Darbouxian Umbilic Points on Smooth Surfaces in $\mathbb{R}^3$

Luis Fernando Mello

Universidade Federal de Itajubá – UNIFEI
E-mail: lfmelo@unifei.edu.br

Texas Christian University, Fort Worth

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Plan of the talk:

1. Introduction: Elementary differential geometry;
2. Darbouxian umbilic points;
3. Open problems.

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1. **Introduction: Elementary differential geometry;**
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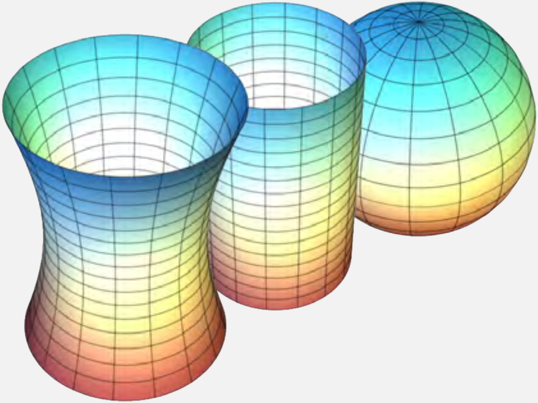
1. Elementary differential geometry

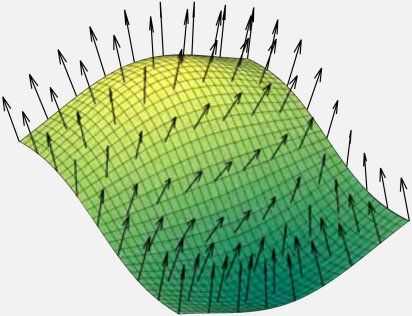
In this talk all the surfaces are regular (smooth).

That is, at each point of a surface it is well–defined a tangent plane.



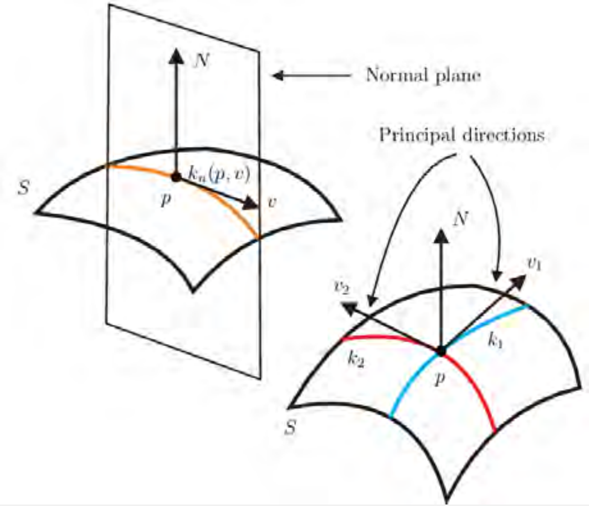
Some surfaces: One–sheet hyperboloid, cylinder and sphere.

The surfaces are orientable and they are oriented by the choice of a unit normal vector field N.



Oriented surface.

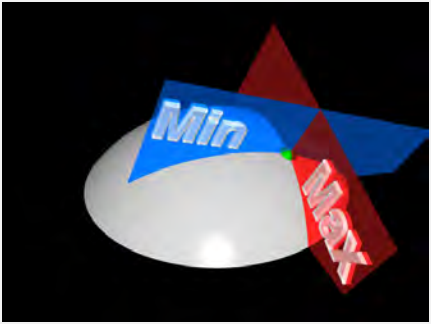
Euler is considered the founder of the curvature theory, defining the **normal curvature** and **principal directions**.



Normal section and principal directions.

1. Elementary differential geometry

At each point on a surface, are well–defined two orthogonal directions in the tangent plane of the surface at this point called **principal directions**, except at the **umbilic points** where the maximum and minimum normal curvatures (**principal curvatures**) coincide.



Orthogonal principal directions.

