Kirillov Theory
TCU GAGA Seminar

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A **representation** of a Lie group $G$ on a Hilbert space $\mathcal{H}$ is a homomorphism

$$\pi : G \rightarrow \text{Aut}(\mathcal{H}) = GL(\mathcal{H})$$

such that $\forall v \in \mathcal{H}$ the map

$$x \mapsto \pi(x)v$$

is continuous.

- If $\pi(x)$ is **unitary** (ie, inner-product preserving) for all $x \in G$, then $\pi$ is a **unitary representation**

- Note that a subspace of $\mathcal{H}$ will always refer to a **closed** subspace of $\mathcal{H}$. 
A subspace $\mathcal{W} \subset \mathcal{H}$ is $G$-invariant iff
\[ \forall x \in G, \pi(x)(\mathcal{W}) \subset \mathcal{W}. \]

A representation $(\pi, \mathcal{H})$ is irreducible iff $\{0\}$ and $\mathcal{H}$ are the only $G$-invariant subspaces of $\mathcal{H}$.

A representation $(\pi, \mathcal{H})$ is completely reducible iff $\mathcal{H}$ is a(n orthogonal) direct sum of irreducible subspaces.

Two (unitary) representations $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are (unitarily) equivalent iff $\exists$ (unitary) isomorphism $T : \mathcal{H} \rightarrow \mathcal{H}'$ such that
\[ \forall x \in G \forall v \in \mathcal{H}, T(\pi(x)v) = \pi'(x)(Tv) \]

ie, $T \circ \pi = \pi' \circ T$. The mapping $T$ is called the intertwining operator.
A Lie algebra \( \mathfrak{g} \) is **nilpotent** iff

\[
\cdots \subset [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]) \subset [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}
\]

eventually ends. A Lie group \( G \) is **nilpotent** iff its Lie algebra is. For any Lie algebra \( \mathfrak{g} \), there is a unique simply connected Lie group \( G \) with Lie algebra \( \mathfrak{g} \).

**Example:** The Heisenberg Lie algebra \( \mathfrak{h} = \text{span}\{X, Y, Z\} \) with Lie bracket \([X, Y] = Z\) and all other basis brackets not determined by skew-symmetry zero. Then \([\mathfrak{h}, \mathfrak{h}] = \text{span}\{Z\}\), and \([\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]] = \{0\}\), so \( \mathfrak{h} \) is **two-step** nilpotent.
Every simply-connected nilpotent Lie group is diffeomorphic to \( \mathbb{R}^n \).

The Lie group exponential \( \exp : g \to G \) is a diffeomorphism that induces a coordinate system on any such \( G \). We denote the inverse of \( \exp \) by \( \log \).
Example: if we use the matrix coordinates given above, which are not the exponential coordinates, then the Lie group exponential is given by

$$\exp(xX + yY + zZ) = e^A,$$

where

$$A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

Note that

$$e^A = \begin{pmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

We then have

$$\log \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) = xX + yY - \frac{1}{2}xyZ$$
The **co-adjoint action** of $G$ on $g^*(=\text{dual of } g)$ is given by

$$x \cdot \lambda = \lambda \circ \text{Ad}(x^{-1})$$

(We need the inverse to make it an action.)

Group actions induce equivalence relations $=$ partitions

So, $g^*$ can be partitioned into **coadjoint orbits**

Note that as sets $\lambda \circ \text{Ad}(G^{-1}) = \lambda \circ \text{Ad}(G)$, so we drop the inverse when computing an entire orbit.
Example: The co-adjoint action of the Heisenberg group. Let \( \{\alpha, \beta, \zeta\} \) be the basis of \( \mathfrak{h}^* \) dual to \( \{X, Y, Z\} \). Let \( \lambda \in \mathfrak{h}^* \).

