ for all subspaces W of (V, ω), so that $\dim W^\omega = \text{codim}W$ for all subspaces of a symplectic vector space V. Then, since $\dim\mathcal{O}_p = \dim G = \text{codim } \mu^{-1}(0) = \dim T_p(\mu^{-1}(0))^\omega$, we have $T_p(\mu^{-1}(0))^\omega = T_p\mathcal{O}_p \subset T_p(\mu^{-1}(0))$, so $\mu^{-1}(0)$ is coisotropic. By the previous lemmas, the result follows. □

For future use, we show the dual Lie algebra $\mathfrak{g}^*$ of a compact, connected Lie group is itself a Poisson manifold (meaning it has a Poisson bracket but not necessarily a symplectic structure), and moreover, it is a union of symplectic submanifolds.

Example 9.17. *Assume that G is compact and connected, and let $\mathcal{O} \subset \mathfrak{g}^*$ denote an orbit under the coadjoint action. Then, note that $\mathcal{O}$ carries a natural symplectic structure. To see this, observe that for $\eta \in \mathcal{O}$*

$$T_\eta\mathcal{O} = \{ad(\xi)^*\eta : \xi \in \mathfrak{g}\},$$

where $ad(\xi_1)\xi_2 = [\xi_1, \xi_2]$ for $\xi_1, \xi_2 \in \mathfrak{g}$. The symplectic form $\omega_\mathcal{O}$ is defined by

$$\omega_{\mathcal{O},\eta}(ad(\xi_1)^*\eta, ad(\xi_2)^*\eta) = \eta([\xi_1, \xi_2]).$$

(One may check that it is closed and nondegenerate.) Thus $\mathfrak{g}^*$ is a union of orbits of the coadjoint action, and each such orbit is a symplectic submanifold of the vector space $\mathfrak{g}^*$. Also, the vector field

$$X_\xi(\eta) = ad(\xi)^* \eta$$

is the Hamiltonian vector field generated by $H_\xi(\eta) = \eta(\xi)$, and this identification makes the coadjoint action on $\mathcal{O}$ Hamiltonian. Thus, the moment map is

$$\mu(\eta) = \eta.$$

The dual $\mathfrak{g}^*$ is actually a Poisson manifold, where the Poisson bracket is defined as follows. The Lie algebra $\mathfrak{g}$ embedds in $C^\infty(\mathfrak{g}^*, \mathbb{R})$ as the subspace of linear functions, and the Poisson bracket applied to these functions is just the regular Lie bracket. This may be extended to a Lie bracket on all of $C^\infty(\mathfrak{g}^*, \mathbb{R})$.

Next, we consider a slightly more general construction of the symplectic quotient. Let $\mathcal{O} \subset \mathfrak{g}^*$ denote an orbit of the coadjoint action. Then $\mu^{-1}(\mathcal{O})$ is G-invariant. Then it turns out that $\mu^{-1}(\mathcal{O}) \diagup G$ is a symplectic manifold under suitable assumptions. In particular, μ is transverse to $\mathcal{O}$ if and only if every point of $\mathcal{O}$ is a regular value of μ.

Proposition 9.18. *Let $\mathcal{O} \subset \mathfrak{g}^*$ denote an orbit of the coadjoint action. Suppose that every point of $\mathcal{O}$ is a regular value of μ and that G acts freely on $\mu^{-1}(\mathcal{O})$. Then*

$$M_{\mathcal{O}} := \mu^{-1}(\mathcal{O}) \diagup G$$

is a symplectic manifold of dimension

$$\dim M_{\mathcal{O}} = \dim M + \dim \mathcal{O} - 2\dim G,$$

with (transverse) symplectic form at $p \in \mu^{-1}(\mathcal{O})$ given by

$$\omega_\mu(v_1, v_2) = \omega(v_1, v_2) - \mu(p)([\xi_1, \xi_2]),$$

for $v_1, v_2 \in T_p\mu^{-1}(\mathcal{O})$ and $\xi_1, \xi_2 \in \}$, where

$$ad(\xi_j)^* \mu(p) + d\mu(p) v_j = 0$$

for $j = 1, 2$.

Proof. Consider the symplectic manifold $M' = M \times \mathcal{O}$ with symplectic structure $\omega' = \omega \times (-\omega_{\mathcal{O}})$, where $\omega_{\mathcal{O}}$ is defined by the example above. This action is Hamiltonian, and the moment map of the product is the sum of the moment maps, so

$$\mu'(p, \eta) = \mu(p) - \eta.$$

Hence, there is a natural diffeomorphism $M \times \mathcal{O} \diagup\diagup G \cong M_{\mathcal{O}}$. The result follows from the previous result. □

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