# Kirillov Theory

## TCU GAGA Seminar

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- A **representation** of a Lie group $G$ on a Hilbert space $\mathcal{H}$ is a homomorphism

$$\pi : G \to \mathrm{Aut}(\mathcal{H}) = GL(\mathcal{H})$$

  such that $\forall v \in \mathcal{H}$ the map

$$x \mapsto \pi(x)v$$

  is continuous.
- If $\pi(x)$ is **unitary** (ie, inner-product preserving) for all $x \in G$, then $\pi$ is a **unitary representation**
- Note that a subspace of $\mathcal{H}$ will always refer to a **closed** subspace of $\mathcal{H}$.

- A subspace $\mathcal{W} \subset \mathcal{H}$ is $G$-**invariant** iff

$$\forall x \in G, \pi(x)(\mathcal{W}) \subset \mathcal{W}.$$

- A representation $(\pi, \mathcal{H})$ is **irreducible** iff $\{0\}$ and $\mathcal{H}$ are the only $G$-invariant subspaces of $\mathcal{H}$.
- A representation $(\pi, \mathcal{H})$ is **completely reducible** iff $\mathcal{H}$ is a(n orthogonal) direct sum of irreducible subspaces.
- Two (unitary) representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are **(unitarily) equivalent** iff $\exists$ (unitary) isomorphism $T : \mathcal{H} \to \mathcal{H}'$ such that

$$\forall x \in G \forall v \in \mathcal{H}, T(\pi(x)v) = \pi'(x)(Tv)$$

ie, $T \circ \pi = \pi' \circ T$. The mapping $T$ is called the **intertwining operator**.

- A Lie algebra $\mathfrak{g}$ is **nilpotent** iff

$$\cdots \subset [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]] \subset [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] \subset [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$$

  eventually ends. A Lie group $G$ is **nilpotent** iff its Lie algebra is. For any Lie algebra $\mathfrak{g}$, there is a unique simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$.
- Example: The Heisenberg Lie algebra $\mathfrak{h} = \mathrm{span}\{X, Y, Z\}$ with Lie bracket $[X, Y] = Z$ and all other basis brackets not determined by skew-symmetry zero. Then $[\mathfrak{h},\mathfrak{h}] = \mathrm{span}\{Z\}$, and $[\mathfrak{h},[\mathfrak{h},\mathfrak{h}]] = \{0\}$, so $\mathfrak{h}$ is **two-step** nilpotent.

- Every simply-connected nilpotent Lie group is diffeomorphic to $\mathbb{R}^n$
- The Lie group exponential $\exp : \mathfrak{g} \to G$ is a diffeomorphism that induces a coordinate system on any such $G$. We denote the inverse of exp by $\log$.

- Example: if we use the matrix coordinates given above, which are not the exponential coordinates, then the Lie group exponential is given by

$$\exp(xX + yY + zZ) = e^A,$$

where

$$A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

- Note that

$$e^A = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

- We then have

$$\log\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) = xX + yY - \frac{1}{2}xyZ$$

- The **co-adjoint action** of $G$ on $\mathfrak{g}^*$(= dual of $\mathfrak{g}$) is given by

$$x \cdot \lambda = \lambda \circ \mathrm{Ad}(x^{-1})$$

- (We need the inverse to make it an action.)
- Group actions induce equivalence relations = partitions
- So, $\mathfrak{g}^*$ can be partitioned into **coadjoint orbits**
- Note that as sets $\lambda \circ \mathrm{Ad}(G^{-1}) = \lambda \circ \mathrm{Ad}(G)$, so we drop the inverse when computing an entire orbit.

- Example: The co-adjoint action of the Heisenberg group. Let $\{\alpha, \beta, \zeta\}$ be the basis of $\mathfrak{h}^*$ dual to $\{X, Y, Z\}$ Let $\lambda \in \mathfrak{h}^*$.
- Note that for $x \in H$ and $U \in \mathfrak{h}$,

$$\mathrm{Ad}(x)(U) = \frac{d}{dt}|_0 x \exp(tU) x^{-1} = U + [\log x, U]$$

- Case 1: If $\lambda(Z) = 0$, then $\lambda \circ \mathrm{Ad}(x) = \lambda, \quad \forall x \in H$
- Case 2: If $\lambda(Z) \neq 0$, then let $\lambda = a\alpha + b\beta + c\zeta$. Let

$$x = \begin{pmatrix} 1 & -b/c & * \\ 0 & 1 & a/c \\ 0 & 0 & 1 \end{pmatrix}$$

  Note that $\log x = -\frac{b}{c}X + \frac{a}{c}Y + *Z$
- Claim: $\lambda \circ \mathrm{Ad}(x) = c\zeta$. Assuming this is true for the moment, this means that the coadjoint orbit of an element in this case is completely determined by its value at $Z$.