- Note that for \( x \in H \) and \( U \in \mathfrak{h} \),

\[
\operatorname{Ad}(x)(U) = \frac{d}{dt}\bigg|_{t=0} \exp(tU)x^{-1} = U + [\log x, U]
\]

- Case 1: If \( \lambda(Z) = 0 \), then \( \lambda \circ \operatorname{Ad}(x) = \lambda \), \( \forall x \in H \)

- Case 2: If \( \lambda(Z) \neq 0 \), then let \( \lambda = a\alpha + b\beta + c\zeta \). Let

\[
x = \begin{pmatrix} 1 & -b/c & * \\ 0 & 1 & a/c \\ 0 & 0 & 1 \end{pmatrix}
\]

Note that \( \log x = -\frac{b}{c}X + \frac{a}{c}Y + *Z \)

- Claim: \( \lambda \circ \operatorname{Ad}(x) = c\zeta \). Assuming this is true for the moment, this means that the coadjoint orbit of an element in this case is completely determined by its value at \( Z \).
The computation:

\[ (\lambda \circ \text{Ad}(x))(X) = \lambda (X + \log x, X) = \lambda (X + [-\frac{b}{c}X + \frac{a}{c}Y + *Z, X]) \]

\[ = \lambda (X) - \frac{a}{c} \lambda (Z) = 0 = c\zeta(0) \]

Likewise

\[ (\lambda \circ \text{Ad}(x))(Y) = \lambda (Y + \log x, Y) = \lambda (Y + [-\frac{b}{c}X + \frac{a}{c}Y + *Z, Y]) \]

\[ = \lambda (Y) - \frac{b}{c} \lambda (Z) = 0 = c\zeta(0) \]

Finally,

\[ (\lambda \circ \text{Ad}(x))(Z) = \lambda (Z + \log x, Z) = \lambda (Z) = c = c\zeta(Z) \]
Let $G$ be a simply connected nilpotent Lie group

Let $\hat{G}$ denote the equivalence classes of irreducible unitary representations of $G$.

Kirillov Theory: $\hat{G}$ corresponds to the co-adjoint orbits of $\mathfrak{g}^*$

(i) $\forall \lambda \in \mathfrak{g}^* \exists$ irred unitary rep $\pi_{\lambda}$ of $G$ that is unique up to unitary equivalence of reps

(ii) $\forall \pi \in \hat{G} \exists \lambda \in \mathfrak{g}^*, \pi \sim \pi_{\lambda}$

(iii) $\pi_{\lambda} \sim \pi_{\mu}$ iff $\mu = \lambda \circ \text{Ad}(x)$ for some $x \in G$
Let $\lambda \in \mathfrak{g}^*$

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is **subordinate to** $\lambda$ iff $\lambda([\mathfrak{k}, \mathfrak{k}]) = 0$. Let $K = \exp(\mathfrak{k})$, the simply connected Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. We also say $K$ is subordinate to $\lambda$.

If $\mathfrak{k}$ is maximal with respect to being subordinate, then $\mathfrak{k}$ (or $K$) is a **polarization** of $\lambda$, or a **maximal subordinate subalgebra** for $\lambda$.

Define a **character** ($= 1$-dim’l rep) of $K = \exp(\mathfrak{k})$ by

$$\bar{\lambda}(k) = e^{2\pi i \lambda \log k} \in \mathbb{C}.$$  

This is a homomorphism.

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**Kirillov Theory**

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Why is this a homomorphism?

\[ \tilde{\lambda}(k) = e^{2\pi i \lambda \log k} \in \mathbb{C}. \]

Recall the Campbell-Baker-Hausdorff formula:

\[ \exp(A) \exp(B) = \exp(A + B + \frac{1}{2}[A, B] + \text{higher powers of bracket}). \]

So \( \tilde{\lambda}(k_1 k_2) = e^{2\pi i \lambda (\log k_1 + \log k_2 + \frac{1}{2}[\log k_1, \log k_2] + \cdots)} \)

\[ = e^{2\pi i \lambda (\log k_1)} e^{2\pi i \lambda (\log k_2)} \text{ since } \lambda([\mathfrak{g}, \mathfrak{g}]) = 0. \]
Example: Consider the Heisenberg group and algebra. Let \( \lambda \in \mathfrak{h}^* \). If \( \lambda(Z) = 0 \), then the polarization \( \mathfrak{k} = \mathfrak{h} \). That is, \( \lambda([\mathfrak{h}, \mathfrak{h}]) = 0 \).

If \( \lambda(Z) \neq 0 \), let \( \mathfrak{k} = \text{span}\{ Y, Z \} \). Then \( \mathfrak{k} \) is abelian, so \( \lambda([\mathfrak{k}, \mathfrak{k}]) = 0 \). This is a polarization, ie, maximal.

There are other polarizations. They are not unique.

