KODAIRA'S THEOREM AND COMPACTIFICATION OF MUMFORD'S MODULI SPACE M_g

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Abstract. This is intended as an appendix to Ken Richardson's recent talks on the Field's medal work of Maryam Mirzhakani [14]. I hope to explain how Wolpert's description of the Weil-Petersson symplectic form and it's extension to a positive line bundle on $\overline{M_{g,n}}$ [17] embeds it into a projective space via the Kodaira embedding theorem (following Lazarsfeld's book on positivity in algebraic geometry [13]). I will also explain the standard algebro-geometric method for compactifying Mumford's moduli space M_g parametrizing isomorphism classes of complex projective smooth connected curves of genus g [15, 7].

1. Motivation

The following theorem was mentioned in Ken's talk:

Theorem 1.1. (Wolpert [17]): *The Weil-Petursson symplectic form extends to a positive line bundle on $\overline{M_{g,n}}$, therefore $\overline{M_{g,n}}$ is a complex projective variety.*

My goals here are to (a) explain various notions of positivity for line bundles, (b) explain Kodaira's embedding theorem, which gives the "therefore" part of the theorem and (c) explain how algebraic geometers compactify the Mumford moduli space.

2. Kähler forms

Let X be a complex manifold. For $x \in X$, let T_xX denote the real tangent space at x with endomorphism $J : T_xX \to T_xX$ determining the complex structure, so that $J^2 = -\text{Id}$. If ω is a C^∞ 2-form on X, we write $\omega(v, w) \in \mathbb{R}$ for the evaluation at real tangent vectors $v, w \in T_xX$.

Definition 2.1. Let ω be a 2-form on a complex manifold X. We say that ω is

- **type** $(1, 1)$ if $\omega(Jv, Jw) = \omega(v, w)$ at each $x \in X$ and for each pair v, w of real tangent vectors.
- **positive** if it has type $(1, 1)$ and $\omega(v, Jv) > 0$ for each $x \in X$ and nonzero v.

A **Kähler form** is positive $(1, 1)$ form ω which is closed, meaning that $d\omega = 0$.

Remark 2.2. A 2-form ω is a positive $(1, 1)$ form if and only if it can be written

$$\omega = \frac{i}{2} \sum h_{\alpha\beta} dz_\alpha \wedge d\overline{z}_\beta$$

in local holomorphic coordinates $z_1, \dots z_n$ with $h_{\alpha\beta}$ a positive definite Hermitian matrix of complex valued functions at each point.

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Example 2.3. The classic example of a Kähler form is the Fubini-Study form on $\mathbb{P}^n$. Following the description given by Arnol'd [2], let $\langle v, w\rangle = v^t \cdot \overline{w}$ be the standard Hermitian inner product on $\mathbb{C}^{n+1}$ and consider the canonical projection $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$. First scale the inner product as

$$H_x(v,w) = \langle \frac{v}{|x|}, \frac{w}{|x|}\rangle \quad \text{for } v, w \in T_x = \mathbb{C}^{n+1}$$

to make it invariant under the $\mathbb{C}^*$ action so that H descends to the Fubini-Study (Hermitian) metric H_{FS} on $\mathbb{P}^n$, which is invariant under the $\text{U}(n+1)$ action on $\mathbb{P}^n$.

The Fubini-Study form is $\omega_{\text{FS}} = -\text{Im} H_{\text{FS}}$. On the standard open set $U_0 \subset \mathbb{P}^n$ defined by $x_0 = 1$, one can calculate in local coordinates that

$$\omega_{\text{FS}} = \frac{i}{2} \cdot \left(\frac{\sum dz_\alpha \wedge d\overline{z}_\alpha}{1 + \sum |z_\alpha|^2} - \frac{(\sum \overline{z}_\alpha dz_\alpha) \wedge (\sum z_\alpha d\overline{z}_\alpha)}{(1 + \sum |z_\alpha|^2)^2} \right).$$

which is a positive $(1,1)$ form by Remark 2.2 or note that the negative imaginary part of a Hermitian matrix is positive definite. Mumford observed that the $\text{U}(n+1)$ invariance gives a quick proof that ω_{FS} is closed [16, Lemma 5.20]: for $p \in \mathbb{P}^n$, choose $\gamma \in \text{U}(n+1)$ with $\gamma(p) = p$ and $d\gamma_p = -\text{Id}$ (take the $(n+1) \times (n+1)$ matrix which 1 along the axis corresponding to x and $-\text{Id}$ on the orthogonal complement). Then for $u, v, w \in T_p\mathbb{P}^n$ we have

$$d\omega_{\text{FS}}(u,v,w) = \gamma^*(d\omega_{\text{FS}})(u,v,w) = d\omega_{\text{FS}}(-u,-v,-w)$$

so that $d\omega_{\text{FS}} = 0$.

