SCISSORS CONGRUENCE

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1. Scissors Congruence in the Plane

Definition 1.1. *A* polygonal decomposition *of a polygon P in the plane is a finite set $\{P_1, P_2, \ldots, P_k\}$ of polygons whose union is P and that pairwise only intersect on their boundaries. We will often write this using the notation $P = P_1 + P_2 + \cdots + P_k$.*

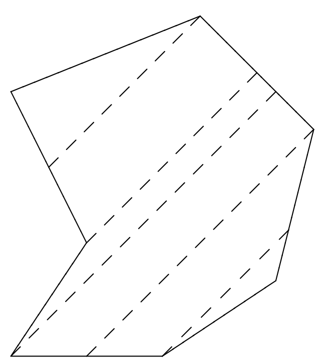
Definition 1.2. *Polygons P and Q are* scissors congruent *if there exist polygonal decompositions $\{P_1, P_2, \ldots, P_k\}$ and $\{Q_1, Q_2, \ldots, Q_k\}$ of P and Q respectively with the property that P_i is congruent to Q_i for $1 \le i \le k$.*

Proposition 1.3. *Scissors congruence is an equivalence relation on the set of polygons.*

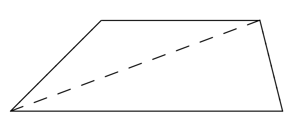
Proposition 1.4. *If two polygons are scissors congruent, then they have equal area.*

Lemma 1.5. *Every polygon has a polygonal decomposition consisting of triangles.*

Proof. Let P be a polygon and choose a slope m that is different from each of the slopes of the sides of P. Lines of slope m through the vertices of P give a polygonal decomposition consisting of triangles and trapezoids:



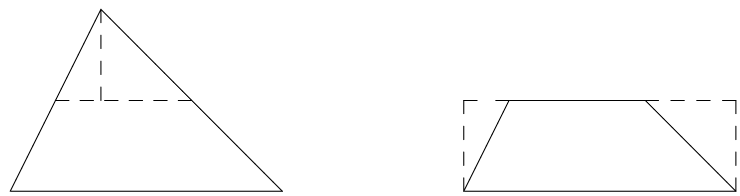
For each trapezoid, make a diagonal cut to decompose it into two triangles.



$\square$

Lemma 1.6. *Every triangle is scissors congruent to a rectangle.*

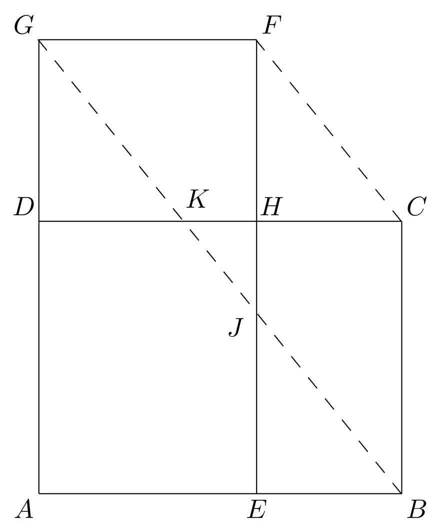
Proof. Given a triangle, construct the altitude at the vertex with the largest angle. Cut the triangle with a line that is perpendicular to the altitude and bisects it. This cuts the triangle into three pieces which can be reassembled into a rectangle.



□

Proposition 1.7. *Any two rectangles with equal area are scissors congruent.*

Proof. A rectangle with base b and height h is clearly scissors congruent to a rectangle with base $2b$ and height $\frac{1}{2}h$. Therefore, given rectangles R_i with base b_i and height h_i, $i = 1, 2$, we may apply the "double the base and halve the height" operation to one or both of R_1 and R_2 to arrange that either (a) $b_1 = b_2$, in which case R_1 and R_2 are congruent, or (b) we have the inequalities $b_1 < b_2 < 2b_1$. For the latter case, consider this picture:



The rectangles $AEFG$ and $ABCD$ are R_1 and R_2 respectively. By elementary Euclidean geometry, we deduce that triangles GDK and JEB are each congruent to FHC and thus congruent to one another. Also, we see that the triangles GJF and KBC are congruent. Therefore

$$ABCD = AEJKD + JEB + KBC$$

is scissors congruent to

$$AEFG = AEJKD + GDK + GJF.$$

□

Corollary 1.8. *If two polygons have equal area, then they are scissors congruent.*

2. Scissors Congruence in 3-Space

Definition 2.1. *A* polyhedral decomposition *of a polyhedron P in 3-space is a finite set $\{P_1, P_2, \ldots, P_k\}$ of polyhedra whose union is P and that pairwise only intersect on their faces and/or edges. In this case we write $P = P_1 + P_2 + \cdots + P_k$.*

Definition 2.2. *Polyhedra P and Q are* scissors congruent *if there exist polyhedral decompositions $\{P_1, P_2, \ldots, P_k\}$ and $\{Q_1, Q_2, \ldots, Q_k\}$ of P and Q respectively with the property that P_i is congruent to Q_i for $1 \le i \le k$.*

Proposition 2.3. *Scissors congruence is an equivalence relation on the set of polyhedra.*

Proposition 2.4. *If two polyhedra are scissors congruent, then they have equal volume.*

Question 2.5 (Hilbert's Third Problem)**.** *Is the converse of Proposition 2.4 true?*

Definition 2.6. *Let A be a set of real numbers. A function $f : A \longrightarrow \mathbb{R}$ is* integrally additive *if whenever a finite sum*

$$n_1\alpha_1 + n_2\alpha_2 + \cdots + n_m\alpha_m = 0$$

for some integers $n_1, n_2, \ldots, n_m$ and some elements $\alpha_1, \alpha_2, \ldots, \alpha_m$ of A, it is also true that

$$n_1 f(\alpha_1) + n_2 f(\alpha_2) + \cdots + n_m f(\alpha_m) = 0.$$

Definition 2.7. *Suppose that P is a polyhedron, let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be the set of dihedral angles of P, and for each $1 \le i \le m$, let ℓ_i be the length (not necessarily an integer!) of the edge of P that forms the vertex of the angle α_i. Given an integrally additive function f on A, the quantity*

$$f(P) := \ell_1 f(\alpha_1) + \ell_2 f(\alpha_2) + \cdots + \ell_m f(\alpha_m)$$

is called the Dehn invariant *of P associated to f.*

Lemma 2.8. *Suppose we have a polyhedral decomposition of a polyhedron P:*

$$P = P_1 + P_2 + \cdots + P_k.$$

Let A be the set containing π and the dihedral angles of $P, P_1, P_2, \ldots, P_k$, and let f be an integrally additive function on A with the property that $f(\pi) = 0$. Then

$$f(P) = f(P_1) + f(P_2) + \cdots + f(P_k). \tag{$\star$}$$

Proof. Our polyhedral decomposition of P divides each edge of P, P_1, P_2, …, P_k into a union of line segments; we will call these line segments *edgitos*. Suppose an edgito has length ℓ and corresponding dihedral angle α. The quantity $\ell f(\alpha)$ is called the *weight* of the edgito. A moment's thought yields that $f(P)$ is the sum of the weights of all the edgitos of P.

Fix an edgito e with length ℓ, let $P_{i_1}, P_{i_2}, \ldots, P_{i_j}$ be the polyhedra in our decomposition that have e as part of one of their edges, and for $1 \le s \le j$, let γ_s be the dihedral angle in P_{i_s} associated to e.

Next, we study the quantity

$$\ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j),$$

which is a "subsum" of $f(P)$. We consider three cases.