So then for all \( (0, y, z) \in H \) (with the obvious correspondence between coordinates)

\[
\bar{\lambda}((0, y, z)) = e^{2\pi i \lambda(yY + zZ)}.
\]
The representation $\pi_\lambda$ of Kirillov Theory is defined as the representation of $G$ induced by the representation $\tilde{\lambda}$ of $K$.

What the heck is an induced representation?
Let $G$ be a Lie group with closed Lie subgroup $K$. Let $(\pi, \mathcal{H})$ be a unitary rep of $H$.

Define the representation space of the induced rep

$$\mathcal{W} := \{ f : G \to \mathcal{H} : f(kx) = \pi(k)(f(x)) \forall k \in K, \forall x \in G \}.$$ 

We also require that $\|f\| \in L^2(K \backslash G, \mu)$. Note that $\pi(k)$ is unitary.

So $\|f(kx)\| = \|\pi(k)f(x)\| = \|f(x)\|$, so $\|f\|$ induces a well-defined map from $K \backslash G$ to $\mathbb{R}$. Can put a right $G$-invariant measure $\mu$ on $K \backslash G$.

$\mathcal{W}$ is a Hilbert space

Define a rep $\tilde{\pi}$ of $G$ on $\mathcal{W}$ by

$$\tilde{\pi}(a)f(x) = f(xa).$$

$\tilde{\pi}$ is a unitary rep of $G$, the unitary rep induced by the unitary rep $\pi$ of $K \subset G$. 

Ruth Gornet Kirillov Theory
Recall: we have $\lambda \in g^*$, a polarization $\mathfrak{k}$ of $\lambda$ and a character
$\bar{\lambda}(k) = e^{2\pi i \lambda(\log k)}$ of $\exp(\mathfrak{k})$.

The representation space of $\pi_{\lambda}$ is then

$$W = \{ f : G \to \mathbb{C} : f(kx) = e^{2\pi i \lambda \log k} f(x) \quad \forall k \in K \}.$$

$G$ acts by right translation on $W$

Kirillov showed that $\pi_{\lambda}$ is unitary and irreducible
Example: The Heisenberg group and algebra. Let $\lambda \in \mathfrak{h}^*$. 

Case 1: $\lambda(Z) = 0$, $\implies K = H$. Then $\bar{\lambda}$ is a character of $H$ that is independent of $Z$,

$$\bar{\lambda}(x, y, z) = e^{2\pi i \lambda(xX + yY)}.$$ 

The induced rep $\pi_{\lambda}$ is unitarily equivalent to $\bar{\lambda}$.

To see this, note that the representation space $\mathcal{W}$ is defined as

$$\mathcal{W} = \{ f : H \rightarrow \mathbb{C} : f(hx) = e^{2\pi i \lambda \log h} f(x) \quad \forall h \in H \forall x \in H \}.$$ 

Letting $x = e$

$$\mathcal{W} = \{ f : H \rightarrow \mathbb{C} : f(h) = e^{2\pi i \lambda \log h} f(e) \quad \forall h \in H \} = \mathbb{C} \bar{\lambda}$$
Case 2: \( \lambda(Z) \neq 0 \implies K = (0, y, z) \)

\( \bar{\lambda}((0, y, z)) = e^{2\pi i \lambda(yY + zZ)} \) So that

\[ \mathcal{W} = \{ f : H \to \mathbb{C} : f(kx) = f(x) \forall k \in K \} \]

\( (x, y, z) = (0, y, z)(x, 0, 0) \), so

\[ f(x, y, z) = f((0, y, z)(x, 0, 0)) = e^{2\pi i \lambda(yY + zZ)} f(x, 0, 0). \]

Note that we can choose \( \lambda = c\zeta \)

This is equivalent to an action on \( \mathcal{W}' = \{ f : \mathbb{R} \to \mathbb{C} \} \)

What does this action look like. \( H \) acts on \( \mathcal{W} \) by right multiplication, so

\[ (\pi'_\lambda((x, y, z)) f)(u) = e^{2\pi i c(z + py)} f(u + x). \]
Let $\Gamma \subset G$ be a cocompact, discrete subgroup of $G$.

