Injectivity of local diffeomorphism and the global asymptotic stability problem

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Plan of the talk:

1. The Global Asymptotic Stability Problem (GASP);
2. GASP: solutions in $\mathbb{R}^2$;
3. GASP: solution in $\mathbb{R}^n$, $n \geq 3$;
4. GASP for piecewise linear vector fields in $\mathbb{R}^2$.

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1. The Global Asymptotic Stability Problem (GASP)

Let

$$\begin{array}{rccc} F: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & X & \longmapsto & F(X) \end{array}$$

be a C^1 vector field.

Consider the initial value problem

$$X' = \frac{dX}{dt} = F(X), \quad X(0) = X_0 \in \mathbb{R}^n, \tag{1}$$

where $t \in \mathbb{R}$ is the independent variable, called here the time.

1. The Global Asymptotic Stability Problem (GASP)

Suppose that:

1. The origin is an equilibrium point of (1), that is
$$F(0) = 0.$$
2. The eigenvalues of the linearization of F at the origin $DF(0)$ have negative real parts.

Then, by the Hartman–Grobman Theorem, the origin is

locally asymptotically stable,

that is there exists an open neighborhood U of the origin such that for every initial condition $X_0 \in U$ the solution $X(t)$ of (1) satisfies:

1. $X(t) \in U$, for all $t \geq 0$.
2. $\lim_{t \to \infty} X(t) = 0$.

The **basin of attraction** of an equilibrium point P, $B_P \subset \mathbb{R}^n$, is the largest open set whose elements $X_0 \in B_P$ satisfy the previous conditions 1 and 2.

The problem of determining the basin of attraction of an equilibrium point of a smooth vector field is of great importance for applications of stability theory of ordinary differential equations.

Denote by $\mathcal{F}^n$ the set of all C^1 vector fields $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that:

(i) The origin is an equilibrium point of (1), that is $F(0) = 0$.

(ii) For every $X \in \mathbb{R}^n$, the eigenvalues of $DF(X)$ have negative real parts.

1. The Global Asymptotic Stability Problem (GASP)

Problem 1 (Global Asymptotic Stability Problem)

Does $F \in \mathcal{F}^n$ imply that $X = 0$ is globally asymptotically stable?

In other words, does the basin of attraction of the equilibrium point $X = 0$ equal to $\mathbb{R}^n$?

1. The Global Asymptotic Stability Problem (GASP)

I will give some historical background for this subject, which nevertheless does not attempt to be exhaustive.

The GASP, also known as **Markuz–Yamabe Conjecture**, was explicitly stated by Markuz and Yamabe in

L. MARKUS, H. YAMABE, *Global stability criteria for differential systems*, Osaka Math. J., **12** (1960), 305–317.

1. The Global Asymptotic Stability Problem (GASP)

Osaka Math. J.
12 (1960), 305–317.

Global Stability Criteria for Differential Systems

By Lawrence MARKUS and Hidehiko YAMABE

Consider the real differential system

$$\mathscr{S}) \qquad \frac{dx^i}{dt} = f^i(x^1, \cdots, x^n) \qquad i = 1, 2, \cdots, n$$

with the real vector-valued function $f(x)$ in class C^1 in the real vector space R^n. The local stability theorem of A. Liapounov [8, and 2, P. 341] states that if the origin is a critical point,

$$f(0) = 0\,,$$

and if the eigenvalues of the Jacobian matrix $J(0)$, where

$$J^i_j(x) = \frac{\partial f^i}{\partial x^j}(x)\,,$$

have negative real parts, then each solution of $\mathscr{S}$) which initiates near the origin must approach the origin $t \to +\infty$. We shall extend this result to a global stability criterion which generalizes a theorem of N. N. Krasovski [3, 4].

For $n = 2$, that is in the plane, the GASP has been solved affirmatively under additional conditions:

- **One of the partial derivatives of F vanishes identically.**

L. MARKUS, H. YAMABE, *Global stability criteria for differential systems*, Osaka Math. J., **12** (1960), 305–317.

• **When** $DF(X) + DF(X)^{\top}$ **is everywhere negative definite**, where the symbol $^{\top}$ means transposition.

P. HARTMAN, *On stability in the large for systems of ordinary differential equations*, Canad. J. Math., **13** (1961), 480–492.

