THE DICTIONARY BETWEEN COMMUTATIVE AND NONCOMMUTATIVE GEOMETRY AND TOPOLOGY

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1. C* algebras

Let $X$ be a compact Hausdorff space. Let $C(X)$ be a $\mathbb{C}$-algebra of continuous $\mathbb{C}$-valued functions on $X$. We have the involution $f^*(x) = \overline{f(x)}$ and the norm $\|f\| = \sup \{|f(x)| : x \in X\}$. Also, $C(X)$ is a Banach $*$-algebra (normed complete $*$-algebra).

There is a contravariant functor $C$ from the category of compact, Hausdorff spaces and continuous functions to Banach $*$-algebras and $*$-homomorphisms. A $C^*$-algebra $A$ is a Banach $*$-algebra where $\|a^*a\| = \|a\|^2$ for all $a \in A$. The space $C(X)$ is a commutative $C^*$-algebra.

**Theorem 1.1.** (Gelfand-Naimark) Every commutative $C^*$-algebra with unit is $*$-isomorphic to $C(X)$ for some compact Hausdorff space $X$.

**Theorem 1.2.** Every closed $*$-ideal of $C(X)$ uniquely has the form $C_0(X \setminus A) := \{f \in C(X) : f(a) = 0 \text{ for all } a \in A\}$ for some unique closed subset $A$.

**Corollary 1.3.** The maximal ideals in $C(X)$ can be identified with points of $X$.

**Corollary 1.4.** The space $X$ can be recovered from $C(X)$.

**Theorem 1.5.** The functor $C$ determines a category equivalence between compact Hausdorff spaces and commutative $C^*$-algebras.

In theory, we could do topology by working with $C^*$-algebras, but in practice this usually does not work well. One good example is as follows. $\{\mathbb{C}$-vector bundles over $X\}$ corresponds to $\{\text{finitely generated projective modules over } C(X)\}$ (Serre-Swan Theorem).

"Non-commutative topology" can be viewed as the study of general unital $C^*$-algebras — ie noncommutative ones. Why can’t you learn more topology from the noncommutative side? There are many maps between topological spaces. However, the $C^*$ condition is very strong, and there is a lot of rigidity: not many $*$-homomorphisms in the noncommutative case.

A more modern idea (Connes): study "bad" topological spaces(i.e. nonHausdorff), by replacing them with "good", but noncommutative, $C^*$-algebras.

**Examples:**

1. Orbit space of a (not necessarily compact) Lie group acting on a compact manifold.
2. Leaf space of a foliation.
3. Space of irreducible representations of a discrete or Lie group on a Hilbert space.
The $C^*$-algebra for a group $G$ acting on a compact manifold $M$ via $\alpha : G \to \text{Aut}(M)$ is $C_c(G, C(M)) = \{\text{continuous fcn}s \phi : G \to C(M) \text{ with compact support}\}$ with convolution product

$$(\phi * \psi)(g) = \int \phi(h) \alpha_h (\psi(h^{-1}g)) \, dh$$

with pointwise addition. We complete $C_c(G, C(M))$ to a $C^*$-algebra.

If $\Gamma$ is a discrete group, $\mathbb{C}\Gamma$ acts on $\ell^2\Gamma$, i.e. $\mathbb{C}\Gamma \subseteq \ell(\ell^2\Gamma)$. Then $C^*_r(\Gamma) = \text{closure of } \mathbb{C}\Gamma \text{ in } \ell(\ell^2\Gamma)$ is known as the reduced group $C^*$-algebra. The simplest case is $C^*_r(\mathbb{Z}) \cong C(\mathbb{T})$ via the Fourier series.

2. Cyclic Homology

Let $A$ be a $\mathbb{C}$-algebra with unit. Let

$$C^\lambda_n(A) = \bigotimes_{n+1} A/ \sim,$$

where

$$a_n \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} \sim (-1)^n a_0 \otimes a_1 \otimes \ldots \otimes a_n.$$

We have the boundary map

$$b : C^\lambda_n(A) \to C^\lambda_{n-1}(A)$$

$$b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 a_1 \otimes \ldots \otimes a_n$$

$$+ \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$$

$$+ (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_n.$$

Then $b^2 = 0$, and the cyclic homology of $A$ is defined to be $H^\lambda_*(A) = \text{homology of } (C^\lambda_*(A), b)$.

