CLASS GROUPS OF ALGEBRAIC VARIETIES

Scott Nollet

1. Complex Algebraic Varieties

The fundamental object of study in algebraic geometry is are the common zeros of polynomial equations: the **Zero Set** of $f_1, f_2, \dots f_r \in \mathbb{C}[x_1, x_2, \dots, x_n]$ is given by

$$Z(f_1, \dots, f_r) = \{a \in \mathbb{C}^n : f_i(a) = 0 \ \ \forall i\}$$

We say that the zero set $X = Z(f_i)$ is an **affine algebraic variety** if the ideal $(f_1, \dots, f_r) \subset \mathbb{C}[x_1, \dots, x_n]$ is a *prime* (or if the radical $\sqrt{(f_1, \dots, f_r)}$ is a prime).

Being defined by a *prime* ideal has consequences:
• The **affine coordinate ring** $A_X = \mathbb{C}[x_1, \dots, x_n]/(f_i)$ is an integral domain.

• The field of fractions of A_X is the **function field of** X.

• Using $K(X)$, we define $\dim X = tr.deg._{\mathbb{C}} K(X)$, the transcendence degree of the field extension $\mathbb{C} \subset K(X)$, the size of a transcendence basis of $K(X)$ over $\mathbb{C}$.

• If $n = \dim X$, then there is an open dense subset $U \subset X$ which has the structure of an n-dimension complex manifold. In particular, X is equidimensional and irreducible.

Example 1. Two quick examples:
(a) The ideal $(xy - 1) \subset \mathbb{C}[x, y]$ is prime and defines the affine variety $X \subset \mathbb{C}^2$, a smooth complex conic.
(b) The zero set $Z(xy(x-1), xy(y-1)) \subset \mathbb{C}^2$ consists of the two coordinate axes and the point $(1, 1)$, so the ideal $(xy(x-1), xy(y-1))$ is not prime.

Loosely speaking, a **complex algebraic variety** is obtained by gluing together affine varieties along Zariski open sets with regular functions, much the way a manifold is obtained by gluing together open balls in Euclidean space. This informal definition was experimented with in the 1940s, but has been subsumed by Grothendieck's scheme language: an (abstract) variety is an integral separated scheme of finite type over an algebraically closed field k.

Example 2. If x, y, z are homogeneous coordinates for $\mathbb{P}^2_{\mathbb{C}}$, it is clear that the open sets U_x, U_y, U_z cover $\mathbb{P}^2_{\mathbb{C}}$, where $U_f = \{(x, y, z) : f \neq 0\}$. Each is isomorphic to the

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complex plane $\mathbb{C}^2$, for example $\mathbb{C}^2 \cong U_x$ via tha map $(a,b) \mapsto (1,a,b)$, so $\mathbb{P}^2_{\mathbb{C}}$ is an algebraic variety. Obviously this construction works for any projective space.

Now consider $X \subset \mathbb{P}^2_{\mathbb{C}}$ defined by the vanishing of the homogeneous $xy - z^2$, i.e. $X = Z(xy - z^2)$. Then via the isomorphisms, $X \cap U_x$ is defined by the prime ideal $(y - z^2)$, $X \cap U_y$ is defined by $x - z^2$ and $X \cap U_z$ is defined by $xy - 1$, so X is a (projective) variety.

Remark 1. *Algebraic varieties are classified up to birational equivalence by their function fields: two varieties have isomorphic function fields if and only if they contain isomorphic Zariski open subsets.*

2. Divisors on Algebraic Varieties

Let X be a complex algebraic variety. A **prime divisor** V on X is a subvariety $V \subset X$ such that $\dim V = \dim X - 1$. The prime divisor $V \subset X$ is **Cartier** if V is locally defined by a single equation X.

Example 3. Two examples:

(a) Taking $X = \mathbb{C}$, the prime divisors are simply points $a \in \mathbb{C}$. Such a divisor is Cartier because it is defined (globally) by the vanishing of $g(x) = x - a$.

(b) Let $X = Z(y^2 - (x^3 + x^2)) \subset \mathbb{C}^2$. The variety X is not smooth, it has a node at the origin. In this case the prime divisors are still points, but the node is **not** Cartier. You might think to define the origin by using the equation $x = 0$, but this gives $x = y^2 = 0$, which defines the origin with multiplicity two. We require that the equation cut out V on X exactly, without multiplicity.

Remark 2. *When X is smooth, every prime divisor is Cartier.*

Remark 3. *It is not so easy to tell when a prime divisor is Cartier along the singular locus, especially in cases where one has an equation that works set-theoretically, but not ideal-theoretically (such as Example 3 (b) above). This usually requires commutative algebra rather than geometry as a tool.*

Example 4 We would like our divisors to be closed under deformation, but this is not true without some adjustment:

(a) The family of curves $X_t \subset \mathbb{P}^2$ given by $xy - z^2 t = 0$ deforms the smooth conic $(t = 1)$ to the union of two lines $xy = 0$ $(t = 0)$, which is not a prime divisor because the ideal (xy) is not prime. One can see that the limit is not irreducible because it consists of two lines.

(b) The family X_t given by $xyt - z^2 = 0$ deforms the smooth conic to the double line $z^2 = 0$, i.e. the line $z = 0$ counted with multiplicity two.