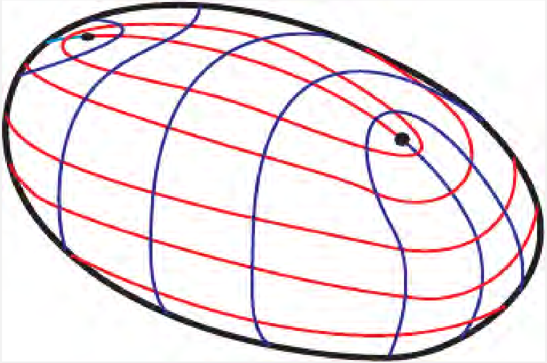
The **lines of curvature** of a surface are the integral curves of the principal direction fields.

The lines of curvature define two orthogonal foliations (**principal foliations**) with **singularities** on a surface, being the singularities the umbilic points.

Monge (1786) was the first to describe the global behavior of the lines of curvature and umbilic points on a surface: the triaxial ellipsoid (three different axes).

G. MONGE, *Sur les lignes de courbure de la surface de l'ellipsoide*, Journ. de l'Ecole Polytech., II cah., (1796).

1. Elementary differential geometry



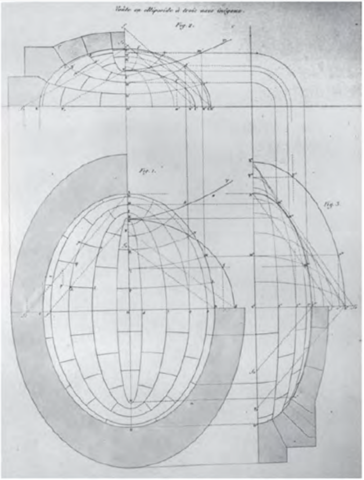
Lines of curvature and umbilic points on a triaxial ellipsoid.

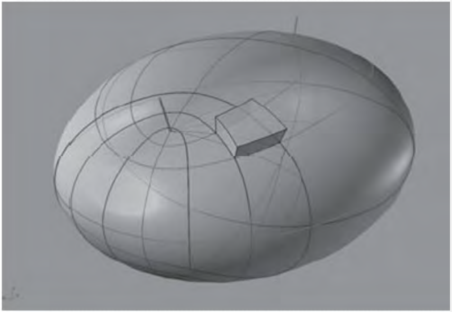
The principal configuration on the triaxial ellipsoid appears in Monge's proposal for the dome of the Legislative Palace for the government of the French Revolution.

The lines of curvature are the guiding curves for the workers to put the stones. The umbilic points, from which were to hang the candle lights, would also be the reference points below which to put the podiums for the speakers.

But, this dome was not built.

1. Elementary differential geometry



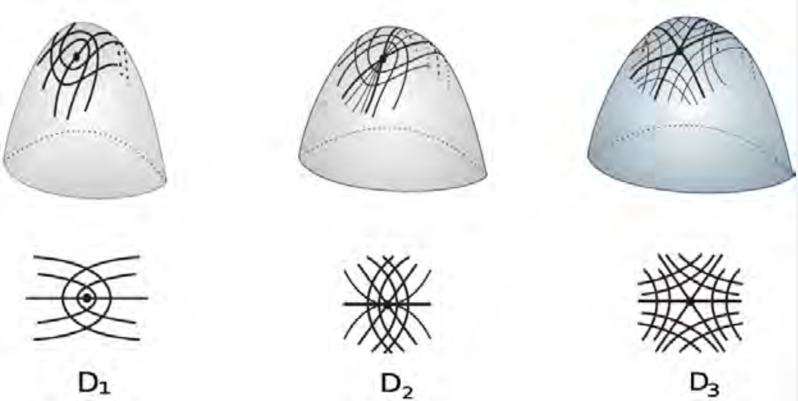


C.F.A. Leroy, Traité de Géométrie Descriptive, Paris, Mallet–Bachelier, 1844.

The study of some aspects of geometry by using the Qualitative Theory of Ordinary Differential Equations started with Darboux.