We also write elements $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ as noncommutative differential forms

$$a_0 da_1 da_2 \ldots da_n.$$

When $A = C^\infty(M)$, this produces isomorphisms

$$H^\lambda_{2n}(C^\infty(M)) \cong H^\text{even}_{dR}(M; \mathbb{C})$$

$$H^\lambda_{2n+1}(C^\infty(M)) \cong H^\text{odd}_{dR}(M; \mathbb{C})$$

for $n$ sufficiently large. So cyclic homology is a way of making sense of differential forms when you don’t have a smooth manifold. More precisely,

$$H^\lambda_k(C^\infty(M)) = \Omega^k(M) / d(\Omega^{k-1}(M)) \oplus H^k_{dR}(M) \oplus H^{k-2}_{dR}(M) \oplus \ldots$$

Other ways of getting cyclic homology are as follows. Question: where do elements of $H^\lambda_*(A)$ come from? Answer: $K$-theory. Let $e$ (determines class in $K_0(A)$) be an idempotent in $M(m, A)$, then

$$\text{Tr} \left( e (de)^n \right) \in H^\lambda_n(A)$$

for $n$ even. Let $u$ (determines class in $K_1(A)$) be an element of $GL(m, A)$. Then

$$\text{Tr} \left( (u^{-1}du)^n \right) \in H^\lambda_n(A)$$
for $n$ odd. Think of $e$ as a projection from a trivial bundle to a vector bundle.

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
da_{11} & da_{12} \\
da_{21} & da_{22}
\end{pmatrix}^n.
\]

Next, consider cyclic cohomology. Let $A$ be a topological algebra with unit. Let $C^n(A)$ be the $A$-module of continuous multilinear maps $A^{n+1} \to \mathbb{C}$. Let

\[
b : C^n(A) \to C^{n+1}(A)
\]

\[
(b\phi)(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \phi \left(a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}\right)
\]

\[
\sum_{i=1}^{n-1} (-1)^i \phi \left(a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes \cdots \otimes a_n\right) + (-1)^n \phi \left(a_n a_0 \otimes a_1 \otimes \cdots \otimes a_n\right).
\]

Then $b^2 = 0$, and cyclic cohomology is defined to be

\[
H^*_A(A) = \text{cohomology of } (C^n(A), b).
\]

There is a pairing

\[
H^*_A(A) \times H^*_A(A) \to \mathbb{C}.
\]

Question: where do interesting elements of cyclic cohomology come from? Answer: From Fredholm modules. A **Fredholm module** over $A$ is a triple $(\mathcal{H}, \pi, F)$, where $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space with grading operator $\varepsilon \ (\varepsilon^2 = 1, \mathcal{H}_+ = 1$-eigenspace, $\mathcal{H}_- = (-1)$-eigenspace; $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation of $A$ on $\mathcal{H}$ that respects the grading:

\[
\pi(a) = \begin{pmatrix}
\pi_+(a) & 0 \\
0 & \pi_-(a)
\end{pmatrix}
\]

$F \in \mathcal{B}(\mathcal{H})$, $F^2 - 1 \in \mathcal{K}(\mathcal{H})$, $F\pi(a) - \pi(a)F \in \mathcal{K}(\mathcal{H})$ for all $a \in A$, $\varepsilon F = -F \varepsilon$,

\[
F = \begin{pmatrix}
0 & * \\
* & 0
\end{pmatrix}.
\]

If $F\pi(a) - \pi(a)F \in \Lambda^p(\mathcal{H})$ (ie $p^{th}$ power is trace class, $p \geq 1$) for all $a \in A$, we say $(\mathcal{H}, \pi, F)$ is $p$-summable. If $F\pi(a) - \pi(a)F \in \Lambda^p(\mathcal{H})$ for all $a \in A \subseteq A$ for a dense subset, we say $(\mathcal{H}, \pi, F)$ is essentially $p$-summable.