- The computation:

$$(\lambda\circ\mathrm{Ad}(x))(X) = \lambda(X+[\log x, X]) = \lambda(X+[-\frac{b}{c}X+\frac{a}{c}Y+*Z, X])$$

$$= \lambda(X) - \frac{a}{c}\lambda(Z) = 0 = c\zeta(0)$$

- Likewise

$$(\lambda\circ\mathrm{Ad}(x))(Y) = \lambda(Y+[\log x, Y]) = \lambda(Y+[-\frac{b}{c}X+\frac{a}{c}Y+*Z, Y])$$

$$= \lambda(Y) - \frac{b}{c}\lambda(Z) = 0 = c\zeta(0)$$

- Finally,

$$(\lambda \circ \mathrm{Ad}(x))(Z) = \lambda(Z + [\log x, Z]) = \lambda(Z) = c = c\zeta(Z)$$

# Kirillov Theory of Unitary Representations

- Let $G$ be a simply connected nilpotent Lie group
- Let $\hat{G}$ denote the equivalence classes of irreducible unitary representations of $G$.
- Kirillov Theory: $\hat{G}$ corresponds to the co-adjoint orbits of $\mathfrak{g}^*$
- (i) $\forall \lambda \in \mathfrak{g}^* \; \exists$ irred unitary rep $\pi_\lambda$ of $G$ that is unique up to unitary equivalence of reps
- (ii) $\forall \pi \in \hat{G} \exists \lambda \in \mathfrak{g}^*, \pi \sim \pi_\lambda$
- (iii) $\pi_\lambda \sim \pi_\mu$ iff $\mu = \lambda \circ \mathrm{Ad}(x)$ for some $x \in G$

- Let $\lambda \in \mathfrak{g}^*$
- A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is **subordinate to** $\lambda$ iff $\lambda([\mathfrak{k}, \mathfrak{k}]) = 0$. Let $K = \exp(\mathfrak{k})$, the simply connected Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. We also say $K$ is subordinate to $\lambda$.
- If $\mathfrak{k}$ is maximal with respect to being subordinate, then $\mathfrak{k}$ (or $K$) is a **polarization** of $\lambda$, or a **maximal subordinate subalgebra** for $\lambda$
- Define a **character**(= 1-dim'l rep) of $K = \exp(\mathfrak{k})$ by

$$\bar{\lambda}(k) = e^{2\pi i \lambda \log k} \in \mathbb{C}.$$

This is a homomorphism.

- Why is this a homomorphism?

$$\bar{\lambda}(k) = e^{2\pi i \lambda \log k} \in \mathbb{C}.$$

- Recall the Campbell-Baker-Hausdorff formula:

$$\exp(A)\exp(B) = \exp(A+B+\frac{1}{2}[A,B]+\text{higher powers of bracket}).$$

- So $\bar{\lambda}(k_1 k_2) = e^{2\pi i \lambda(\log k_1 + \log k_2 + \frac{1}{2}[\log k_1, \log k_2] + \cdots)}$
- $= e^{2\pi i \lambda(\log k_1)} e^{2\pi i \lambda(\log k_2)}$ since $\lambda([\mathfrak{k}, \mathfrak{k}]) = 0$.

- Example: Consider the Heisenberg group and algebra. Let $\lambda \in \mathfrak{h}^*$. If $\lambda(Z) = 0$, then the polarization $\mathfrak{k} = \mathfrak{h}$. That is, $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$.
- If $\lambda(Z) \neq 0$, let $\mathfrak{k} = \text{span}\{Y, Z\}$. Then $\mathfrak{k}$ is abelian, so $\lambda([\mathfrak{k}, \mathfrak{k}]) = 0$. This is a polarization, ie, maximal.
- There are other polarizations. They are not unique.
- So then for all $(0, y, z) \in H$ (with the obvious correspondence between coordinates)

$$\bar{\lambda}((0, y, z)) = e^{2\pi i \lambda(yY + zZ)}.$$

- The representation $\pi_\lambda$ of Kirillov Theory is defined as the representation of $G$ induced by the representation $\bar{\lambda}$ of $K$.
- What the heck is an induced representation?

