Remarks on my GAGA talks on quantum motion on complete manifolds

F. Xavier, Feb 2019

The lectures were based on an old unpublished manuscript of mine, "*Quantum motion on symmetric spaces of non-positive curvature*". Let (M^n, g) be a complete Riemannian manifold. By analogy with the dichotomy between Newtonian and quantum mechanics, the "classical motion" on M is the dynamics given by moving along geodesics, whereas the "quantum motion" is given by solving the Schrodinger equation

$$\frac{d}{dt}\psi = -iH\psi,$$

relative to the unique self-adjoint extension H of $\Delta|C_0^\infty(M^n)$ in $L^2(M^n)$, where Δ is the intrinsic Laplacian associated to g.

We would like to investigate to what extent one can realize the naive expectation that if the classical motion is "free", i.e. any geodesic spends only a finite time on a given compact set $K \subset M^n$, then the quantum motion is also "free". The latter means that the expected time

$$E(K, \psi) = \int_{-\infty}^{\infty} \int_K |e^{-itH}\psi|^2 d\mu dt \tag{0.1}$$

that a normalized state $\Psi \in L^2(M)$ remains in K should be finite.

It is elementary to show that the classical motion is free if M^n carries a strictly convex function. Somewhat surprisingly, it turns out that under suitable additional assumptions on the convex function the quantum motion is free as well, i.e. the integral (0.1) is finite. This is much harder, and the necessary technical machinery comes from scattering theory. In both cases, one obtains quite explicit estimates for the occupation time in K, in terms of certain quantities associated to K and the convex function.

Studying when the motion is "free" (both in the classical and quantum senses) is more than a mere pseudo-physical curiosity, as the resulting estimates can be used to prove that the Laplacian has no singular spectrum, and also to establish the existence of spectrally stable Riemannian manifolds.

The latter concept, which I introduced in the aforementioned manuscript, means that the Laplacians associated to the nearby metrics are all unitarily equivalent. This is a new phenomenon, which never occurs for compact manifolds.

Ordinarily, one studies a Riemannian manifold in an "inward" way, looking at its curvature and how it relates to the topology and analysis of the manifold, etc. In Riemannian spectral stability the point of view is different: one is interested in how "neighborly" the manifold is, in the sense that the metric should be indistinguishable, from the point of view of the L^2 theory of Δ, from its neighbors.

The central geometric result thus far (obtained in collaboration with H. Donnelly, Math. Z, 2006) states that all globally symmetric spaces of non-compact type are spectrally stable. It has been many years since I have worked on this topic, but there are still several interesting open problems (see the list below).

Due to the inherent instability of the singular spectrum (at least from an abstract operator-theoretic point of view), the central issue in the theory of Riemannian spectral stability is to show that the spectral measures of the Laplacian are absolutely continuous. In our context, this is accomplished using a certain integral identity involving strictly convex functions f for which the biLaplacian of f satisfies $\Delta^2 f \leq 0$. The general framework of scattering theory (wave operators, Kato's theory of H-smooth perturbations, positive commutators, the Kato-Birman theory, etc.) can in principle be applied, but the heart of the matter is to construct convex functions with certain additional properties, and this is where finer aspects of Riemannian geometry come into play.

Although I didn't lecture on this, similar ideas can also be applied to a problem in the interface of ergodic theory and the geodesic flow of compact manifolds of negative curvature (see our joint work with V. Nitica: *Schrodinger operators and topological pressure on manifolds of negative curvature*, Proc. Symposia of AMS, Smooth ergodic theory and Applications, edited by A. Katok, vol. 69, 2001).

1 Open problems in Riemannian spectral stability

Problem 1. Show that if (M, g_0) is spectrally stable, then the self-adjoint realization H_{g_0} of Δ_{g_0} has to be absolutely continuous. For abstract operators, in the presence of singular spectrum even rank one perturbations can yield a non-unitarily equivalent operator. But it is not clear, assuming that H_{g_0} has some singular spectrum, how to perturb H_{g_0} *through Laplacians of metrics* in order to obtain an H_g that is not unitarily equivalent to H_{g_0}. A positive solution to this problem would imply, via the Kato-Birman theory, a most satisfying criterion for Riemannian spectral stability: (M, g_0) is spectrally stable if and only if H_g is absolutely continuous whenever $g = g_0$ or g is sufficiently close to g_0 (*persistence* of absolute continuity), in a suitable Whitney topology in the space of Riemannian metrics.

Problem 2. Compact manifolds are certainly unstable. Must the same be true if (M, g) is complete and has finite volume? Notice that 0 is a persistent eigenvalue.

Problem 3. If (M, g) is the universal cover of a compact manifold of negative curvature, does it support a strictly convex solution of $\Delta f = 1$? An affirmative answer would show that H_g is absolutely continuous and, most likely, that (M, g) is spectrally stable. This kind of question is potentially relevant in the approach to the Chern-Hopf conjecture (namely, that $(-1)^n\chi(M^{2n}) > 0$ should hold whenever the compact manifold M^{2n} carries a metric of negative sectional curvature) based on Atyiah's L^2 index theorem (see, e.g., Donnelly-Xavier, Amer. J. Math. (1984); Gromov, JDG (1991); Cao-Xavier, Math. Ann. (2001); Jost-Zuo, Comm. Anal. Geom (2000)). Notice that the notion of Riemannian spectral stability is, of course, meaningful for the L^2 Laplacian acting on differential forms as well.

Problem 4. Keeping in mind the heuristics behind classical versus quantum free motion, is it true that every spectrally unstable complete manifold must have a "recurrent" geodesic, i.e. one that visits some compact set infinitely often, along a sequence of times tending to infinity?

Problem 5. a) Classify the unitary equivalence classes of spectrally stable complete Riemannian manifolds. b) Does every such class contain a globally symmetric space of non-positive curvature (including the ones that contain flat factors)?

Problem 6. In the presence of a smooth potential V, most the ideas discussed in the talks work for the Schrodinger operator $\Delta + V$ as well. In fact, they provide a "coordinate-free" approach to Lavine's theorem on *repulsive* interactions in quantum mechanics (see Reed and Simon's "Methods of Modern Mathematical Physics", vol.4, p.159). Can Lavine's theorem be improved so that whenever the Newtonian motion $x'' = -\nabla V$ is free (and not just when $\frac{\partial V}{\partial r} \leq 0$), then the quantum motion is also free, i.e. (0.1) holds? Here our manifold is $\mathbb{R}^n$, $n \geq 3$, and we assume the existence of a Lyapunov function in the phase space $\mathbb{R}^n \times \mathbb{R}^n$ for $x'' = -\nabla V$. It is unclear whether such a grand generalization would further one's understanding of the physics underlying the transition from classical to quantum mechanics, but from a purely mathematical standpoint the aesthetic appeal would be undeniable.