Darboux studied the structure of the lines of curvature in a neighborhood of a generic umbilic point using the theory developed by Poincaré for equilibrium points.

G. Darboux, *Sur la forme des lignes de courbure dans la voisinage d'un ombilic*, Note 07, Leçons sur la théorie de surfaces, vol. IV, Gauthier Villars, Paris, 1896.



Darbouxian umbilic points: D_1, D_2 and D_3.

To finish this Introduction it is worth to mention the book

I. PORTEOUS, Geometric Differentiation (for the intelligence of curves and surfaces), Cambridge University Press, 1994.

It is noted in Porteous (page 271) that a detailed study of umbilic points had been undertaken by the Swedish physician Allvar Gullstrand starting in 1904, with the aim of understanding the human eye.

In his work on the accommodation of the eye lens, for which he was awarded the Nobel Prize in 1911, Gullstrand studied umbilical points of surfaces in detail (1904). After describing in detail the patterns of lines of curvature and ridges around ordinary umbilics of a surface he proceeds to detail what happens at umbilics with rectangular symmetry. In the final section of his paper he also gives examples of surfaces having lines of umbilical points, with both variable and constant radius of curvature.

Porteous' book, page 271.

A. GULLSTRAND, *Zur Kenntnis der Kreispunkte*, Acta Math., **29** (1904), 59–100.

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Darbouxian umbilic points

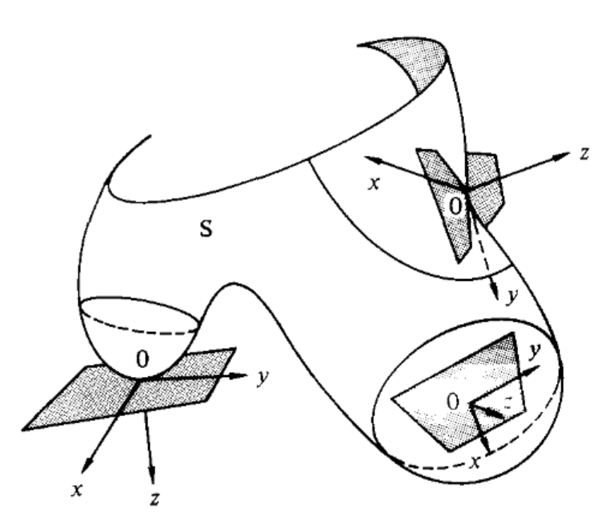
Let $P \in S$ be an isolated umbilic point, where S is a regular oriented surface.

Without loss of generality, assume that the surface S is given in a **Monge chart**:

- $P = (0, 0, 0)$;
- The tangent plane of S at P is the xy–plane;
- The surface S is locally the graph of a smooth function

$$z = \alpha(x, y), \quad (x, y) \in U \subset \mathbb{R}^2, \quad (0, 0) \in U.$$

Darbouxian umbilic points



S is the graph of a smooth function in a neighborhood of a point $P = 0$.

Darbouxian umbilic points

Under our assumptions,

$$\alpha(x, y) = \frac{k}{2}(x^2 + y^2) + \frac{a}{6}x^3 + \frac{\bar{b}}{2}x^2 y + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + O(4),$$

where $k = k_1(P) = k_2(P)$ is the principal curvature at the umbilic point P.

By a rotation on the xy–plane it is possible to take $\bar{b} = 0$.

So, in the Monge chart the surface S is given by

$$X(x,y) = (x, y, \alpha(x,y)), \quad (x,y) \in U,$$

where

$$\alpha(x,y) = \frac{k}{2}(x^2 + y^2) + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + O(4). \qquad (1)$$

