1. Scissors Congruence in the Plane

**Definition 1.1.** A polygonal decomposition of a polygon $P$ in the plane is a finite set $\{P_1, P_2, \ldots, P_k\}$ of polygons whose union is $P$ and that pairwise only intersect on their boundaries. We will often write this using the notation $P = P_1 + P_2 + \cdots + P_k$.

**Definition 1.2.** Polygons $P$ and $Q$ are scissors congruent if there exist polygonal decompositions $\{P_1, P_2, \ldots, P_k\}$ and $\{Q_1, Q_2, \ldots, Q_k\}$ of $P$ and $Q$ respectively with the property that $P_i$ is congruent to $Q_i$ for $1 \leq i \leq k$.

**Proposition 1.3.** Scissors congruence is an equivalence relation on the set of polygons.

**Proposition 1.4.** If two polygons are scissors congruent, then they have equal area.

**Lemma 1.5.** Every polygon has a polygonal decomposition consisting of triangles.

*Proof.* Let $P$ be a polygon and choose a slope $m$ that is different from each of the slopes of the sides of $P$. Lines of slope $m$ through the vertices of $P$ give a polygonal decomposition consisting of triangles and trapezoids:

For each trapezoid, make a diagonal cut to decompose it into two triangles.
Lemma 1.6. Every triangle is scissors congruent to a rectangle.

Proof. Given a triangle, construct the altitude at the vertex with the largest angle. Cut the triangle with a line that is perpendicular to the altitude and bisects it. This cuts the triangle into three pieces which can be reassembled into a rectangle.

Proposition 1.7. Any two rectangles with equal area are scissors congruent.

Proof. A rectangle with base $b$ and height $h$ is clearly scissors congruent to a rectangle with base $2b$ and height $\frac{1}{2}h$. Therefore, given rectangles $R_i$ with base $b_i$ and height $h_i$, $i = 1, 2$, we may apply the “double the base and halve the height” operation to one or both of $R_1$ and $R_2$ to arrange that either (a) $b_1 = b_2$, in which case $R_1$ and $R_2$ are congruent, or (b) we have the inequalities $b_1 < b_2 < 2b_1$. For the latter case, consider this picture:

The rectangles $AEFG$ and $ABCD$ are $R_1$ and $R_2$ respectively. By elementary Euclidean geometry, we deduce that triangles $GDK$ and $JEB$ are each congruent to $FHC$ and thus congruent to one another. Also, we see that the triangles $GJF$ and $KBC$ are congruent. Therefore

$$ABCD = AEJKD + JEB + KBC$$

is scissors congruent to

$$AEFG = AEJKD + GDK + GJF.$$

Corollary 1.8. If two polygons have equal area, then they are scissors congruent.
2. Scissors Congruence in 3-Space

**Definition 2.1.** A polyhedral decomposition of a polyhedron $P$ in 3-space is a finite set $\{P_1, P_2, \ldots, P_k\}$ of polyhedra whose union is $P$ and that pairwise only intersect on their faces and/or edges. In this case we write $P = P_1 + P_2 + \cdots + P_k$.

**Definition 2.2.** Polyhedra $P$ and $Q$ are scissors congruent if there exist polyhedral decompositions $\{P_1, P_2, \ldots, P_k\}$ and $\{Q_1, Q_2, \ldots, Q_k\}$ of $P$ and $Q$ respectively with the property that $P_i$ is congruent to $Q_i$ for $1 \leq i \leq k$.

**Proposition 2.3.** Scissors congruence is an equivalence relation on the set of polyhedra.

**Proposition 2.4.** If two polyhedra are scissors congruent, then they have equal volume.

**Question 2.5** (Hilbert’s Third Problem). Is the converse of Proposition 2.4 true?

**Definition 2.6.** Let $A$ be a set of real numbers. A function $f : A \to \mathbb{R}$ is integrally additive if whenever a finite sum

$$n_1 \alpha_1 + n_2 \alpha_2 + \cdots + n_m \alpha_m = 0$$

for some integers $n_1, n_2, \ldots, n_m$ and some elements $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $A$, it is also true that

$$n_1 f(\alpha_1) + n_2 f(\alpha_2) + \cdots + n_m f(\alpha_m) = 0.$$

**Definition 2.7.** Suppose that $P$ is a polyhedron, let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be the set of dihedral angles of $P$, and for each $1 \leq i \leq m$, let $\ell_i$ be the length (not necessarily an integer) of the edge of $P$ that forms the vertex of the angle $\alpha_i$. Given an integrally additive function $f$ on $A$, the quantity

$$f(P) := \ell_1 f(\alpha_1) + \ell_2 f(\alpha_2) + \cdots + \ell_m f(\alpha_m)$$

is called the Dehn invariant of $P$ associated to $f$.

