Igor's Lecture: Functional Determinants

Motivation from Physics

In statistical physics one has to calculate Gaussian integrals. A simple example of such an integral is the partition function

$$\int \dots \int \exp(-\langle Ax, x\rangle) \frac{dx_1 \dots dx_n}{\pi^{n/2}},$$

where A is an $n \times n$ symmetric matrix with positive eigenvalues, and $\langle \bullet, \bullet \rangle$ denotes the Euclidean scalar product. After an orthogonal change of coordinates which diagonalizes A, this integral becomes

$$\int \dots \int \exp\left(-\sum_{i=1}^{n} \lambda_i x_i^2\right) \frac{dx_1 \dots dx_n}{\pi^{n/2}} = \frac{1}{(\lambda_1 \dots \lambda_n)^{1/2}} = \frac{1}{\sqrt{\det A}}.$$

In quantum field theory, they instead compute

$$Z = \int \exp(-I(\phi))\, [D\phi],$$

where integration may take place over an infinite-dimensional vector space of smooth (or continuous) functions on a compact manifold or of smooth maps between manifolds. Here

$$\begin{aligned} [D\phi] &= \text{measure on this space}, \\ I(\phi) &= \text{positive definite quadratic form representing the Lagrangian}. \end{aligned}$$

See, for example, Witten's talk at the 1996 ICM in Berkeley. The replacements we have made are:

$$\begin{aligned} x &\mapsto \phi : M \to N \\ \exp(-\langle Ax, x\rangle) &\mapsto \exp(-I(\phi)) \\ \frac{dx_1 \dots dx_n}{\pi^{n/2}} &\mapsto D(\phi). \end{aligned}$$

If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, then

$$\det A = \lambda_1 \dots \lambda_n.$$

This is hard to generalize, unless we use a Fredholm determinant, ie

$$A = \lim_{n \to \infty} A_n,$$

where each A_n is finite dimensional. Interesting example for physicists:

Let $M = S^1$, we integrate over the space of zero-average real-valued functions on M:

$$\begin{aligned} Z(S^1) &= \int \exp(-I(\phi))[D\phi]. \\ [D\phi] &= \text{Gaussian measure} \\ I(\phi) &= \int_{S^1} |\phi'(t)|^2\, dt. \end{aligned}$$

After integration by parts:

$$I(\phi) = \int_{S^1} \left\langle -\frac{d^2}{dt^2}\phi, \phi \right\rangle.$$

Computation of $Z(S^1)$ leads to computing

$$\det\left(-\frac{d^2}{dt^2}\right).$$

The expression

$$\frac{1}{\sqrt{\det\left(-\frac{d^2}{dt^2}\right)}}$$

could serve as a definition of $Z(S^1)$!

Goal: extend definition of determinant to a class of infinite-dimensional operators. Consider a positive $n \times n$ symmetric matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$\det(A) = \lambda_1 \ldots \lambda_n.$$

In infinite dimensional space, we wish to get a number.

One simple case is that of the Fredholm determinant. For bounded operators $A : \mathcal{H} \to \mathcal{H}$ which are limits of finite-dimensional operators, we say

$$\det_{Fr}(A) = \lim_{n\to\infty} \det(A_n).$$

If A is bounded and trace class, then

$$\det_{Fr}(e^A) = e^{TrA},$$

and also

$$\det_{Fr}(AB) = \det_{Fr}(A)\det_{Fr}(B).$$

In general, for any $\lambda > 0$,

$$\frac{d}{ds}\Big|_{s=0} \frac{1}{\lambda^s} = -\ln(\lambda).$$

Then

$$\begin{aligned} \ln(\det A) &= \sum_{i\geq 1} \ln(\lambda_i) \\ &= -\frac{d}{ds}\Big|_{s=0} \sum_{i\geq 1} \lambda_i^{-s} = -\zeta'_A(0), \end{aligned}$$

where

$$\zeta_A(s) = Tr(A^{-s}) = \sum_{i\geq 1} \lambda_i^{-s}$$

is the zeta function of A. For a positive symmetric matrix, this works, and the zeta function is analytic on the whole complex plane. This was first introduced by Ray and Singer in 1971.

$$\det A := \exp(-\zeta_A'(0)).$$

This formula generalizes to the case of self-adjoint unbounded operators on Hilbert space.

Assume that the eigenvalues of A are positive and satisfy

$$\lambda_n = c(n^k) + O\left(n^{k-\frac{1}{2}}\right), k > 0, c > 0$$

as $n \to \infty$. Then the infinite sum

$$\zeta_A(s) = \sum_{i \geq 1} \lambda_i^{-s}$$

converges for $Re(s)$ sufficiently large (say $s > k - 1$). In many interesting cases, $\zeta(s)$ can be meromorphically continued to the whole complex s-plane so that it is regular at $s = 0$. The poles and residues are known.