Case 1: The edgito e lines entirely (except possibly for one or both endpoints) in the interior of P. Then

$$\gamma_1 + \gamma_2 + \cdots + \gamma_j - 2\pi = 0,$$

whence

$$\begin{aligned} 0 &= f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) - 2f(\pi) \\ &= f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) \\ &= \ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j). \end{aligned}$$

Case 2: The edgito e lies on a face of P, but not on an edge. Then

$$\gamma_1 + \gamma_2 + \cdots + \gamma_j - \pi = 0,$$

whence

$$\begin{aligned} 0 &= f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) - f(\pi) \\ &= f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) \\ &= \ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j). \end{aligned}$$

Case 3: The edgito e lies on an edge of P. If α is the dihedral angle of P associated to this edge, then the quantity $\gamma_1 + \gamma_2 + \cdots + \gamma_j$ must equal either α or $\alpha - \pi$. In either event, the requirement that $f(\pi) = 0$ implies that

$$f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) = f(\alpha),$$

and hence

$$\ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j) = \ell f(\alpha),$$

which is the weight of the edgito e in P. Therefore, if we sum over all edgitos in our polyhedral decomposition of P, we obtain the right hand side of $(\star)$. $\square$

Lemma 2.9. *Let f be an integrally additive function on a set A and let $\widetilde{\alpha}$ be a real number not in A. Then f can be extended to an integrally additive function on $A \cup \{\widetilde{\alpha}\}$.*

Proof. For notational convenience, we use the notation

$$\sum_{\alpha \in A}{}' n_\alpha \alpha$$

to denote a sum of integer multiples of elements of A with all but finitely many of the integers n_α equal to zero. We consider two cases:

Case 1: There exists no equation

$$\sum_{\alpha \in A}{}' n_\alpha \alpha + n_{\widetilde{\alpha}} \widetilde{\alpha} = 0$$

with $n_{\widetilde{\alpha}} \neq 0$. In this case we can define $f(\widetilde{\alpha})$ to be any real number.

Case 2: There does exist an equation

$$\sum_{\alpha \in A}{}' n_\alpha \alpha + n_{\widetilde{\alpha}} \widetilde{\alpha} = 0 \tag{1}$$

with $n_{\widetilde{\alpha}} \neq 0$. Choose one such equation and define

$$f(\widetilde{\alpha}) = -{\sum_{\alpha \in A}}' \tfrac{n_\alpha}{n_{\widetilde{\alpha}}} f(\alpha).$$

We must show that f is integrally additive on $\widetilde{A}$. Suppose we have an integral linear dependence

$$(2) \qquad {\sum_{\alpha \in A}}' m_\alpha \alpha + m_{\widetilde{\alpha}} \widetilde{\alpha} = 0.$$

If $m_{\widetilde{\alpha}} = 0$, the lemma follows immediately, so suppose that $m_{\widetilde{\alpha}} \neq 0$. Multiply Equation (1) by $m_{\widetilde{\alpha}}$, multiply Equation (2) by $n_{\widetilde{\alpha}}$, and subtract to eliminate $\widetilde{\alpha}$:

$$(3) \qquad {\sum_{\alpha \in A}}' (n_{\widetilde{\alpha}} m_\alpha - n_\alpha m_{\widetilde{\alpha}}) \alpha = 0.$$

Then

$$(4) \qquad {\sum_{\alpha \in A}}' (n_{\widetilde{\alpha}} m_\alpha - n_\alpha m_{\widetilde{\alpha}}) f(\alpha) = 0.$$

Next, multiply the equation

$$(5) \qquad {\sum_{\alpha \in A}}' n_\alpha f(\alpha) + n_{\widetilde{\alpha}} f(\widetilde{\alpha}) = 0$$

by $m_{\widetilde{\alpha}}$ and subtract Equation (4) to obtain

$$(5) \qquad {\sum_{\alpha \in A}}' n_{\widetilde{\alpha}} m_\alpha f(\alpha) + n_{\widetilde{\alpha}} m_{\widetilde{\alpha}} f(\widetilde{\alpha}) = 0$$

The fact that $n_{\widetilde{\alpha}} \neq 0$ yields

$$(7) \qquad {\sum_{\alpha \in A}}' m_\alpha f(\alpha) + m_{\widetilde{\alpha}} f(\widetilde{\alpha}) = 0,$$

as desired. □

Theorem 2.10. *Let P and Q be polyhedra and define*

$$M = \{\pi\} \cup A_P \cup A_Q,$$

where A_P and A_Q are the sets of dihedral angles of P and Q respectively. Suppose there exists an integrally additive function f on M such that $f(\pi) = 0$. If P and Q are scissors congruent, then $f(P) = f(Q)$.