1. The Global Asymptotic Stability Problem (GASP)

- **When F is a polynomial vector field.**

G. Meisters, C. Olech, *Solution of the global asymptotic stability Jacobian conjecture for the polynomial case*, Analyse Mathématique et Applications, 373–381, Gauthier-Villars, Montrouge, 1988.

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2. GASP: solutions in $\mathbb{R}^2$

The Global Asymptotic Stability Problem in $\mathbb{R}^2$ was solved independently by:

- R. FESSLER, *A proof of the two–dimensional Markus–Yamabe stability conjecture and a generalization*, Ann. Polon. Math., **62** (1995), 45–74.
- C. GUTIERREZ, *A solution to the bidimensional global asymptotic stability conjecture*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **12** (1995), 627–671.
- A.A. GLUTSYUK, *The asymptotic stability of the linearization of a vector field on the plane with a singular point implies global stability*, translation in Funct. Anal. Appl., **29** (1996), 238–247.

Remark 1

All the three authors proved the **injectivity** *of* $F \in \mathcal{F}^2$ *and used it in the following equivalent problems.*

2. GASP: solutions in $\mathbb{R}^2$

Problem 2 (Global Injectivity)

Does $F \in \mathcal{F}^2$ *imply that the map* $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *is globally injective?*

Problem 3

Does $F \in \mathcal{F}^2$ *imply that there is a positive integer* K *such that for each* $Y \in \mathbb{R}^2$ *the number of solutions of* $F(X) = Y$ *is bounded by* K*?*

2. GASP: solutions in $\mathbb{R}^2$

Problem 4

Does $F \in \mathcal{F}^2$ imply that there are two positive constants ρ and r such that

$$||F(X)|| \geq \rho > 0, \quad \textit{for} \quad ||X|| \geq r > 0\ ?$$

Problem 5

Does $F \in \mathcal{F}^2$ imply that

$$\int_0^\infty \left[\min_{||X||=r} ||F(X)|| \right] dr = \infty\ ?$$

We mean that Problem *i* is equivalent to Problem *j*, for $i \neq j$ when:

Problem *i* has an affirmative answer for all $F \in \mathcal{F}^2$
if and only if
Problem *j* has an affirmative answer for all $F \in \mathcal{F}^2$.

The proof that Problems 1 and 2 are equivalent was given by Olech.

C. Olech, *On the global stability of an autonomous system on the plane*, Contributions to Differential Equations, **1** (1963), 389–400.

The proofs that Problems 1, 2, 3, 4 and 5 are equivalent were given by Gasull, Llibre and Sotomayor.

A. GASULL, J. LLIBRE, J. SOTOMAYOR, *Global asymptotic stability of differential equations in the plane*, J. Differential Equations, **91** (1991), 327–335.

Sketch of the proof:

$$P4 \Rightarrow P5 \Rightarrow P1 \Rightarrow P2 \Rightarrow P3 \Rightarrow P4.$$

Gutierrez solution of GASP in $\mathbb{R}^2$.

Let $\rho \in [0, \infty)$ and let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a C^1 map. We say that F satisfies the **ρ–eigenvalue condition** if:

(a) For every $X \in \mathbb{R}^2$, the determinant of $DF(X)$ is positive.

(b) For every $X \in \mathbb{R}^2$ with $||X|| \geq \rho$, the spectrum of $DF(X)$ is disjoint of $[0, \infty)$.

We have the following theorem.

Theorem 1 (Gutierrez Theorem)

If $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a C^1 map satisfying the ρ–eigenvalue condition, for some $\rho \in [0, \infty)$, then F is injective.

C. GUTIERREZ, *A solution to the bidimensional global asymptotic stability conjecture*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **12** (1995), 627–671.

In a more sophisticated setting, ten years later, Fernandes, Gutierrez and Rabanal proved the following remarkable theorem.

Theorem 2

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^1 map. If, for some $\varepsilon > 0$,

$$\mathrm{Spec}\,(DF(X)) \cap [0, \varepsilon) = \emptyset,$$

then F is injective.

A. Fernandes, C. Gutierrez, R. Rabanal, *Global asymptotic stability for differentiable vector fields of $\mathbb{R}^2$*, J. Differential Equations, **206** (2004), 470–482.