Now we study the differential equation of the lines of curvature in the above Monge chart:

$$(Ef - eF)\,dx^2 + (Eg - eG)\,dx\,dy + (Fg - fG)\,dy^2 = 0, \qquad (2)$$

where

$$E = E(x,y), \quad F = F(x,y), \quad G = G(x,y),$$

$$e = e(x,y), \quad f = f(x,y), \quad g = g(x,y),$$

are the coefficients of the first and second fundamental forms of S

$$I = Edx^2 + 2Fdx\,dy + Gdy^2, \quad II = edx^2 + 2fdx\,dy + gdy^2.$$

The coefficients of the first fundamental form in the Monge chart are given by:

$$E(x, y) = \langle X_x(x, y), X_x(x, y)\rangle = 1 + O(2),$$

$$F(x, y) = \langle X_x(x, y), X_y(x, y)\rangle = O(2),$$

$$G(x, y) = \langle X_y(x, y), X_y(x, y)\rangle = 1 + O(2),$$

Darbouxian umbilic points

The coefficients of the second fundamental form in the Monge chart are given by:

$$e(x,y) = \langle X_{xx}(x,y), N(x,y)\rangle = \frac{k + ax + O(2)}{\sqrt{1 + {\alpha_x}^2 + {\alpha_y}^2}},$$

$$f(x,y) = \langle X_{xy}(x,y), N(x,y)\rangle = \frac{by + O(2)}{\sqrt{1 + {\alpha_x}^2 + {\alpha_y}^2}},$$

$$g(x,y) = \langle X_{yy}(x,y), N(x,y)\rangle = \frac{k + bx + cy + O(2)}{\sqrt{1 + {\alpha_x}^2 + {\alpha_y}^2}}.$$

Therefore, the differential equation of the lines of curvature (2) has the form

$$A(x,y)\,dx^2 + B(x,y)\,dx\,dy + C(x,y)\,dy^2 = 0, \tag{3}$$

where

$$\begin{aligned} A(x,y) &= by + O(2), \\ B(x,y) &= (b-a)x + cy + O(2), \\ C(x,y) &= -by + O(2). \end{aligned}$$

The **discriminant** of (3) is defined by

$$\Delta(x,y) = B^2(x,y) - 4A(x,y)C(x,y). \tag{4}$$

By construction,

$$\Delta(x,y) \geq 0,$$

for all $(x,y) \in U$ and

$$\Delta(x,y) = 0,$$

if and only if $(x,y) = (0,0)$.

Taking $p = dy/dx$ in the left hand side of (3) we obtain

$$F(x,y,p) = [-by + O(2)]p^2 + [(b-a)x + cy + O(2)]\, p + [by + O(2)]. \quad (5)$$

Consider the surface $M = F^{-1}(0)$ defined in $U \times \mathbb{R}P^1$.

It follows that

$$F(0,0,p) = 0, \quad \forall\, p,$$

that is, the axis $(0,0,p)$, the **projective line**, is contained in M.

We have the following theorem.

Theorem 1

The following statements hold:

1. *The surface M is a double covering of $U \backslash \{(0,0)\}$;*
2. *The surface M is regular on the projective line if and only if $b(b-a) \neq 0$;*
3. *The projection $\pi : M \longrightarrow U$, $\pi(x, y, p) = (x, y)$ is a local diffeomorphism out of the projective line.*

Proof of item 1. It is immediate since F is quadratic in the variable p.

Proof of item 2. The gradient of F on the projective line is given by

$$F_x(0,0,p) = (b-a)p, \quad F_y(0,0,p) = -bp^2+cp+b, \quad F_p(0,0,p) = 0.$$

It follows that the gradient of F on the projective line vanishes if and only if the resultant of $(b-a)p$ and $-bp^2+cp+b$ vanishes, that is, if and only if

$$\mathrm{Res}(F_x(0,0,p), F_y(0,0,p)) = b(b-a)^2 = 0.$$

It completes the proof of item 2.

Proof of item 3. The projection $\pi : M \longrightarrow U$, $\pi(x, y, p) = (x, y)$ is not a local diffeomorphism when

$$F(x, y, p) = 0, \quad F_p(x, y, p) = 0.$$

But, the resultant of F and F_p gives exactly the discriminant (4) of F, that is

$$\mathrm{Res}(F, F_p) = \Delta(x, y),$$

which vanishes if and only if $(x, y) = (0, 0)$.