**Lemma 2.8.** Suppose we have a polyhedral decomposition of a polyhedron $P$:

$$P = P_1 + P_2 + \cdots + P_k.$$

Let $A$ be the set containing $\pi$ and the dihedral angles of $P, P_1, P_2, \ldots, P_k$, and let $f$ be an integrally additive function on $A$ with the property that $f(\pi) = 0$. Then

$$(\star) \quad f(P) = f(P_1) + f(P_2) + \cdots + f(P_k).$$

**Proof.** Our polyhedral decomposition of $P$ divides each edge of $P, P_1, P_2, \ldots, P_k$ into a union of line segments; we will call these line segments edgitos. Suppose an edgito has length $\ell$ and corresponding dihedral angle $\alpha$. The quantity $\ell f(\alpha)$ is called the weight of the edgito. A moment’s thought yields that $f(P)$ is the sum of the weights of all the edgitos of $P$.

Fix an edgito $e$ with length $\ell$, let $P_1, P_{i_2}, \ldots, P_{i_j}$ be the polyhedra in our decomposition that have $e$ as part of one of their edges, and for $1 \leq s \leq j$, let $\gamma_s$ be the dihedral angle in $P_{i_s}$ associated to $e$.

Next, we study the quantity

$$\ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j),$$

which is a “subsum” of $f(P)$. We consider three cases.
Case 1: The edge $e$ lines entirely (except possibly for one or both endpoints) in the interior of $P$. Then
\[ \gamma_1 + \gamma_2 + \cdots + \gamma_j - 2\pi = 0, \]
whence
\[ 0 = f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) - 2f(\pi) \]
\[ = f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) \]
\[ = \ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j). \]

Case 2: The edge $e$ lies on a face of $P$, but not on an edge. Then
\[ \gamma_1 + \gamma_2 + \cdots + \gamma_j - \pi = 0, \]
whence
\[ 0 = f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) - f(\pi) \]
\[ = f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) \]
\[ = \ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j). \]

Case 3: The edge $e$ lies on an edge of $P$. If $\alpha$ is the dihedral angle of $P$ associated to this edge, then the quantity $\gamma_1 + \gamma_2 + \cdots + \gamma_j$ must equal either $\alpha$ or $\alpha - \pi$. In either event, the requirement that $f(\pi) = 0$ implies that
\[ f(\gamma_1) + f(\gamma_2) + \cdots + f(\gamma_j) = f(\alpha), \]
and hence
\[ \ell f(\gamma_1) + \ell f(\gamma_2) + \cdots + \ell f(\gamma_j) = \ell f(\alpha), \]
which is the weight of the edge $e$ in $P$. Therefore, if we sum over all edges in our polyhedral decomposition of $P$, we obtain the right hand side of (⋆).

Lemma 2.9. Let $f$ be an integrally additive function on a set $A$ and let $\tilde{\alpha}$ be a real number not in $A$. Then $f$ can be extended to an integrally additive function on $A \cup \{\tilde{\alpha}\}$.

Proof. For notational convenience, we use the notation
\[ \sum_{\alpha \in A}^{'} n_\alpha \alpha \]
to denote a sum of integer multiples of elements of $A$ with all but finitely many of the integers $n_\alpha$ equal to zero. We consider two cases:

Case 1: There exists no equation
\[ \sum_{\alpha \in A}^{'} n_\alpha \alpha + n_{\tilde{\alpha}} \tilde{\alpha} = 0 \]
with $n_{\tilde{\alpha}} \neq 0$. In this case we can define $f(\tilde{\alpha})$ to be any real number.

Case 2: There does exist an equation
(1) \[ \sum_{\alpha \in A}^{'} n_\alpha \alpha + n_{\tilde{\alpha}} \tilde{\alpha} = 0 \]
with \( n_\tilde{\alpha} \neq 0 \). Choose one such equation and define

\[
f(\tilde{\alpha}) = -\sum_{\alpha \in A} \frac{n_\alpha}{n_\tilde{\alpha}} f(\alpha).
\]

We must show that \( f \) is integrally additive on \( \tilde{A} \). Suppose we have an integral linear dependence

\[
\sum_{\alpha \in A}' m_\alpha \alpha + m_\tilde{\alpha} \tilde{\alpha} = 0.
\]

If \( m_\tilde{\alpha} = 0 \), the lemma follows immediately, so suppose that \( m_\tilde{\alpha} \neq 0 \). Multiply Equation (1) by \( m_\tilde{\alpha} \), multiply Equation (2) by \( n_\tilde{\alpha} \), and subtract to eliminate \( \tilde{\alpha} \):

\[
\sum_{\alpha \in A}' (n_\tilde{\alpha} m_\alpha - n_\alpha m_\tilde{\alpha}) \alpha = 0.
\]

Then

\[
\sum_{\alpha \in A}' (n_\tilde{\alpha} m_\alpha - n_\alpha m_\tilde{\alpha}) f(\alpha) = 0.
\]

Next, multiply the equation

\[
\sum_{\alpha \in A}' n_\alpha f(\alpha) + n_\tilde{\alpha} f(\tilde{\alpha}) = 0
\]

by \( m_\tilde{\alpha} \) and subtract Equation (4) to obtain

\[
\sum_{\alpha \in A}' n_\tilde{\alpha} m_\alpha f(\alpha) + n_\tilde{\alpha} m_\tilde{\alpha} f(\tilde{\alpha}) = 0
\]

The fact that \( n_\tilde{\alpha} \neq 0 \) yields

\[
\sum_{\alpha \in A}' m_\alpha f(\alpha) + m_\tilde{\alpha} f(\tilde{\alpha}) = 0,
\]

as desired. \(\Box\)

**Theorem 2.10.** Let \( P \) and \( Q \) be polyhedra and define

\[
M = \{\pi\} \cup A_P \cup A_Q,
\]

where \( A_P \) and \( A_Q \) are the sets of dihedral angles of \( P \) and \( Q \) respectively. Suppose there exists an integrally additive function \( f \) on \( M \) such that \( f(\pi) = 0 \). If \( P \) and \( Q \) are scissors congruent, then \( f(P) = f(Q) \).