2. GASP: solutions in $\mathbb{R}^2$

In short:

if $F \in \mathcal{F}^2$, then

F satisfies the hypotheses of Theorems 1 and 2,

which implies that is F is globally injective.

By the equivalence of Problems 1 and 2, the equilibrium point at the origin is globally asymptotically stable, solving the Global Asymptotic Stability Problem in $\mathbb{R}^2$.

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3. GASP: solution in $\mathbb{R}^n$, $n \geq 3$

For $F \in \mathcal{F}^n$, $n \geq 3$, the Global Asymptotic Stability Problem was solved by Cima, van den Essen, Gasull, Hubbers and Mañosas with the following theorem.

A. Cima, A. van den Essen, A. Gasull, E. Hubbers, F. Mañosas, *A polynomial counterexample to the Markus–Yamabe conjecture*, Adv. Math., **131** (1997), 453–457.

Theorem 3

Let $n \geq 3$, $X = (x_1, \ldots, x_n)$ *and* $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be given by*

$$F(X) = \left(-x_1 + x_3(d(X))^2, -x_2 - (d(X))^2, -x_3, \ldots, -x_n\right),$$

where

$$d(X) = x_1 + x_2 x_3.$$

Then

1. $F \in \mathcal{F}^n$ *and*
2. *there is a solution of* (1) *which tends to infinity when* t *tends to infinity.*

Proof of Theorem 3.

It is immediate that $F(0) = 0$. For every $X \in \mathbb{R}^n$ the eigenvalues of $DF(X)$ are -1, that is all the eigenvalues of $DF(X)$ have negative real parts. So $F \in \mathcal{F}^n$.

Consider the initial value problem

$$X' = F(X), \quad X(0) = (18, -12, 1, 1, \ldots, 1).$$

The solution of the above IVP, given by

$$X : \mathbb{R} \longrightarrow \mathbb{R}^n, \quad X(t) = \left(18e^t, -12e^{2t}, e^{-t}, \ldots, e^{-t}\right),$$

has the following property

$$\lim_{t \to \infty} ||X(t)|| = \infty.$$

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4. **GASP for piecewise linear vector fields in $\mathbb{R}^2$.**

Piecewise linear differential systems with two zones in the plane are generally defined by

4. GASP for piecewise linear vector fields in $\mathbb{R}^2$

$$X' = \frac{dX}{dt} = \begin{cases} A^- X + B^-, & \mathcal{H}(X) \le 0, \\ A^+ X + B^+, & \mathcal{H}(X) \ge 0, \end{cases} \tag{2}$$

where

• the prime denotes derivative with respect to the independent variable t, called here the time,

• $X = (x, y) \in \mathbb{R}^2$,

• $A^\pm$ are 2×2 real matrices,

• $B^\pm$ are 2×1 real matrices,

• the function $\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}$ is at least continuous,

• the set $\Sigma = \mathcal{H}^{-1}(0)$ divides the plane in two unbounded components (zones) Σ^+ and Σ^- where $\mathcal{H}$ is positive and negative, respectively.

Usually, the points on the separation boundary Σ are classified as crossing (or sewing), sliding, escaping or tangency points.

A point $X_0 = (x_0, y_0) \in \Sigma = \mathcal{H}^{-1}(0)$ is a **crossing point** if

$$\left(\left(A^- X_0 + B^-\right) \cdot \nabla\mathcal{H}(X_0)\right)\left(\left(A^+ X_0 + B^+\right) \cdot \nabla\mathcal{H}(X_0)\right) > 0.$$

4. GASP for piecewise linear vector fields in $\mathbb{R}^2$

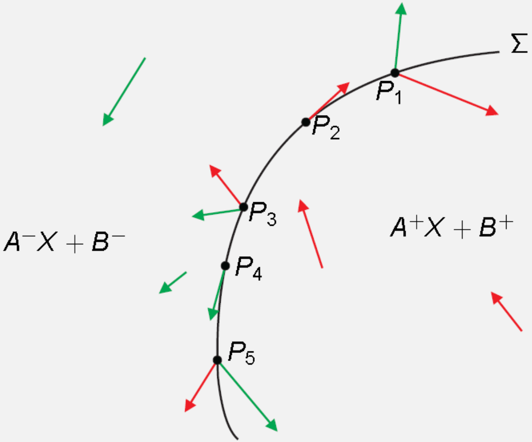


Figure : P_1 is an *escaping point*, P_3 is a *crossing point*, P_5 is a *sliding point*, P_2 and P_4 are *tangency points*.