■

Assume that the surface M is regular on the projective line, that is the condition T below is satisfied:

Definition 1

$$T: \qquad b(b-a) \neq 0.$$

Definition 2

Define the Lie–Cartan vector field in $U \times \mathbb{R}P^1$ by

$$\mathcal{X}(x, y, p) = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}, \tag{6}$$

where

$$F_p = F_p(x, y, p), \quad F_x = F_x(x, y, p), \quad F_y = F_y(x, y, p).$$

We have the following theorem.

Theorem 2

The following statements hold:

1. *The Lie–Cartan vector field is tangent to M;*
2. *The projections of the integral curves of the Lie–Cartan vector field $\mathcal{X}$ on M give the lines of curvature on U.*

Proof of item 1. Taking the derivative of F in the direction of the Lie–Cartan vector field we have

$$\begin{aligned}\mathcal{X}\,F &= \langle \mathcal{X}(x,y,p), \nabla F(x,y,p)\rangle \\ &= \langle (F_p, pF_p, -(F_x + pF_y)), (F_x, F_y, F_p)\rangle \\ &= F_p\,F_x + pF_p\,Fy - (F_x + pF_y)\,F_p = 0,\end{aligned}$$

for all $(x,y,p) \in M$.

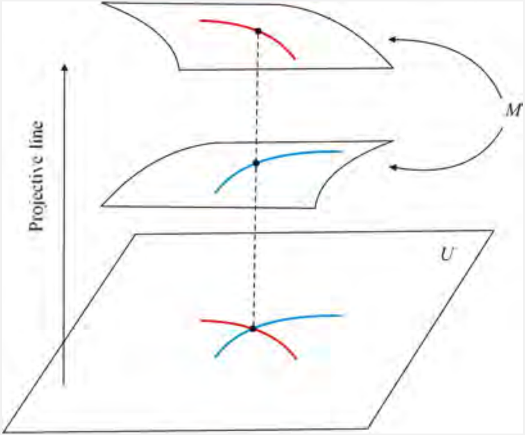
Proof of item 2. Consider the projection $\pi : M \longrightarrow U$, $\pi(x, y, p) = (x, y)$. It is easy to see that the slope of

$$D\pi(x, y, p)\mathcal{X}(x, y, p) = (F_p(x, y, p), pF_p(x, y, p))$$

is exactly p.

$\blacksquare$

Darbouxian umbilic points



The surface *M* and the projections of the integral curves of the Lie–Cartan vector field.

Now we study the equilibrium points of the Lie–Cartan vector field on the projective line.

We have the following theorem.

Theorem 3

The hyperbolic equilibrium points of the Lie–Cartan vector field on the projective line are given by:

1. *One saddle if*
$$c^2 + 4b(2b - a) < 0;$$
2. *One node and two saddles if*
$$c^2 + 4b(2b - a) > 0, \quad 1 < a/b \neq 2;$$
3. *Three saddles if*
$$a/b < 1.$$

I will prove only the first case of Theorem 3. The proofs of the two other cases are similar. From now on, I will omit the symbol $O(k)$ for the high order terms.

Recall that the Lie–Cartan vector field (6) is given by

$$\mathcal{X}(x,y,p) = (F_p(x,y,p), pF_p(x,y,p), -(F_x(x,y,p)+pF_y(x,y,p))),$$

where

$$F(x,y,p) = [-by+O(2)]p^2+[(b-a)x+cy+O(2)]p+[by+O(2)].$$

So, we obtain

$$
\begin{aligned}
F_p(x,y,z) &= -2pby + (b-a)x + cy, \\
pF_p(x,y,z) &= -2p^2by + (b-a)px + cpy, \\
-\left(F_x(x,y,p) + pF_y(x,y,p)\right) &= p\left(bp^2 - cp + a - 2b\right).
\end{aligned}
$$

The Lie–Cartan vector field on the projective line has the form

$$
\mathcal{X}(0,0,p) = \left(0, 0, p\left(bp^2 - cp + a - 2b\right)\right).
$$

Therefore, the equilibrium points of the Lie–Cartan vector field on the projective line are given by $(0, 0, p)$ where p is a real root of the cubic polynomial

$$p(\underbrace{bp^2 - cp + a - 2b}_{\gamma(p)}).$$

The discriminant of the polynomial γ is given by

$$\delta = c^2 + 4b(2b - a) < 0,$$

by the assumption in the first case of the theorem. So, $p_0 = 0$ is the unique real root of the above cubic polynomial.