Proof. Decompose P and Q into polyhedra

$$P = P_1 + P_2 + \cdots + P_k$$
$$Q = Q_1 + Q_2 + \cdots + Q_k$$

with the property that $P_i \cong Q_i$ for $1 \leq i \leq k$. Using Lemma 2.9, we can extend f to be an integrally additive function on the set consisting of M and all the dihedral angles of our subpolyhedra. Obviously $f(P_i) = f(Q_i)$ for $1 \leq i \leq k$, and so we have

$$f(P) = f(P_1) + f(P_2) + \cdots + f(P_k) = f(Q_1) + f(Q_2) + \cdots + f(Q_k) = f(Q)$$

by Lemma 2.8. □

Lemma 2.11. *The number $\frac{1}{\pi} \arccos\left(\frac{1}{3}\right)$ is irrational.*

Proof. Using induction and the trig identity

$$\cos((n+1)\theta) = 2\cos\theta\cos n\theta - \cos((n-1)\theta)$$

we see that $\cos n\theta = T_n(\cos\theta)$, where T_n is an n-th degree polynomial with integer coefficients and whose leading term is 2^{n-1}.

Set $\phi = \arccos\left(\frac{1}{3}\right)$ and suppose that $\phi = \frac{p}{q}\pi$ for some positive integers p and q. Then

$$T_q\left(\frac{1}{3}\right) = T_q\left(\cos\left(\frac{p}{q}\pi\right)\right) = T_q\left(\cos\left(p\frac{\pi}{q}\right)\right) = \cos(p\pi) = \pm 1,$$

whence $\frac{1}{3}$ is a root of a polynomial with integer coefficients and leading term 2^{q-1} for some q. We deduce from Rational Root Test that 3 divides 2^{q-1} for some q, a contradiction. □

Theorem 2.12. *A regular tetrahedron and a cube of equal volume are not scissors congruent.*

Proof. Let T and C be the regular tetrahedron and cube of volume 1 respectively. Obviously each dihedral angle of C is $\frac{\pi}{2}$ and an easy exercise in analytic geometry shows that each dihedral angle ϕ of T equals $\arccos\left(\frac{1}{3}\right)$. Let A be the set $\left\{\pi, \frac{\pi}{2}, \phi\right\}$ and define $f : A \longrightarrow \mathbb{R}$ by setting

$$f(\pi) = 0, \qquad f\left(\frac{\pi}{2}\right) = 0, \qquad f(\phi) = 1.$$

I claim that f is integrally additive. Suppose that

$$n_1\pi + n_2 \cdot \frac{\pi}{2} + n_3\phi = 0$$

for some integers n_1, n_2, and n_3. If $n_3 = 0$, we immediately see that

$$n_1 f(\pi) + n_2 \cdot \frac{\pi}{2} + n_3\phi = n_1 f(\pi) + n_2 f\left(\frac{\pi}{2}\right) = 0.$$

On the other hand, if $n_3 \neq 0$, then

$$\frac{\phi}{\pi} = -\frac{2n_1 + n_2}{2n_3},$$

contradicting Lemma 2.11.

Now we compute the Dehn invariants of C and T associated to f. The length of each side of C is 1, so

$$f(C) = 12 \cdot 1 \cdot f\left(\frac{\pi}{2}\right) = 0.$$

Let m be the length of each side of T. From the volume formula for a regular tetrahedron

$$V = \frac{\sqrt{2}}{12}m^3$$

we compute

$$m = \left(\frac{12}{\sqrt{2}}\right)^{1/3} \approx 2.04 \neq 0$$

and thus

$$f(T) = 6mf(\phi) = 6\left(\frac{12}{\sqrt{2}}\right)^{1/3} \neq 0.$$

Applying (the contrapositive of) Theorem 2.10, we see that T and C are not scissors congruent. □

Theorem 2.13 (Sydler, 1965)**.** *If P and Q are polyhedra that have equal volumes and have the property that $f(P) = f(Q)$ for every integrally additive function satisfying $f(\pi) = 0$, then P and Q are scissors congruent.*