Example (Sturm - Liouville)

Let $M = S^1$, $A\phi = -\phi''$, $\phi(0) = \phi(2\pi)$, $\phi'(0) = \phi'(2\pi)$. One has

$$\lambda_0 = 0, \lambda_n = n^2 \text{ for } n \geq 1$$

with multiplicity two (eigenfunctions $\sin(n\theta)$, $\cos(n\theta)$). We have

$$\zeta_A(s) = 2\sum \frac{1}{n^{2s}} = 2\zeta(2s),\ Re(s) > \frac{1}{2},$$

where $\zeta(s)$ is the Riemann zeta function.

We have

$$\zeta_A'(0) = 4\zeta'(0) = -2\log(2\pi).$$

Thus

$$\det A = \exp(-\zeta_A'(0)) = \exp(2\log(2\pi)) = 4\pi^2.$$

Also, it can be shown that

$$\zeta_A(0) = -\frac{1}{2}.$$

Example (Harmonic Oscillator)

Let $H =$ Hamiltonian for quantum mechanical harmonic oscillator.

$$H\psi = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)\psi,$$

where ψ is a smooth L^2 function on $\mathbb{R}$. The eigenvalues are

$$\lambda_n = n + \frac{1}{2}, n = 0, 1, 2, 3, \ldots,$$

and the eigenfunctions are

$$\psi_n = \exp\left(-\frac{x^2}{2}\right)H_n(x),$$

where $H_n(x)$ are Hermite polynomials. We have

$$\zeta_H(s) = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{-s}, \; Re(s) > 1.$$

This operator zeta function is a particular case of the Hurwitz zeta function

$$\zeta_{H,a}(s) = \sum_{n=0}^{\infty} (n+a)^s, \; Re(s) > 1.$$

Then it turns out that

$$\det(H) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}\right)} = \sqrt{2}.$$

This is a particular case of the Hurwitz zeta function

$$\zeta_H(s,a) := \sum_{n\geq 0} (n+a)^{-s}.$$

Example (Laplacian on Torus)

$$Af = \Delta f = -f_{xx} - f_{yy},$$

where f is doubly periodic, i.e.

$$f(x + 2\pi a, y + 2\pi b) = f(x,y)$$

for specific $a, b \in \mathbb{R}$. The eigenvalues of this operator are

$$\frac{m^2}{a^2} + \frac{n^2}{b^2}, m, n \in \mathbb{Z},$$

corresponding to eigenfunctions

$$\sin\left(\frac{mx}{a}\right) \cos\left(\frac{ny}{b}\right)$$

and other similar combinations. The zeta function of this operator is related to the Epstein zeta function of number theory.

Remarks

There is more than one way to regularize an infinite product of positive numbers. Physicists adopt the regularization which leads to the right answer in the cases when the answer is known from other considerations.

Zeta function regularization leads to a multiplicative anomaly. We have

$$\det(AB) \neq \det(A)\det(B)$$

in some situations where both sides of the equation make sense. Kontsevich and Vishik showed a formula describing this in 1993. For simple examples, see Elizalde 1999.

Applications to Riemannian Geometry

Question: does a smooth compact manifold carry a best or a family of best Riemannian structure(s)? For surfaces, the uniformization theorem shows that any compact surface admits a metric of constant Gauss curvature. Moreover, on a given surface, these metrics are completely classified by Teichmüller theory, forming a finite-dimensional moduli space. The dimension is $6G - 6$, where $G \geq 2$ is the genus of the surface. How about higher dimensions? Perelman used this to prove the geometrization conjecture for 3-manifolds.

Possible general scheme of attack: Consider a suitable numerical functional $F(g)$ defined on the space of all metrics g. Search for all g for which the functional is maximum (or at least critical). In other words, g is critical if the derivative of $F(g)$ at g is zero for any variation of the metric. If the functional is scale-dependent, renormalize it. One such functional is the determinant of the Laplacian.

The Laplacian on functions looks like this in local coordinates. Let M be a closed compact manifold with Riemannian metric $g = (g_{ij})$. Then the Laplacian Δ onfunctions is

$$\Delta = -\sum_{j,k} \frac{1}{\sqrt{\det g}} \partial_j \left(\sqrt{\det g}\, g^{jk} \partial_k \right),$$

where g^{jk} is the inverse of the metric matrix. Then the equation

$$\Delta\varphi = \lambda\varphi$$

has solutions $\lambda = \lambda_k$ for

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

On a manifold of dimension n, the eigenvalues λ_k grow as $k^{2/n}$. Thus we can define a zeta function and the determinant of the Laplacian as before:

$$\zeta_\Delta(s) = \sum_{k=1}^{\infty} \lambda_k^{-s},$$

$$\det\Delta = \exp(-\zeta'_\Delta(0)).$$

Theorem (Osgood-Phillips-Sarnak 87) Let M be a closed surface. Among all metrics of constant area and in a given conformal class (ie of the type $g = exp(-\varphi) g_0$ with fixed metric g_0 of constant curvature and function φ), the constant curvature metric has **maximum** determinant.