# Inducing Representations

- Let $G$ be a Lie group with closed Lie subgroup $K$. Let $(\pi, \mathcal{H})$ be a unitary rep of $H$.
- Define the representation space of the induced rep

$$\mathcal{W} := \{f : G \to \mathcal{H} : f(kx) = \pi(k)(f(x)) \forall k \in K, \forall x \in G\}.$$

- We also require that $||f|| \in L^2(K \backslash G, \mu)$. Note that $\pi(k)$ is unitary.
- So $||f(kx)|| = ||\pi(k)f(x)|| = ||f(x)||$, so $||f||$ induces a well-defined map from $K \backslash G$ to $\mathbb{R}$. Can put a right $G$-invariant measure $\mu$ on $K \backslash G$.
- $\mathcal{W}$ is a Hilbert space
- Define a rep $\tilde{\pi}$ of $G$ on $\mathcal{W}$ by

$$(\tilde{\pi}(a)f)(x) = f(xa).$$

- $\tilde{\pi}$ is a unitary rep of $G$, the unitary rep **induced** by the unitary rep $\pi$ of $K \subset G$.

- Recall: we have $\lambda \in \mathfrak{g}^*$, a polarization $\mathfrak{k}$ of $\lambda$ and a character $\bar{\lambda}(k) = e^{2\pi i \lambda(\log k)}$ of $\exp(\mathfrak{k})$.
- The representation space of $\pi_\lambda$ is then

$$\mathcal{W} = \{f : G \to \mathbb{C} : f(kx) = e^{2\pi i \lambda \log k} f(x) \quad \forall k \in K\}.$$

- $G$ acts by right translation on $\mathcal{W}$
- Kirillov showed that $\pi_\lambda$ is unitary and irreducible

- Example: The Heisenberg group and algebra. Let $\lambda \in \mathfrak{h}^*$.
- Case 1: $\lambda(Z) = 0, \implies K = H$. Then $\bar{\lambda}$ is a character of $H$ that is independent of $Z$,

$$\bar{\lambda}(x, y, z) = e^{2\pi i \lambda(xX + yY)}.$$

  The induced rep $\pi_\lambda$ is unitarily equivalent to $\bar{\lambda}$.
- To see this, note that the representation space $\mathcal{W}$ is defined as

$$\mathcal{W} = \{f : H \to \mathbb{C} : f(hx) = e^{2\pi i \lambda \log h} f(x) \quad \forall h \in H \forall x \in H\}.$$

- Letting $x = e$

$$\mathcal{W} = \{f : H \to \mathbb{C} : f(h) = e^{2\pi i \lambda \log h} f(e) \quad \forall h \in H\} = \mathbb{C}\bar{\lambda}$$

- Case 2: $\lambda(Z) \neq 0 \implies K = (0, y, z)$ $\bar{\lambda}((0, y, z)) = e^{2\pi i \lambda(yY + zZ)}$ So that

$$\mathcal{W} = \{f : H \to \mathbb{C} : f(kx) = f(x) \forall k \in K\}$$

- $(x, y, z) = (0, y, z)(x, 0, 0)$, so $f(x, y, z) = f((0, y, z)(x, 0, 0)) = e^{2\pi i \lambda(yY + zZ)} f(x, 0, 0)$.
- note that we can choose $\lambda = c\zeta$
- This is equivalent to an action on $\mathcal{W}' = \{f : \mathbb{R} \to \mathbb{C}\}$
- What does this action look like. $H$ acts on $\mathcal{W}$ by right multiplication, so $(\pi'_\lambda((x, y, z))f)(u) = e^{2\pi i c(z + py)} f(u + x)$.

- Let $\Gamma \subset G$ be a cocompact, discrete subgroup of $G$.
- Example: Recall that the Heisenberg group can be realized as the set of matrices

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

- A cocompact (ie, $\Gamma \backslash G$ compact) discrete subgroup of $H$ is given by

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

- (The existence of a cocompact, discrete subgroup places some restrictions on $\mathfrak{g}$, and it also implies that $G$ is unimodular.)