Example: Recall that the Heisenberg group can be realized as the set of matrices

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

A cocompact (i.e., $\Gamma \backslash G$ compact) discrete subgroup of $H$ is given by

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

(The existence of a cocompact, discrete subgroup places some restrictions on $g$, and it also implies that $G$ is unimodular.)
The **right action** $\rho$ of $G$ on $L^2(G)$ is a representation of $G$ on $\mathcal{H} = L^2(G)$:

$$ (\rho(a)f)(x) = f(xa) \quad \forall a \in G, x \in G $$

The **quasi-regular representation** $\rho_\Gamma$ of $G$ on $\mathcal{H} = L^2(\Gamma \backslash G)$ is given by

$$ (\rho_\Gamma(a)f)(x) = f(xa) \quad \forall a \in G, x \in \Gamma \backslash G $$

We generally view functions $f \in L^2(\Gamma \backslash G)$ as left-$\Gamma$ invariant functions on $G$, ie

$$ f(\gamma x) = f(x) \quad \forall \gamma \in \Gamma \forall x \in G $$

Both $\rho$ and $\rho_\Gamma$ are unitary.
Of interest to spectral geometry is determining the decomposition of the quasi-regular representation $\rho_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$.

To see why, we consider left invariant metrics on the Lie group $G$.

A left invariant metric on $G$ corresponds to a choice of inner product $\langle \ , \ \rangle$ on $g$. 
Let $f \in C^\infty(M)$

Recall that

$$(\Delta f)(p) = - \sum_j ((E_j(p))^2 + \nabla_{E_j(p)}E_j(p))f(p)$$

Claim: On $\Gamma \backslash G$, with Riemannian metric induced from $\langle \ , \rangle$ on $g$, $\Delta = - \sum_j E_j^2$, where $\{E_1, \ldots, E_n\}$ is an ONB of $g$.

From the standard proof of uniqueness of the Levi-Civita connection

$$2 \langle \nabla_X Y, W \rangle = X \langle Y, W \rangle + Y \langle X, W \rangle - W \langle X, Y \rangle$$

$$+ \langle [X, Y], W \rangle + \langle [W, X], Y \rangle - \langle [Y, W], X \rangle$$

But if $X, Y, W$ are left-invariant, then

$$\langle \nabla_X Y, W \rangle = \frac{1}{2} (\langle [X, Y], W \rangle + \langle [W, X], Y \rangle - \langle [Y, W], X \rangle)$$
Claim: $\sum_j \nabla E_j E_j = 0$

Proof: $\left\langle \sum_j \nabla E_j E_j, U \right\rangle = \sum_j \left\langle \nabla E_j E_j, U \right\rangle$

$= \frac{1}{2} \sum_j \left\langle [U, E_j], E_j \right\rangle + \left\langle [U, E_j], E_j \right\rangle + \left\langle [E_j, E_j], U \right\rangle$

$= \sum_j \left\langle \text{ad}(U)E_j, E_j \right\rangle = \text{tr}(\text{ad}U)$.

Since $G$ is unimodular, $\text{tr}(\text{ad}U) = 0$ for all $U \in g$.

See, eg, the Springer Encyclopedia of Mathematics (online) entry on unimodular.
A representation $\pi$ of a Lie algebra $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$ is a linear map

$$\pi : \mathfrak{g} \to \text{End}_\mathbb{R}(\mathcal{H})$$

such that $\pi([X, Y]) = [\pi(X), \pi(Y)]$

Let $(\pi, \mathcal{H})$ be a representation of $G$. Define

$$\mathcal{H}^\infty_\pi = \{ v \in \mathcal{H} : x \mapsto \pi(x)v \text{ is smooth} \},$$

the smooth vectors of $\mathcal{H}$ with respect to $\pi$.

$\mathcal{H}^\infty_\pi$ is $G$-invariant and dense
The derived representation $\pi_*$ of $\mathfrak{g}$ associated to the representation $(\pi, \mathcal{H})$ of $G$ is defined as, for $X \in \mathfrak{g}$

$$\pi_*(X)v = \frac{d}{dt}|_0\pi(\exp(tX))v,$$

where $\pi_*(X) : \mathcal{H}_\pi^\infty \to \mathcal{H}_\pi^\infty$

If $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ are unitarily equivalent, so are their derived representations.
If $E \in \mathfrak{g}$, then
\[ E(x) = \frac{d}{ds} |_0 x \cdot \exp(sE). \]

Let $f \in C^\infty(\Gamma \backslash G)$, then
\[ Ef(x) = \frac{d}{ds} |_0 f(x \cdot \exp(sE)) \]
\[ = \frac{d}{ds} |_0 \rho_\Gamma(\exp(sE)f)(x) \]
\[ = (\rho_{\Gamma^*}(E)f)(x) \]
So we extend $\Delta$ to $\mathcal{H}^\infty$ by
\[ \Delta f = - \sum_j \rho_{\Gamma^*}(E_j)^2 \]
Kirillov theory says that $L^2(\Gamma \backslash G)$ can be decomposed into the orthogonal sum of various $\pi_\lambda$, for $\lambda \in \mathfrak{g}^*$, each $\pi_\lambda$ occurring with finite multiplicity.

