The Covering Spectrum and Isospectrality

AMS Special Session: Inverse Problems in Geometry

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- Let X be a length space. A δ-**cover** is the covering obtained by using the open covering of all open balls of radius δ. We denote this covering by $\tilde{X}^\delta$.
- That is, $\tilde{X}^\delta := X_{\mathcal{U}_\delta}$, where $\mathcal{U}_\delta = \{B(\delta,p) : p \in X\}$.

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  the generator corresponding to $S^1(2)$ does not unravel until δ is at or below $\frac{2}{2} = 1$.
- Much of this behavior generalizes: the δ-covers are always monotone, and $\tilde{X}^\delta = \tilde{X}^{\delta - \epsilon}$ for some $\epsilon > 0$.

Definition: Covering Spectrum

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- Note that $\mathrm{CovSpec}(S^1(3) \times S^1(2)) = \{1, \frac{3}{2}\}$.
- Properties:
  If X is its own universal cover, $\mathrm{CovSpec}(X) = \emptyset$.
  If X is a compact length space, $\mathrm{CovSpec}(X) \subset (0, \mathrm{diam}(X))$, the covering spectrum is discrete, and its closure is contained in $\mathrm{CovSpec}(X) \cup \{0\}$.

Recall the Sunada method for producing isospectral manifolds

- Let H, K be subgroups of G with the property, $\forall x \in G$

$$\#(H \cap [x]) = \#(K \cap [x])$$

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- The Riemannian manifolds (M_H, g_H) and (M_K, g_K) are then isospectral.

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  - Is the covering spectrum a Sunada isospectral invariant?
- Most of the "usual suspects" of Gassmann-Sunada triples produce manifolds with the same covering spectrum.

- Let M be a compact Riemannian manifold.Define the minimum marked length map $m : \pi_1(M) \to \mathbb{R}$ by:

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- The mapping m has the following properties:
  $m(g) = 0$ if and only if $g = e$,
  $m(hgh^{-1}) = m(g)$ for all $h \in \pi_1(M)$,
  $m(g) = m(g^{-1})$ for all $g \in \pi_1(M)$.

We have the following algorithm for computing $\mathrm{CovSpec}(M)$:

- $\delta_1 := \min\{m(h)/2 : h \in \pi_1(M), h \neq e\} = \mathrm{systol}(M)/2$
  $S_1 := \{h \in \pi_1(M) : m(h) = 2\delta_1\}$
  $G_1 := \langle S_1 \rangle$
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- $\delta_{k+1} := \min\{m(h)/2 : h \in \pi_1(M), h \notin G_k\}$
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  $\ldots$
- stops when $G_{k_0} = \pi_1(M)$
  $\mathrm{CovSpec}(M) = \{\delta_1, \delta_2, \ldots, \delta_{k_0}\}$

Let G be a group.

- A **weighting** of G is a map $w : G \to \mathbb{R}^+ \cup \{0, \infty\}$ such that
  $w(e) = 0$
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- The **jump set** of w is

$$\text{jump}(w) := \{ r \geq 0 : r \text{ is a jump } \}.$$

- Proposition: Let M be a compact Riemannian manifold with minimum marked length map $m : \pi_1(M) \to \mathbb{R}^+ \cup \{0\}$. Then

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- Recall that $m : \pi_1(M) \to \mathbb{R}$ maps g to the length of the shortest representative of the free homotopy class of M corresponding to g.

Let (G, H, K) be a triple of finite groups with $H, K \subset G$ (not necessarily Sunada).

- We say a subset S of G is **stable** if $xsx^{-1} \in S$ and $s^{-1} \in S$ whenever $s \in S, x \in G$.

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- **ECS1** if for every stable subset S of G,

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- **ECS2** if for every pair of stable subsets S, T of G,

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- **ECS3** if for each weighting w on G we have

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- Proposition: $(ECS1) \implies (ECS2) \iff (ECS3)$.

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- Theorem: Let (G, H, K) be a triple of finite groups satisfying $(ECS2)$. Let M_0 be a closed manifold such that $\pi_1(M_0) = G$. Then $\text{CovSpec}(M_H, g_H) = \text{CovSpec}(M_K, g_K)$. If in addition (G, H, K) is a Gassmann-Sunada triple, (M_H, g_H) and (M_K, g_K) are isospectral.

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- Consequently, (G, H, K) is a Gassmann-Sunada triple not satisfying condition (ECS2).
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- In fact, $S = [x]$ for any element x of order 2 in G.

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- We obtain isospectral Riemann surfaces M_H, M_K using the Sunada setup.

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- However, for M_K, $K_1 := \langle K \cap [x] \rangle$ is index 2 in K, hence $G_1 = F^{-1}(K_1) \neq \pi_1(M_K)$. We conclude that the covering spectrum must have at least 2 elements.

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- Sunada isospectral Riemann surfaces need not have the same covering spectrum.

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- For any Gassmann-Sunada triple not satisfying (ECS2) with a generator of order 2, we can use this method while adjusting the metric on M_0 to obtain Sunada isospectral 4 manifolds with different covering spectrum.

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- For any Gassmann-Sunada triple not satisfying (ECS2) with a generator of order 2, we can use this method while adjusting the metric on M_0 to obtain Sunada isospectral 4 manifolds with different covering spectrum.
- We have a more straightforward albeit higher order example with the same properties.

ONLY ONE MORE SLIDE

- For any Gassmann-Sunada triple not satisfying (ECS2) with a generator of order 2, we can use this method while adjusting the metric on M_0 to obtain Sunada isospectral 4 manifolds with different covering spectrum.
- We have a more straightforward albeit higher order example with the same properties.
- Some of the Conway-Sloane isospectral flat 4-tori have different covering spectrum.

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- It is then clear that the covering spectrum of the flat torus $\mathbb{R}^5/\mathcal{L}'$ will have six elements in it.