Idea of proof: Use a variational formula for the determinant (as a function of the conformal parameter φ), given by Polyakov-Ray-Singer in 1981 by

$$\Delta_g = e^{\varphi}\Delta_{g_0},$$

$$\log\det\Delta_g - \log\det\Delta_{g_0} = \frac{1}{12}\int_M K_0\varphi\, dA_0 - \frac{1}{48\pi}\int_M |\nabla_0\varphi|^2\, dA_0$$
$$+ \log A - \log A_0,$$

where

$$\begin{aligned} K_0 &= \text{Gauss curvature for metric } g_0, \\ \nabla_0 &= \text{gradient for } g_0, \\ dA_0 &= \text{area element for } g_0, \\ A, A_0 &= \text{area of } M \text{ in metrics } g, g_0. \end{aligned}$$

Application:

Theorem (Osgood-Phillips-Sarnak 87) Isospectral sets of surfaces are compact in the smooth topology.

The proof uses the heat invariants and $\det \Delta$. Note that the heat invariants are the coefficients in the short time asymptotic expansion of the trace of the heat kernel $\exp(-t\Delta)$.

There are analogous results for manifolds with boundary.

Recently Pollicott-Rocha (1997) derived an explicit formula for $\det(\Delta)$ on a surface of constant negative curvature in terms of lengths of closed geodesics.

Theorem (Policott, Rocha 1997) Let M be a compact surface of constant negative curvature of genus g_0. Then there exists a constant C (depending only on the genus such that

$$det\,\Delta_M = C\sum_{n=1}^{\infty} b_n\,,$$

where the summation converges uniformly. Furthermore, the terms b_n tend to zero faster than any exponential. The numbers b_n satisfy

$$b_n = \sum(-1)^{r+1}\frac{\ell(\gamma_1)+\ldots+\ell(\gamma_r)}{(e^{\ell(\gamma_1)}-1)\ldots(e^{\ell(\gamma_r)}-1)},$$

where the sum is over all closed geodesics $\gamma_1,\ldots,\gamma_r$ with

$$|\gamma_1|+\ldots+|\gamma_r| = n.$$

Determinants in higher dimensions

The conformal Laplacian for the metric g on M is defined to be

$$\Delta^{conf} = \Delta + \frac{n-2}{4(n-1)}R,$$

where R is the scalar curvature. In 1991 Branson and Orsted showed that under conformal variations of the metric on a closed compact manifold, there is a local variation formula for $\log\det\Delta^{conf}$ analogous to the Polyakov-Ray-Singer variational formula. Note that

$$\Delta^{conf}_{e^{2\varphi}g} = e^{\left(-\frac{n}{2}-1\right)\varphi}\Delta^{conf}_g e^{\left(\frac{n}{2}-1\right)\varphi}.$$

Theorem (Branson-Chang-Yang 1992) On the standard 4-sphere, the standard metric **minimizes** $log\ det\,\Delta^{conf}$ among all conformal metrics of fixed volume.

Theorem (Branson 1996) On the standard 6-sphere, the standard metric **maximizes** $log\ det\,\Delta^{conf}$ among all conformal metrics of fixed volume.

Theorem (Morpurgo 1994) Among all metrics on S^2 with the same area as the standard metric, the standard metric **maximizes** $tr(exp(-t\Delta))$.

Ordinary Laplacian in Higher Dimensions

Theorem (Richardson 1994) The standard metric on S^3 is a local maximum for $det(\Delta)$ among constant volume, conformal variations.

Theorem (Richardson 1994) If g_0 is a critical point for $det\,\Delta$ for constant volume conformal variations of the metric of a closed 3-manifold, and if $-\zeta(1)\lambda_1 \geq 5$, then g_0 is a local maximum for the determinant (under conformal variations).

Theorem (Chiu 1995) The determinant of the Laplacian on the space of flat 3-tori of volume 1 has a local **maximum** at the torus corresponding to the face-centered cubic lattice.

General variations of the determinant of the Laplacian

Theorem (Okikiolu 1997) The standard metric on S^3 is a local **maximum** for $det\,\Delta$ under deformations of the metric which fix the volume.

Theorem (Okikiolu 1997) Let g_0 be a local extremal point of $det\,\Delta$ under deformations which fix the total volume. If $n = 3, 7, 11, \ldots$ then g_0 is a local **maximum**. If $n = 5, 9, 13, \ldots$ then g_0 is a local **minimum**.