This implies that $(0, 0, 0)$ the unique equilibrium point of the Lie–Cartan vector field on the projective line.

The Jacobian matrix of the Lie–Cartan vector field at the equilibrium $(0,0,0)$ is given by

$$D\mathcal{X}(0,0,0) = \begin{pmatrix} b-a & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a-2b \end{pmatrix}.$$

The eigenvalues of the above matrix are

$$\lambda_1 = 0, \quad \lambda_2 = b-a \neq 0, \quad \lambda_3 = a-2b \neq 0.$$

We need to study the signs of λ_2 and λ_3.

By assumption, $c^2 + 4b(2b - a) < 0$. So,

$$c^2 + 4b(2b - a) < 0 \Rightarrow 4b(2b - a) < -c^2 \Rightarrow b(2b - a) < -\frac{c^2}{4}.$$

Thus

$$-b(2b - a) > \frac{c^2}{4} \Rightarrow b(a - 2b) > \frac{c^2}{4} \geq 0.$$

Therefore,

$$b(a - 2b) > 0.$$

We have two possibilities:

- $b > 0$ and $a - 2b > 0$.
  From the second inequality we obtain $a > 2b > b$. Thus

$$\lambda_2 \cdot \lambda_3 = \underbrace{(b - a)}_{<0}\underbrace{(a - 2b)}_{>0} < 0.$$

  Thus λ_2 and λ_3 have opposite signs.

- $b < 0$ and $a - 2b < 0$.
  From the second inequality we obtain $a < 2b < b$. Thus

$$\lambda_2 \cdot \lambda_3 = \underbrace{(b - a)}_{>0}\underbrace{(a - 2b)}_{<0} < 0.$$

  Thus λ_2 and λ_3 have opposite signs.

From the above analysis, the equilibrium point $(0, 0, 0)$ of the Lie–Cartan vector field is a saddle.

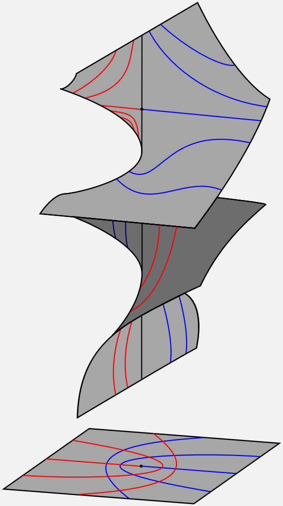
By an easy calculation, the eigenvectors associated to the eigenvalues λ_2 and λ_3 are, respectively

$$V_2 = (1, 0, 0), \quad V_3 = (0, 0, 1).$$

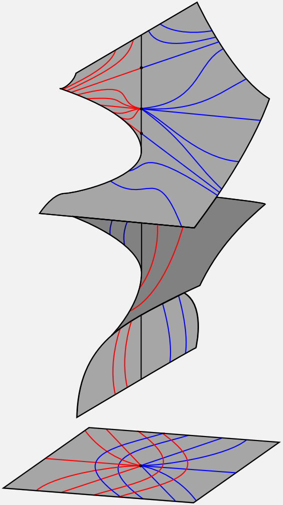
So, one separatrix is contained in the projective line and the other is transversal to it.
The integral curves of the Lie–Cartan vector field in a neighborhood of the projective line is depicted in the next figure.

In short, Theorem 3 is proved.