4. GASP for piecewise linear vector fields in $\mathbb{R}^2$

For a crossing point $P \in \Sigma$, the solution of (2) by this point is just the concatenation of the solutions on Σ^+ and Σ^-.

In

E. FREIRE, E. PONCE, F. RODRIGO, F. TORRES, *Bifurcation sets of continuous piecewise linear systems with two zones*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **8** (1998) 2073–2097,

was studied the following case.

H1. A^+ and A^- are Hurwitz matrices (the real parts of all eigenvalues are negative);

H2. $B^+ = B^- = 0$;

H3. the separation boundary Σ is a straight line that contains the unique equilibrium point at the origin;

H4. the vector fields A^+X and A^-X are continuous on $\Sigma \backslash \{(0,0)\}$.

With the above hypotheses, Freire, Ponce, Rodrigo and Torres proved that the unique equilibrium point is globally asymptotically stable.

Now consider the previous hypotheses except H4, that is:

H1. A^+ and A^- are Hurwitz matrices (the real parts of all eigenvalues are negative);

H2. $B^+ = B^- = 0$;

H3. the separation boundary Σ is a straight line that contains the unique equilibrium point at the origin;

H4'. The points on $\Sigma \backslash \{(0,0)\}$ are of crossing type.

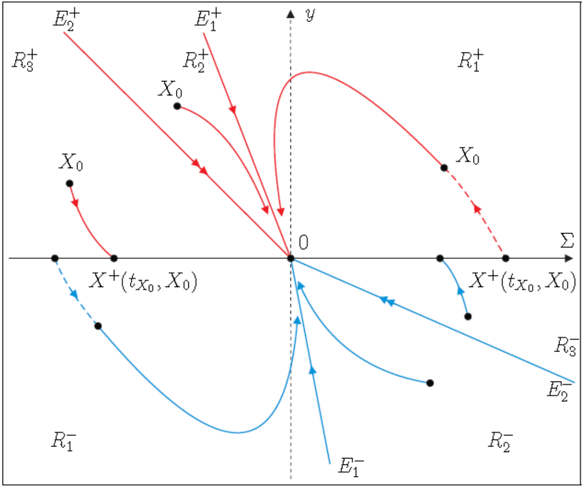
We have the following theorem.

Theorem 4

*Consider system (2) with the hypotheses **H1**, **H2**, **H3** and **H4'**. Then the unique equilibrium point at the origin is globally asymptotically stable.*

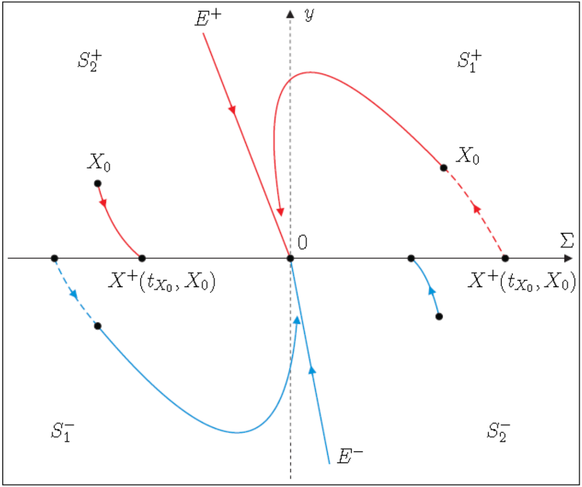
D.C. Braga, A.F. Fonseca, L.F. Mello, *The matching of two stable sewing linear systems in the plane*, J. Math. Anal. Appl., **433** (2016), 1142–1156.

4. GASP for piecewise linear vector fields in $\mathbb{R}^2$



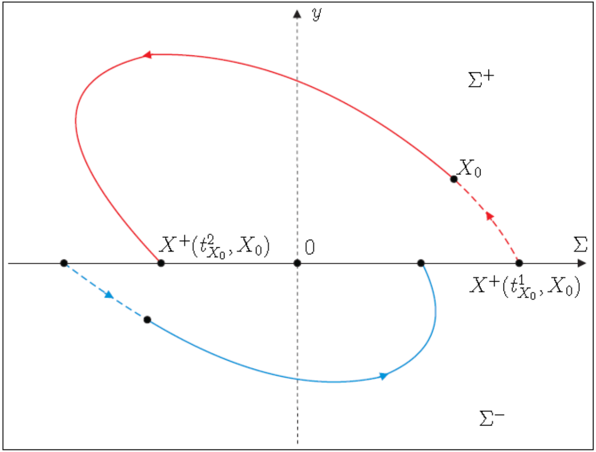
The origin is an attracting node with distinct eigenvalues. The curves in red (blue) are orbits of $X' = A^+X$ ($X' = A^-X$).