We seek a condition that says when $\pi_\lambda$ occurs, and with what multiplicity.
A rational Lie algebra is a Lie algebra defined over \( \mathbb{Q} \) rather than \( \mathbb{R} \). If we take \( g_\mathbb{Q} \otimes \mathbb{R} \), we obtain a real Lie algebra.

A choice of cocompact, discrete subgroup of \( G \) determines a rational structure. In particular, the existence of \( \Gamma \) implies that we can pick a basis of \( g \) from the set \( \log \Gamma \), which implies that the structure constants are rational on this basis.

Then \( g_\Gamma = \text{span}_\mathbb{Q}\{\log \Gamma\} \) is a rational Lie algebra.

A subalgebra \( \mathfrak{k} \subset g \) is a rational Lie subalgebra iff there exists subalgebra \( \mathfrak{k}_\mathbb{Q} \subset g_\Gamma \) such that \( \mathfrak{k} = \mathfrak{k}_\mathbb{Q} \otimes \mathbb{R} \). That is, there exists a basis of \( \mathfrak{k} \) contained in \( g_\Gamma \).
If $\mathfrak{k}$ is a rational subalgebra of $\mathfrak{g}$ (with respect to $\Gamma$), then $\Gamma \cap \exp(\mathfrak{k})$ is a cocompact, discrete subgroup of $K = \exp(\mathfrak{k})$.

To obtain a multiplicity formula, we must consider $\lambda \in \mathfrak{g}$ that have rational polarizations, and such that $\bar{\lambda}(\Gamma \cap \exp(\mathfrak{k})) = 1$. Thus $\bar{\lambda}$ is really a mapping on $\Gamma \cap K \backslash K$.

We call the pair $(\bar{\lambda}, \mathfrak{k})$ and integral point iff $\mathfrak{k}$ is rational (with respect to the rational structure induced by $\Gamma$) and $\bar{\lambda}(\Gamma \cap \exp(\mathfrak{k})) = 1$. 
Consider the set $F = \text{all pairs } (\bar{\lambda}, \xi) \text{ where } \bar{\lambda} \text{ is the character of } \exp(\xi) \text{ determined by } \lambda \in g, \text{ and } \xi \text{ is a polarization of } \lambda.$

$G$ acts by conjugation on $F$:

$$x \cdot (\bar{\lambda}, \xi) = (\bar{\lambda} \circ I_x, \text{Ad}(x^{-x})(\xi)),$$

for all $x \in G$.

Fact: If $(\bar{\lambda}, \xi) \in F$, then $x \cdot (\bar{\lambda}, \xi) \in F$. The isotropy subgroup of the point $(\bar{\lambda}, \xi)$ is $\exp(\xi)$.

Fact: $\Gamma$ maps integral points of $F$ to integral points of $F$.
Theorem: (L. Richardson and R. Howe) Let $\lambda \in \mathfrak{g}^*$ and let $(\bar{\lambda}, \kappa)$ induce $\pi_{\lambda}$. Then $\pi_{\lambda}$ occurs in the rep $\rho_\Gamma$ of $G$ on $L^2(\Gamma \backslash G)$ iff the $G$-orbit of $(\bar{\lambda}, \kappa)$ contains an integral point. The multiplicity of $\pi_{\lambda}$ is equal to the number of $\Gamma$-orbits on the set of integral points in the $G$-orbit of $(\bar{\lambda}, \kappa)$.

Restated:

$$m(\pi_{\lambda}, \rho_\Gamma) = \# \{ \Gamma \backslash \lambda(\text{Ad}(G))_\Gamma \},$$

where $\lambda(\text{Ad}(G))_\Gamma$ is the set of integral points of the co-adjoint action of $G$. 