**Proof.** Decompose \( P \) and \( Q \) into polyhedra

\[
P = P_1 + P_2 + \cdots + P_k
\]

\[
Q = Q_1 + Q_2 + \cdots + Q_k
\]

with the property that \( P_i \cong Q_i \) for \( 1 \leq i \leq k \). Using Lemma 2.9, we can extend \( f \) to be an integrally additive function on the set consisting of \( M \) and all the dihedral angles of our subpolyhedra. Obviously \( f(P_i) = f(Q_i) \) for \( 1 \leq i \leq k \), and so we have

\[
f(P) = f(P_1) + f(P_2) + \cdots + f(P_k) = f(Q_1) + f(Q_2) + \cdots + f(Q_k) = f(Q)
\]

by Lemma 2.8. \(\Box\)

**Lemma 2.11.** The number \( \frac{1}{\pi} \arccos \left( \frac{1}{3} \right) \) is irrational.
Proof. Using induction and the trig identity
\[ \cos((n + 1)\theta) = 2 \cos \theta \cos n\theta - \cos((n - 1)\theta) \]
we see that \( \cos n\theta = T_n(\cos \theta) \), where \( T_n \) is an \( n \)-th degree polynomial with integer coefficients and whose leading term is \( 2^{n-1} \).

Set \( \phi = \arccos \left( \frac{1}{3} \right) \) and suppose that \( \phi = \frac{p}{q} \pi \) for some positive integers \( p \) and \( q \). Then
\[ T_q \left( \frac{1}{3} \right) = T_q \left( \cos \left( \frac{p}{q} \pi \right) \right) = T_q \left( \cos \left( \frac{\pi}{q} \right) \right) = \cos(p\pi) = \pm 1, \]
whence \( \frac{1}{3} \) is a root of a polynomial with integer coefficients and leading term \( 2^{q-1} \) for some \( q \). We deduce from Rational Root Test that \( 3 \) divides \( 2^{q-1} \) for some \( q \), a contradiction. □

Theorem 2.12. A regular tetrahedron and a cube of equal volume are not scissors congruent.

Proof. Let \( T \) and \( C \) be the regular tetrahedron and cube of volume 1 respectively. Obviously each dihedral angle of \( C \) is \( \frac{\pi}{2} \) and an easy exercise in analytic geometry shows that each dihedral angle \( \phi \) of \( T \) equals \( \arccos \left( \frac{1}{3} \right) \). Let \( A \) be the set \( \{\pi, \frac{\pi}{2}, \phi\} \) and define \( f : A \rightarrow \mathbb{R} \) by setting
\[ f(\pi) = 0, \quad f \left( \frac{\pi}{2} \right) = 0, \quad f(\phi) = 1. \]
I claim that \( f \) is integrally additive. Suppose that
\[ n_1 \pi + n_2 \cdot \frac{\pi}{2} + n_3 \phi = 0 \]
for some integers \( n_1, n_2, \) and \( n_3 \). If \( n_3 = 0 \), we immediately see that
\[ n_1 f(\pi) + n_2 \cdot \frac{\pi}{2} + n_3 \phi = n_1 f(\pi) + n_2 f \left( \frac{\pi}{2} \right) = 0. \]
On the other hand, if \( n_3 \neq 0 \), then
\[ \phi = -\frac{2n_1 + n_2}{2n_3}, \]
contradicting Lemma 2.11.

Now we compute the Dehn invariants of \( C \) and \( T \) associated to \( f \). The length of each side of \( C \) is 1, so
\[ f(C) = 12 \cdot 1 \cdot f \left( \frac{\pi}{2} \right) = 0. \]
Let \( m \) be the length of each side of \( T \). From the volume formula for a regular tetrahedron
\[ V = \frac{\sqrt{2}}{12} m^3 \]
we compute
\[ m = \left( \frac{12}{\sqrt{2}} \right)^{1/3} \approx 2.04 \neq 0 \]
and thus
\[ f(T) = 6mf(\phi) = 6 \left( \frac{12}{\sqrt{2}} \right)^{1/3} \neq 0. \]
Applying (the contrapositive of) Theorem 2.10, we see that \( T \) and \( C \) are not scissors congruent. □
Theorem 2.13 (Sydler, 1965). If $P$ and $Q$ are polyhedra that have equal volumes and have the property that $f(P) = f(Q)$ for every integrally additive function satisfying $f(\pi) = 0$, then $P$ and $Q$ are scissors congruent.