■

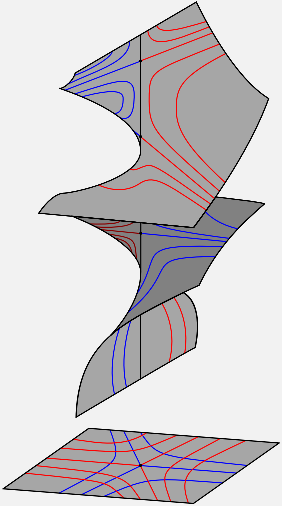


Resolution of the Lie–Cartan vector field near the projective line: one saddle.



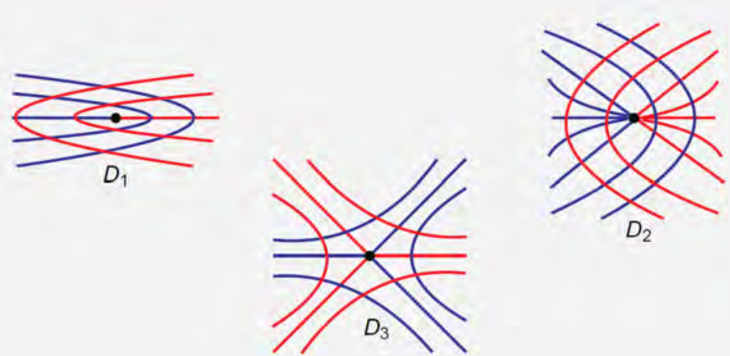
Resolution of the Lie–Cartan vector field near the projective line: one node and two saddle.

Darbouxian umbilic points



Resolution of the Lie–Cartan vector field near the projective line: three saddles.

Darbouxian umbilic points



Darbouxian umbilic points: D_1, D_2 and D_3.

Darbouxian umbilic points

Definition 3

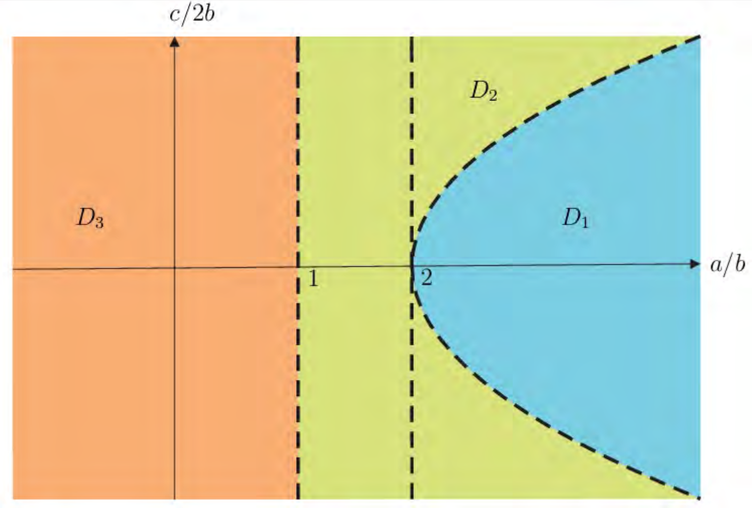
In a Monge chart, we define Darbouxian umbilic points by:

a) Transversality condition (T): $\quad b(b-a) \neq 0.$

b) Darbouxian condition (D_i, $i = 1, 2, 3$):

- D_1: $\quad \left(\dfrac{c}{2b}\right)^2 + 2 < \dfrac{a}{b};$
- D_2: $\quad \left(\dfrac{c}{2b}\right)^2 + 2 > \dfrac{a}{b} > 1, \quad a \neq 2b;$
- D_3: $\quad \dfrac{a}{b} < 1.$

Darbouxian umbilic points



Planar distribution of Darbouxian umbilic points.