3. Metrically positive line bundles and Kodaira's theorem

3.1. **Curvature form associated to a line bundle.** Fix a Hermitian metric h on a holomorphic line bundle L over the complex manifold X. The Hermitian line bundle (L, h) gives rise to the **curvature form**

$$\Theta(L,h) \in C^\infty(X, \Lambda^{1,1}T^*X).$$

It is a closed $(1,1)$ form defined as follows: The exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

on X gives rise to the cohomology sequence fragment

$$\text{Pic}\, X \cong H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

The Picard group classifies line bundles over X, which are also determined by cocycles which gives the isomorphism on the left: thinking of a line bundle $L \in \text{Pic}\, X$, the image under the boundary map on cohomology is the first Chern class $c_1(L) \in H^2(X,\mathbb{Z})$. On the other hand, Hodge theory gives

$$H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

and via this isomorphism the image $c_1(L)$ of a complex analytic line bundle L lands in $H^{1,1}(X)$ and is thus associated to a global 2-form of type $(1,1)$ - this is a theorem of Lefschetz and Hodge, also of Kodaira and Spencer [12]. In the non-Kähler case it was proven by Dolbeault [4, 5].

This approach shows that the curvature form is independent of the choice of metric, but using the metric h there is also a local description: writing $|\ \ |_h$ for the length function on the fibers of L, if s is a holomorphic section of L which is non-vanishing on the open set U, then as a 2-form $\Theta(L,h)$ is defined locally by $\Theta = -\partial\overline{\partial}\text{log}|s|_h^2$ independent of s.

$$\Theta(L,h) = \partial\overline{\partial}\log < s, s >_h$$

is independent of the local nonvanishing section s of the line bundle L with hermitian metric h. Any other such section would be of the form fs, where f is a holomorphic function that also does not vanish. Plugging into the formula,

$$\partial\overline{\partial}\log < fs, fs >_h = \partial\overline{\partial}\log(f\overline{f} < s, s >_h) = \partial\overline{\partial}(\log f + \log\overline{f} + \log < s, s >_h)$$

which simplifies to $\partial\overline{\partial}\log < s, s >_h$ because ∂ annihilates the antiholomorphic function $\log\overline{f}$ and $\overline{\partial}$ annihilates the holomorphic function $\log f$.

3.2. **Metrically positive line bundles.** A holomorphic line bundle L on a complex manifold X is **positive** (in the sense of Kodaira) if it carries a Hermitian metric h for which $\frac{i}{2\pi}\Theta(L,h)$ is a Kähler form.

Example 3.1. The classic example is the hyperplane bundle $\mathcal{O}(1)$ on $X = \mathbb{P}^n$. The standard Hermitian inner product on $\mathbb{C}^{n+1}$ gives a Hermitian metric on the trivial bundle $\mathbb{C} \times \mathbb{P}^n$ which corresponds in sheaf language to $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$, the sheaf of holomorphic functions. Multiplying by a linear form represents $\mathcal{O}(-1)$ as a subbundle of $\mathcal{O}$, so it inherits the corresponding metric. Finally $\mathcal{O}(1)$ is the dual bundle to $\mathcal{O}(-1)$ and as such inherits a canonical metric. Using local coordinates, Griffiths and Harris explicitly compute that

$$\frac{i}{2\pi}\Theta(\mathcal{O}(1),h) = \frac{1}{\pi}\omega_{\text{FS}}$$

so that $\mathcal{O}(1)$ is positive in the sense of Kodaira.

Example 3.2. If X is any complex manifold for which there is a holomorphic embedding

$$\varphi : X \hookrightarrow \mathbb{P}^n \tag{1}$$

Then X is a compact Kähler manifold (meaning it carries a Kähler form) and the pull-back $M = \varphi^*(\mathcal{O}(1))$ is a holomorphic line bundle which is positive in the sense of Kodaira, as can be seen by pulling back the Fubini-Study metric.

Remark 3.3. If L is a positive line bundle, then so is any tensor power $M = L^{\otimes m}$. Conversely, if $M = L^{\otimes m}$ for some positive line bundle M and $m > 0$, then L itself is positive. In other words, positive line bundles in $\operatorname{Pic} X$ are closed under taking mth powers and "mth roots". The reason is that if L, M are line bundles, then $c_1(L \otimes M) = c_1(L) + c_1(M)$ as cohomology classes (or as 2-forms).