- The **right action** $\rho$ of $G$ on $L^2(G)$ is a representation of $G$ on $\mathcal{H} = L^2(G)$ :

$$(\rho(a)f)(x) = f(xa) \qquad \forall a \in G, x \in G$$

- The **quasi-regular representation** $\rho_\Gamma$ of $G$ on $\mathcal{H} = L^2(\Gamma\backslash G)$ is given by

$$(\rho_\Gamma(a)f)(x) = f(xa) \qquad \forall a \in G, x \in \Gamma\backslash G$$

- We generally view functions $f \in L^2(\Gamma\backslash G)$ as left-$\Gamma$ invariant functions on $G$, ie

$$f(\gamma x) = f(x) \qquad \forall \gamma \in \Gamma \forall x \in G$$

- Both $\rho$ and $\rho_\Gamma$ are unitary.

- Of interest to spectral geometry is determining the decomposition of the quasi-regular representation $\rho_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$.
- To see why, we consider left invariant metrics on the Lie group $G$
- A left invariant metric on $G$ corresponds to a choice of inner product $\langle \quad , \quad \rangle$ on $\mathfrak{g}$.

- Let $f \in C^\infty(M)$
- Recall that

$$(\Delta f)(p) = -\sum_j ((E_j(p)^2 + \nabla_{E_j(p)} E_j(p)) f)(p)$$

- Claim: On $\Gamma \backslash G$, with Riemannian metric induced from $\langle \quad , \quad \rangle$ on $\mathfrak{g}$, $\Delta = -\sum_j E_j^2$, where $\{E_1, \dots, E_n\}$ is an ONB of $\mathfrak{g}$.
- From the standard proof of uniqueness of the Levi-Civita connection

$$2 \langle \nabla_X Y, W \rangle = X \langle Y, W \rangle + Y \langle X, W \rangle - W \langle X, Y \rangle$$
$$+ \langle [X, Y], W \rangle + \langle [W, X], Y \rangle - \langle [Y, W], X \rangle$$

- But if $X, Y, W$ are left-invariant, then

$$\langle \nabla_X Y, W \rangle = \frac{1}{2} (\langle [X, Y], W \rangle + \langle [W, X], Y \rangle - \langle [Y, W], X \rangle)$$

- Claim: $\sum_j \nabla_{E_j} E_j = 0$
- Proof: $\left\langle \sum_j \nabla_{E_j} E_j, U \right\rangle = \sum_j \langle \nabla_{E_j} E_j, U \rangle$
- $= \frac{1}{2} \sum_j \langle [U, E_j], E_j] \rangle + \langle [U, E_j], E_j] \rangle + \langle [E_j, E_j], U] \rangle$
- $= \sum_j \langle \mathrm{ad}(U) E_j, E_j \rangle = \mathrm{tr}(\mathrm{ad} U)$.
- Since $G$ is unimodular, $\mathrm{tr}(\mathrm{ad} U) = 0$ for all $U \in \mathfrak{g}$. $\square$
- See, eg, the Springer Encyclopedia of Mathematics (online) entry on unimodular.

- A **representation** $\pi$ of a Lie algebra $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$ is a linear map

$$\pi : \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{R}}(\mathcal{H})$$

  such that $\pi([X, Y]) = [\pi(X), \pi(Y)]$
- Let $(\pi, \mathcal{H})$ be a representation of $G$. Define

$$\mathcal{H}_\pi^\infty = \{v \in \mathcal{H} : x \mapsto \pi(x)v \text{ is smooth}\},$$

  the **smooth vectors of $\mathcal{H}$ with respect to $\pi$.**
- $\mathcal{H}_\pi^\infty$ is $G$-invariant and dense

- The **derived representation** $\pi_*$ of $\mathfrak{g}$ associated to the representation $(\pi, \mathcal{H})$ of $G$ is defined as, for $X \in \mathfrak{g}$

$$\pi_*(X)v = \frac{d}{dt}|_0 \pi(\exp(tX))v,$$

where $\pi_*(X) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty$
- If $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are unitarily equivalent, so are their derived representations.