Darbouxian umbilic points

Definition 4

In a general chart, we define Darbouxian umbilic points by:

a) Transversality condition (T): The surface $M \subset U \times \mathbb{R}P^1$ is regular on the projective line.

b) Darbouxian condition (D_i, $i = 1, 2, 3$):

- *D_1: The Lie–Cartan vector field has one saddle on the projective line;*
- *D_2: The Lie–Cartan vector field has one node and two saddles on the projective line;*
- *D_3: The Lie–Cartan vector field has three saddles on the projective line.*

Consider the differential equation of the lines of curvature (3)

$$A(x, y)dx^2 + B(x, y)dx\,dy + C(x, y)dy^2 = 0,$$

where

$$A = Ef - eF, \quad B = Eg - eG, \quad C = Fg - fG.$$

Suppose that $(x, y) = (0, 0)$ is an isolated umbilic point of S, $p = dy/dx$ and

$$F(x, y, p) = C(x, y)p^2 + B(x, y)p + A(x, y).$$

Let $H = H(x, y)$, $K = K(x, y)$ be the mean and Gaussian curvature of S.

We have the following theorem.

Darbouxian umbilic points

Theorem 4

The following statements are equivalent:

1. *The surface* $M = F^{-1}(0)$ *is regular on the projective line;*
2. *The discriminant function (4) given by* $\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y)$ *has a Morse singularity at* $(x, y) = (0, 0)$*;*
3. *The function* $U(x, y) = H^2(x, y) - K(x, y)$*, whose zeros give the umbilic points, has a Morse singularity at* $(x, y) = (0, 0)$*;*
4. *The curves* $A^{-1}(0)$ *and* $B^{-1}(0)$*, whose intersections give the umbilic points, are regular and have transversal intersection at* $(x, y) = (0, 0)$*.*

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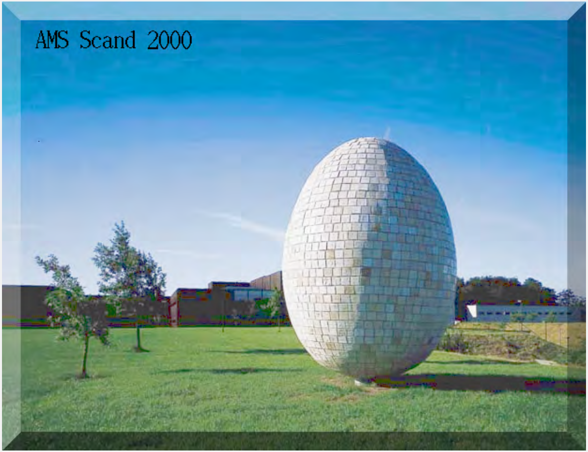
An **ovaloid** is a compact and convex surface.



Odense Campus Sculpture: Peter Tybjerg.

Open problems

Examples of ovaloids:

- Sphere: all the points are umbilic points;
- Ellipsoid of revolution: two umbilic points at the poles;
- Triaxial ellipsoid (Monge): four umbilic points.

Conjecture 1 (Carathéodory Conjecture)

All ovaloid has at least **two** *umbilic points.*

Open problems

At an isolated umbilic point it is well–defined the **index** of the principal foliations at this singular point, which has the form $n/2$, $n \in \mathbb{Z}$.

The Euler characteristic of an ovaloid is 2 (diffeomorphic to $\mathbb{S}^2$).

By the Poincaré–Hopf theorem an ovaloid has at least one umbilic point and the sum of its indices must be equal 2, if the number of umbilic points is finite.

So an affirmative answer to Carathéodory Conjecture follows from an affirmative answer to the following conjecture.

Conjecture 2 (Loewner Conjecture)

The index of an isolated umbilic point is at most 1.

Open problems

The index of an isolated umbilic point p, $I(p)$, can be calculated by (Bendixson formula)

$$I(p) = 1 + \frac{\#e - \#h}{2},$$

where

- $\#e$ is the number of elliptic sectors;
- $\#h$ is the number of hyperbolic sectors

of one principal foliation in a neighborhood of an isolated umbilic point p.

So an affirmative answer to Loewner Conjecture follows from an affirmative answer to the following conjecture.

Conjecture 3 (Elliptic Sector Conjecture)

At an isolated umbilic in a smooth surface in $\mathbb{R}^3$ *the principal foliations cannot have an elliptic sector.*

References

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