3.3. **Kodaira's embedding theorem.** The surprise here is that all positive line bundles occur as in Example 3.2:

Theorem 3.4. (Kodaira, 1954 [10]) *Let X be a compact Kähler manifold and let L be a positive holomorphic line bundle on X. Then for some $m > 0$ there is a holomorphic embedding $\varphi : X \hookrightarrow \mathbb{P}^N$ for which $\varphi^*\mathcal{O}(1) = L^{\otimes m}$.*

Remark 3.5. Chow's theorem [3] says that a complex analytic submanifold of $\mathbb{P}^n$ is in fact algebraic, the zero locus of polynomial equations. Applying this we find in the theorem above that X has the structure of a complex projective algebraic variety.

Remark 3.6. Furthermore, in the context of Kodaira's theorem one can use typical topological methods as in the Whitney embedding theorem to embed into a smaller dimensional space: suppose $\varphi : X \hookrightarrow \mathbb{P}^n$ and $p \in \mathbb{P}^n$ is a point lying on no secant line through of X nor lying on any tangent line to X. Then projection from p gives an embedding into to $\mathbb{P}^{n-1}$:

$$\begin{array}{ccc} \varphi : X & \stackrel{\varphi}{\hookrightarrow} & \mathbb{P}^n - \{p\} \\ & \searrow & \downarrow \pi_p \\ & & \mathbb{P}^{n-1} \end{array}$$

Since the secant variety is the closure of all secant lines to X in $\mathbb{P}^n$, it has dimension at most $\leq 2\dim X + 1$ (the one-dimensional lines are parametrized by pairs of points on X) and the union of tangent planes has dimension at most $2\dim X$, we can continue to project from points in this fashion until we obtain an embedding into $\mathbb{P}^m$ with $m = 2\dim X + 1$.

Remark 3.7. Given a line bundle M on X as above, how does M actually give rise to the embedding φ? I.e. the embedding given in the theorem is stated abstractly, but one would like to understand how this works. The vector space $H^0(X, M)$ of global holomorphic sections of M is finite dimensional, so choose a basis $s_0, \dots s_n$. Taking homogeneous coordinates x_i for $\mathbb{P}^n$, we produce a map $\varphi : X \to \mathbb{P}^n$ for which $\varphi^* x_i = s_i$ as follows: Each s_i vanishes on a closed subset, let $U_i = \{x_i \neq 0\}$ be the open complement. Similarly x_i vanishes on a closed subset in $\mathbb{P}^n$ with open complement $U_i \cong \mathbb{A}^n$ whose coordinates are $y_j = x_j/x_i$ with y_i omitted. Since $s_i \neq 0$ on X_i, it gives an isomorphism

$$s_i : \mathcal{O}_X \xrightarrow{\sim} L$$

which allows us to write s_j/s_i as a holomorphic function in $\mathcal{O}_X(X_i)$, which we can evaluate at p to obtain the map

$$p \mapsto (s_0/s_i(p), \dots s_{i-1}/s_i(p), s_{i+1}/s_i(p), \dots s_n/s_i(p))$$

Gluing together these maps over the open cover gives a global map on X. Perhaps it is easier to exploit the equivalence relation on $\mathbb{P}^n$ and write

$$p \mapsto (s_0(p), \dots s_n(p)) \in \mathbb{P}^n.$$

but this description should go in quotation marks as the evaluation is not so clear here.

An important ingredient in Kodaira's embedding theorem is the famous Kodaira vanishing theorem:

Theorem 3.8. (Kodaira, 1953 [11]**, Akizuki and Nakano, 1954** [1]**)** *Let X be a complex Kähler manifold and L a positive holomorphic line bundle on X. Then the cohomology groups $H^p(X, \Omega_X^p \otimes L^{-1})$ vanish for $p + q < n$.*

Example 3.9. Let X be a compact Riemann surface, i.e. a complex algebraic curve carrying a line bundle L. If $H^0(X, L) \neq 0$, then any section $s \in H^0(X, L)$ vanishes at a finite number of points, which is constant if counted with multiplicity: this number is the **degree** of L. If $H^0(X, L) = 0$, one can use the Riemann-Roch theorem and twist L until it obtains nonzero global sections, and use this number to write a well-defined degree of L, written $\deg L$. The Riemann-Roch theorem can be used to show that $L = \varphi^*\mathcal{O}(1)$ for some holomorphic embedding $\varphi : X \hookrightarrow \mathbb{P}^n$ if $\deg L > 2g+1$, where g is the genus of X. Thus L is positive in the sense of Kodaira if and only if $\deg L > 0$.

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