- If $E \in \mathfrak{g}$, then

$$E(x) = \frac{d}{ds}|_0 x \cdot \exp(sE).$$

- Let $f \in C^\infty(\Gamma \backslash G)$, then

$$Ef(x) = \frac{d}{ds}|_0 f(x \cdot \exp(sE))$$

- $= \frac{d}{ds}|_0 \rho_\Gamma(\exp(sE)f)(x)$
- $= (\rho_{\Gamma *}(E)f)(x)$
- So we extend $\Delta$ to $\mathcal{H}^\infty$ by

$$\Delta f = -\sum_j \rho_{\Gamma *}(E_j)^2$$

- Kirillov theory says that $L^2(\Gamma \backslash G)$ can be decomposed into the orthogonal sum of various $\pi_\lambda$, for $\lambda \in \mathfrak{g}^*$, each $\pi_\lambda$ occuring with finite multiplicity.
- We seek a condition that says when $\pi_\lambda$ occurs, and with what multiplicity.

- A **rational Lie algebra** is a Lie algebra defined over $\mathbb{Q}$ rather than $\mathbb{R}$. If we take $\mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$, we obtain a real Lie algebra.
- A choice of cocompact, discrete subgroup of $G$ determines a rational structure. In particular, the existence of $\Gamma$ implies that we can pick a basis of $\mathfrak{g}$ from the set $\log \Gamma$, which implies that the structure constants are rational on this basis.
- Then $\mathfrak{g}_{\Gamma} = \mathrm{span}_{\mathbb{Q}}\{\log \Gamma\}$ is a rational Lie algebra.
- A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a **rational Lie subalgebra** iff there exists subalgebra $\mathfrak{k}_{\mathbb{Q}} \subset \mathfrak{g}_{\Gamma}$ such that $\mathfrak{k} = \mathfrak{k}_{\mathbb{Q}} \otimes \mathbb{R}$. That is, there exists a basis of $\mathfrak{k}$ contained in $\mathfrak{g}_{\Gamma}$.

- If $\mathfrak{k}$ is a rational subalgebra of $\mathfrak{g}$ (with respect to $\Gamma$), then $\Gamma \cap \exp(\mathfrak{k})$ is a cocompact, discrete subgroup of $K = \exp(\mathfrak{k})$.
- To obtain a multiplicity formula, we must consider $\lambda \in \mathfrak{g}$ that have rational polarizations, and such that $\bar{\lambda}(\Gamma \cap \exp(\mathfrak{k})) = 1$. Thus $\bar{\lambda}$ is really a mapping on $\Gamma \cap K \backslash K$.
- We call the pair $(\bar{\lambda}, \mathfrak{k})$ and integral point iff $\mathfrak{k}$ is rational (with respect to the rational structure induced by $\Gamma$) and $\bar{\lambda}(\Gamma \cap \exp(\mathfrak{k})) = 1$.

- Consider the set $F =$ all pairs $(\bar{\lambda}, \mathfrak{k})$ where $\bar{\lambda}$ is the character of $\exp(\mathfrak{k})$ determined by $\lambda \in \mathfrak{g}$, and $\mathfrak{k}$ is a polarization of $\lambda$.
- $G$ acts by conjugation on $F$ :

$$x \cdot (\bar{\lambda}, \mathfrak{k}) = (\bar{\lambda} \circ I_x, Ad(x^{-x})(\mathfrak{k})),$$

  for all $x \in G$.
- Fact: If $(\bar{\lambda}, \mathfrak{k}) \in F$, then $x \cdot (\bar{\lambda}, \mathfrak{k}) \in F$. The isotropy subgroup of the point $(\bar{\lambda}, \mathfrak{k})$ is $\exp(\mathfrak{k})$.
- Fact: $\Gamma$ maps integral points of $F$ to integral points of $F$

- Theorem: (L Richardson and R. Howe) Let $\lambda \in \mathfrak{g}^*$ and let $(\bar{\lambda}, \mathfrak{k})$ induce $\pi_\lambda$. Then $\pi_\lambda$ occurs in the rep $\rho_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$ iff the $G$-orbit of $(\bar{\lambda}, \mathfrak{k})$ contains an integral point. The multiplicity of $\pi_\lambda$ is equal to the number of $\Gamma$-orbits on the set of integral points in the $G$-orbit of $(\bar{\lambda}, \mathfrak{k})$.
- Restated:

$$m(\pi_\lambda, \rho_\Gamma) = \# \left\{ \Gamma \backslash \lambda(\mathrm{Ad}(G))_\Gamma \right\},$$

where $\lambda(\mathrm{Ad}(G))_\Gamma$ is the set of integral points of the co-adjoint action of $G$.