CLASS GROUPS OF ALGEBRAIC VARIETIES

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1. Complex Algebraic Varieties

The fundamental object of study in algebraic geometry is the common zeros of polynomial equations: the Zero Set of \( f_1, f_2, \ldots, f_r \in \mathbb{C}[x_1, x_2, \ldots, x_n] \) is given by

\[
Z(f_1, \ldots, f_r) = \{ a \in \mathbb{C}^n : f_i(a) = 0 \ \forall i \}
\]

We say that the zero set \( X = Z(f_i) \) is an affine algebraic variety if the ideal \( (f_1, \ldots, f_r) \subset \mathbb{C}[x_1, \ldots, x_n] \) is a prime (or if the radical \( \sqrt{(f_1, \ldots, f_r)} \) is a prime).

Being defined by a prime ideal has consequences:

• The affine coordinate ring \( A_X = \mathbb{C}[x_1, \ldots, x_n]/(f_i) \) is an integral domain.

• The field of fractions of \( A_X \) is the function field of \( X \).

• Using \( K(X) \), we define \( \dim X = \text{tr.deg.}_\mathbb{C}K(X) \), the transcendence degree of the field extension \( \mathbb{C} \subset K(X) \), the size of a transcendence basis of \( K(X) \) over \( \mathbb{C} \).

• If \( n = \dim X \), then there is an open dense subset \( U \subset X \) which has the structure of an \( n \)-dimensional complex manifold. In particular, \( X \) is equidimensional and irreducible.

Example 1. Two quick examples:

(a) The ideal \( (xy - 1) \subset \mathbb{C}[x, y] \) is prime and defines the affine variety \( X \subset \mathbb{C}^2 \), a smooth complex conic.

(b) The zero set \( Z(xy(x - 1), xy(y - 1)) \subset \mathbb{C}^2 \) consists of the two coordinate axes and the point \( (1, 1) \), so the ideal \( (xy(x - 1), xy(y - 1)) \) is not prime.

Loosely speaking, a complex algebraic variety is obtained by gluing together affine varieties along Zariski open sets with regular functions, much the way a manifold is obtained by gluing together open balls in Euclidean space. This informal definition was experimented with in the 1940s, but has been subsumed by Grothendieck’s scheme language: an (abstract) variety is an integral separated scheme of finite type over an algebraically closed field \( k \).

Example 2. If \( x, y, z \) are homogeneous coordinates for \( \mathbb{P}^2_\mathbb{C} \), it is clear that the open sets \( U_x, U_y, U_z \) cover \( \mathbb{P}^2_\mathbb{C} \), where \( U_f = \{(x, y, z) : f \neq 0 \} \). Each is isomorphic to the...
complex plane $\mathbb{C}^2$, for example $\mathbb{C}^2 \cong U_x$ via the map $(a, b) \mapsto (1, a, b)$, so $\mathbb{P}_\mathbb{C}^2$ is an algebraic variety. Obviously this construction works for any projective space.

Now consider $X \subset \mathbb{P}_\mathbb{C}^2$ defined by the vanishing of the homogeneous $xy - z^2$, i.e. $X = Z(xy - z^2)$. Then via the isomorphisms, $X \cap U_x$ is defined by the prime ideal $(y - z^2)$, $X \cap U_y$ is defined by $x - z^2$ and $X \cap U_z$ is defined by $xy - 1$, so $X$ is a (projective) variety.

**Remark 1.** Algebraic varieties are classified up to birational equivalence by their function fields: two varieties have isomorphic function fields if and only if they contain isomorphic Zariski open subsets.

### 2. Divisors on Algebraic Varieties

Let $X$ be a complex algebraic variety. A prime divisor $V$ on $X$ is a subvariety $V \subset X$ such that $\dim V = \dim X - 1$. The prime divisor $V \subset X$ is Cartier if $V$ is locally defined by a single equation.

**Example 3.** Two examples:

(a) Taking $X = \mathbb{C}$, the prime divisors are simply points $a \in \mathbb{C}$. Such a divisor is Cartier because it is defined (globally) by the vanishing of $g(x) = x - a$.

(b) Let $X = Z(y^2 - (x^3 + x^2)) \subset \mathbb{C}^2$. The variety $X$ is not smooth, it has a node at the origin. In this case the prime divisors are still points, but the node is not Cartier. You might think to define the origin by using the equation $x = 0$, but this gives $x = y^2 = 0$, which defines the origin with multiplicity two. We require that the equation cut out $V$ on $X$ exactly, without multiplicity.

**Remark 2.** When $X$ is smooth, every prime divisor is Cartier.

**Remark 3.** It is not so easy to tell when a prime divisor is Cartier along the singular locus, especially in cases where one has an equation that works set-theoretically, but not ideal-theoretically (such as Example 3 (b) above). This usually requires commutative algebra rather than geometry as a tool.

**Example 4** We would like our divisors to be closed under deformation, but this is not true without some adjustment:

(a) The family of curves $X_t \subset \mathbb{P}^2$ given by $xy - z^2t = 0$ deforms the smooth conic ($t = 1$) to the union of two lines $xy = 0$ ($t = 0$), which is not a prime divisor because the ideal $(xy)$ is not prime. One can see that the limit is not irreducible because it consists of two lines.

(b) The family $X_t$ given by $xyt - z^2 = 0$ deforms the smooth conic to the double line $z^2 = 0$, i.e. the line $z = 0$ counted with multiplicity two.