Thus we could like to include unions of prime divisors and count them with multiplicities, which leads to the following groups:

Definition. Let X be an algebraic variety. A **Weil divisor** on X is an element of the free abelian group $\mathrm{Div} X$ generated by prime divisors. The divisor $D = \sum n_i V_i$ is

effective if $n_i \geq 0$.

The group Div$X$ is not very interesting, it doesn't capture any geometry of the variety X. It becomes interesting after we mod out be a certain equivalent relation.

3. Linear equivalence and the class group

For a meromorphic function f on a complex curve, one often considers the divisor associated to f given by the zeros of f minus the poles of f. We can make an analogous definition here. For a variety X and $0 \neq f \in K(X)$, $f = \frac{g}{h}$ locally, which gives two effective Cartier divisors $Z(g)$ and $Z(h)$ on X and we define

$$(f)_0 = Z(g) - Z(h) \in \mathrm{Div}X,$$

the **principal divisor** associated to f. It is clear that the principal divisors form a subgroup Prin$X \subset$ Div$X$.

Definition. Two divisors $D, E \in \mathrm{Div}X$ are **linearly equivalent** if $D - E = (f)_0$ for some $0 \neq f \in K(X)$.

Example 5. Let $X \subset \mathbb{P}^3$ be defined by the equation $xy - z^2$. Geometrically X is the cone over the smooth conic curve $xy - z^2 = 0$ in the plane $w = 0$ with vertex $p = (0, 0, 0, 1)$. An easy way to produce linearly equivalent divisors on X is to intersect X with various planes $H \subset \mathbb{P}^3$.

(a) Let H_1 be the plane $w = 0$. Then $D_1 = H_1 \cap X$ is the smooth plane conic over which X is the cone. It is defined by the equation $w = 0$ on X.

(b) Let H_2 be the plane $z = 0$. Then $D_2 = H_2 \cap X$ is the union of two lines $x = z = 0$ and $y = z = 0$ and is defined by the equation $z = 0$ on X.

(c) Let H_3 be the plane $x = 0$. Then $D_3 = H_3 \cap X$ is a Cartier divisor defined by $x = 0$ on X, and consists of a doubling of the ruling $x = z = 0$ on X (technically it is given by $x = z^2 = 0$).

It is clear that each of the divisors D_i are pairwise linearly equivalent, for example $D_1 - D_2 = (\frac{w}{z})_0$.

Remark 4. *In general, if $X \subset \mathbb{P}^n$ is a projective variety, the hyperplane sections $H \cap X$ yield linearly equivalent divisors on X. When $n = 2$, the hyperplanes $H \subset \mathbb{P}^2$ are actually LINES, hence the term "linear equivalence".*

Definition. For an algebraic variety X, the **class group** is $\mathrm{Cl}X = \dfrac{\mathrm{Div}X}{\mathrm{Prin}X}$.

Definition The Cartier class group of X is defined as follows. A *Cartier divisor* is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$, where $\mathcal{K}$ is the sheaf locally given by the function field on X and $\mathcal{O}$ is the sheaf of regular functions on X. The principal divisors are the image of the map $H^0(\mathcal{K}^*) \to H^0(\mathcal{K}^*/\mathcal{O}^*)$ and CaClX is the corresponding quotient group.

Remark 5. *For a variety X, we can compare various groups:*

(1) In general, the group $\mathrm{CaCl}X$ *is isomorphic to the Picard group* $\mathrm{Pic}X$ *of isomorphism classes of line bundles on X (with tensor product as group operation).*

(2) If X is a locally factorial variety (meaning that the local ring at each point is a unique factorization domain), then there is an isomorphism $\mathrm{Cl}X \cong \mathrm{CaCl}X$*. In particular, this holds if X is smooth.*

(3) The Picard group need NOT be generated by the Cartier prime divisors, so the definition of $\mathrm{Pic}X$ *I gave on March 25th was not correct.*

Example 6. $\mathrm{Cl}\ \mathbb{C}^n = 0$. Indeed, if $V_1, V_2 \subset \mathbb{C}^n$ are Cartier prime divisors, they are given by the vanishing of respective equations $f_1 = 0$ and $f_2 = 0$ on $\mathbb{C}^n$, hence $V_1 - V_2 = (\frac{f_1}{f_2})_0$.

Example 7. $\mathrm{Cl}\ \mathbb{P}^n \cong \mathbb{Z}$, generated by a hyperplane $H \subset \mathbb{P}^n$. The point here is that each prime divisor $V \subset \mathbb{P}^n$ is defined by the vanishing of a single homogeneous polynomial f of some degree d and we can define $\deg V = \deg f$. Extending by linearity gives a surjective group homomorphism $\deg : \mathrm{Cl}\ \mathbb{P}^n \to \mathbb{Z}$. If $\deg(D) = 0$, then we can write $D = D_1 - D_2$ as a difference of effective divisors of the same degree d. Since D_i is effective, it is given by the vanishing of a polynomial f_i of degree d (the prime divisors with multiplicities correspond to irreducible factors of f_i with appropriate powers), when $D = (\frac{f_1}{f_2})_0 \in \mathrm{Prin}X$, so the kernel consists of principal divisors. Note that $\frac{f_1}{f_2}$ really is a well-defined rational function on $\mathbb{P}^n$ because both f_i are homogeneous of the same degree.