4. GASP for piecewise linear vector fields in $\mathbb{R}^2$



The origin is an attracting non–diagonalizable node. The curves in red (blue) are orbits of $X' = A^+X$ ($X' = A^-X$).

4. GASP for piecewise linear vector fields in $\mathbb{R}^2$



The origin is an attracting focus. The curves in red (blue) are orbits of $X' = A^+X$ ($X' = A^-X$).

Consider the function $\mathcal{H}_\rho$ defined by

$$X \mapsto \mathcal{H}_\rho(X) = y - \psi_\rho(x),$$

$$x \mapsto \psi_\rho(x) = \begin{cases} 0, & x \leq 0, \\ \rho x, & x \geq 0, \end{cases} \qquad \rho \geq 0. \tag{3}$$

In this case, the separation boundary

$$\Sigma_\rho = \left\{ X \in \mathbb{R}^2 : \mathcal{H}_\rho = 0 \right\} \tag{4}$$

is a polygonal line for each $\rho > 0$.

Now consider the previous hypotheses except H3, that is:

H1. A^+ and A^- are Hurwitz matrices (the real parts of all eigenvalues are negative);

H2. $B^+ = B^- = 0$;

H3'. The separation boundary Σ_ρ is a polygonal line that contains the unique equilibrium point at the origin;

H4'. The points on $\Sigma_\rho \backslash \{(0,0)\}$ are of crossing type.

We have the following theorem.

Theorem 5

*Consider system (2) with the hypotheses **H1**, **H2**, **H3'** and **H4'**. There are Hurwitz matrices A^+ and A^- such that the unique equilibrium point at the origin is either a stable focus, or a center, or an unstable focus.*

D.C. Braga, A.F. Fonseca, L.F. Mello, *The matching of two stable sewing linear systems in the plane*, J. Math. Anal. Appl., **433** (2016), 1142–1156.

For example, consider $\rho = 6$ and the following Hurwitz matrices A^+ and A^-

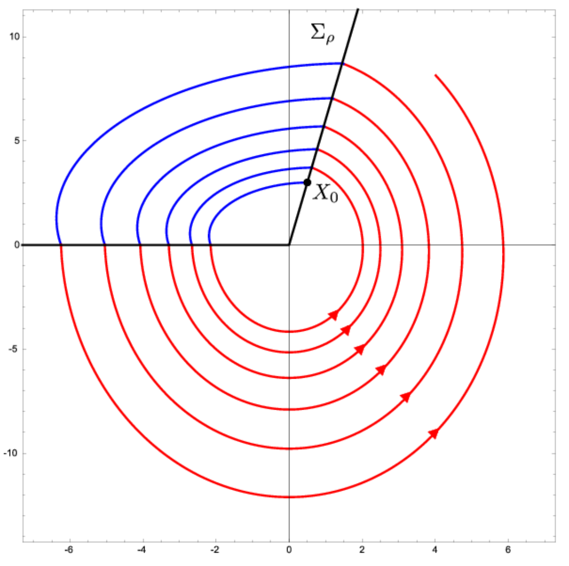
$$A^- = \begin{pmatrix} -\dfrac{1}{50} & -\dfrac{2501}{10000} \\ 1 & 0 \end{pmatrix}, \qquad A^+ = \begin{pmatrix} -\dfrac{1}{5} & -1 \\ 1 & -\dfrac{1}{5} \end{pmatrix},$$

whose eigenvalues, respectively, are

$$\lambda^- = -\frac{1}{100} \pm \frac{i}{2}, \quad \lambda^+ = -\frac{1}{5} \pm i.$$

The phase portrait of this system is illustrated in the following picture.

4. GASP for piecewise linear vector fields in $\mathbb{R}^2$



Thank you very much for your attention.