Thus we could like to include unions of prime divisors and count them with multiplicities, which leads to the following groups:

**Definition.** Let $X$ be an algebraic variety. A Weil divisor on $X$ is an element of the free abelian group $\text{Div} X$ generated by prime divisors. The divisor $D = \sum n_i V_i$ is
effective if \( n_i \geq 0 \).

The group \( \text{Div}X \) is not very interesting, it doesn’t capture any geometry of the variety \( X \). It becomes interesting after we mod out be a certain equivalent relation.

3. Linear equivalence and the class group

For a meromorphic function \( f \) on a complex curve, one often considers the divisor associated to \( f \) given by the zeros of \( f \) minus the poles of \( f \). We can make an analogous definition here. For a variety \( X \) and \( 0 \neq f \in K(X) \), \( f = \frac{g}{h} \) locally, which gives two effective Cartier divisors \( Z(g) \) and \( Z(h) \) on \( X \) and we define

\[
(f)_0 = Z(g) - Z(h) \in \text{Div}X,
\]

the principal divisor associated to \( f \). It is clear that the principal divisors form a subgroup \( \text{Prin}X \subset \text{Div}X \).

**Definition.** Two divisors \( D, E \in \text{Div}X \) are linearly equivalent if \( D - E = (f)_0 \) for some \( 0 \neq f \in K(X) \).

**Example 5.** Let \( X \subset \mathbb{P}^3 \) be defined by the equation \( xy - z^2 \). Geometrically \( X \) is the cone over the smooth conic curve \( xy - z^2 = 0 \) in the plane \( w = 0 \) with vertex \( p = (0,0,0,1) \). An easy way to produce linearly equivalent divisors on \( X \) is to intersect \( X \) with various planes \( H \subset \mathbb{P}^3 \).

(a) Let \( H_1 \) be the plane \( w = 0 \). Then \( D_1 = H_1 \cap X \) is the smooth plane conic over which \( X \) is the cone. It is defined by the equation \( w = 0 \) on \( X \).

(b) Let \( H_2 \) be the plane \( z = 0 \). Then \( D_2 = H_2 \cap X \) is the union of two lines \( x = z = 0 \) and \( y = z = 0 \) and is defined by the equation \( z = 0 \) on \( X \).

(c) Let \( H_3 \) be the plane \( x = 0 \). Then \( D_3 = H_3 \cap X \) is a Cartier divisor defined by \( x = 0 \) on \( X \), and consists of a doubling of the ruling \( x = z = 0 \) on \( X \) (technically it is given by \( x = z^2 = 0 \)).

It is clear that each of the divisors \( D_i \) are pairwise linearly equivalent, for example \( D_1 - D_2 = (w)_0 \).

**Remark 4.** In general, if \( X \subset \mathbb{P}^n \) is a projective variety, the hyperplane sections \( H \cap X \) yield linearly equivalent divisors on \( X \). When \( n = 2 \), the hyperplanes \( H \subset \mathbb{P}^2 \) are actually LINES, hence the term “linear equivalence”.

**Definition.** For an algebraic variety \( X \), the class group is \( \text{Cl}X = \frac{\text{Div}X}{\text{Prin}X} \).

**Definition** The Cartier class group of \( X \) is defined as follows. A Cartier divisor is a global section of the sheaf \( \mathcal{K}^*/\mathcal{O}^* \), where \( \mathcal{K} \) is the sheaf locally given by the function field on \( X \) and \( \mathcal{O} \) is the sheaf of regular functions on \( X \). The principal divisors are the image of the map \( H^0(\mathcal{K}^*) \to H^0(\mathcal{K}^*/\mathcal{O}^*) \) and \( \text{CaCl}X \) is the corresponding quotient group.
**Remark 5.** For a variety $X$, we can compare various groups:

(1) In general, the group $\text{CaCl}X$ is isomorphic to the Picard group $\text{Pic}X$ of isomorphism classes of line bundles on $X$ (with tensor product as group operation).

(2) If $X$ is a locally factorial variety (meaning that the local ring at each point is a unique factorization domain), then there is an isomorphism $\text{Cl}X \cong \text{CaCl}X$. In particular, this holds if $X$ is smooth.

(3) The Picard group need NOT be generated by the Cartier prime divisors, so the definition of $\text{Pic}X$ I gave on March 25th was not correct.

**Example 6.** $\text{Cl } \mathbb{C}^n = 0$. Indeed, if $V_1, V_2 \subset \mathbb{C}^n$ are Cartier prime divisors, they are given by the vanishing of respective equations $f_1 = 0$ and $f_2 = 0$ on $\mathbb{C}^n$, hence $V_1 - V_2 = (\frac{f_1}{f_2})_0$.

**Example 7.** $\text{Cl } \mathbb{P}^n \cong \mathbb{Z}$, generated by a hyperplane $H \subset \mathbb{P}^n$. The point here is that each prime divisor $V \subset \mathbb{P}^n$ is defined by the vanishing of a single homogeneous polynomial $f$ of some degree $d$ and we can define $\deg V = \deg f$. Extending by linearity gives a surjective group homomorphism $\deg : \text{Cl } \mathbb{P}^n \to \mathbb{Z}$. If $\deg(D) = 0$, then we can write $D = D_1 - D_2$ as a difference of effective divisors of the same degree $d$. Since $D_i$ is effective, it is given by the vanishing of a polynomial $f_i$ of degree $d$ (the prime divisors with multiplicities correspond to irreducible factors of $f_i$ with appropriate powers), when $D = (\frac{f_1}{f_2})_0 \in \text{Prin}X$, so the kernel consists of principal divisors. Note that $\frac{f_1}{f_2}$ really is a well-defined rational function on $\mathbb{P}^n$ because both $f_i$ are homogeneous of the same degree.