Prototypical example: $A = C(M)$, $M$ smooth compact manifold, $\mathcal{H} = L^2(M, E)$, with $E$ a $\mathbb{Z}_2$-graded Hermitian vector bundle over $M$, and $A$ acts on $\mathcal{H}$ by pointwise multiplication. Then $D$ is an elliptic (pseudo)differential operator on $E$ of the form $\begin{pmatrix}
0 & * \\
* & 0
\end{pmatrix}$. On $\mathbb{T}^2$,

\[
D = \begin{pmatrix}
0 & \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \\
-\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} & 0
\end{pmatrix}
\]

is an example. Let $F$ = positive spectral projection of $D$ if $D$ is essentially self-adjoint, or

\[
F = D \left(1 + D^2\right)^{-1/2},
\]

so that $F^2 - I \in \mathcal{K}$. For example,

\[
F \left(\sum_{n \in \mathbb{Z}} a_n e^{in\theta}\right) = \sum_{n \geq 0} a_n e^{in\theta}.
\]

Note that $(\mathcal{H}, \pi, F)$ is essentially $p$-summable for $p > \dim M$. 
3. **Answer to Igor’s Question**

Let \( A = C^\infty (M) \), \( M \) a smooth compact manifold. Consider the double complex:

\[
\begin{array}{cccc}
B & \downarrow^b & A \otimes A \otimes A & \downarrow^b \\
\downarrow^b & & \downarrow^b & \\
A \otimes A & \downarrow^b & A & \\
\downarrow^b & & & \\
A & & & \\
\end{array}
\]

Let

\[
B(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} \left[ (-1)^{ni} (1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}) - (-1)^{n(i-1)} (a_{i-1} \otimes 1 \otimes a_i \otimes \ldots \otimes a_{i-2}) \right]
\]

Then \( B^2 = 0 \), \( Bb + bB = 0 \), where

\[
b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0a_1 \otimes \ldots \otimes a_n
\]

\[
+ \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes \ldots \otimes a_n
\]

\[
+ (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_n.
\]

This complex is called \( B(A) \), and \( \text{Tot}(B(A)) \) is the complex obtained by taking direct sums on the diagonal. You can do the same thing with the Cech-de Rham complex.

**Theorem 3.1.** \( H_\ast (\text{Tot}(B(A))) \cong H_\ast^A(A) \).

The truncated de Rham complex is

\[
\begin{array}{cccc}
\downarrow^d & \downarrow^0 & \downarrow^0 & \\
\Omega^2(M) & \Omega^1(M) & \Omega^0(M) & \\
\downarrow^0 & \downarrow^0 & \downarrow^0 & \\
\Omega^1(M) & \Omega^0(M) & \\
\downarrow^0 & \\
\Omega^0(M) & \\
\end{array}
\]

One can check that \( d^2 = 0 \), \( 0^2 = 0 \), \( 0d + d0 = 0 \). Call this complex \( D(M) \).

**Theorem 3.2.** \( H^\ast (\text{Tot}(D(M))) \cong H^\ast_{dR}(M) \).

Define \( \pi_n : \bigotimes_{n+1} A \to \Omega^n(M) \) by

\[
\pi_n(a_0 \otimes \ldots \otimes a_n) = a_0 da_1 \ldots da_n.
\]

Then \( \{ \frac{1}{n!} \pi_n \} \) determines a map from \( B(A) \) to \( D(M) \) that induces an isomorphism.

4. **More Fun with Fredholm Modules**

Recall: a Fredholm module over a unital \( C \)-algebra \( A \) is a triple \((\mathcal{H}, \pi, F)\), where

- \( \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \) is a \( \mathbb{Z}_2 \)-graded Hilbert space with grading operator \( \varepsilon \) (\( \varepsilon^2 = 1 \)).
• \( \pi : A \to B(\mathcal{H}) \) is a representation of \( A \) on \( \mathcal{H} \) and respects the grading, i.e.

\[
\pi(a) = \begin{pmatrix}
\pi_+(a) & 0 \\
0 & \pi_-(a)
\end{pmatrix}
\]

• \( F \in B(\mathcal{H}) \), \( F^2 - I \) is compact, \( F \) reverses the grading, and \([F, \pi(a)]\) is compact for each \( a \in A \).

\[
F = \begin{pmatrix}
0 & P \\
Q & 0
\end{pmatrix}
\]

Note that compact means a (operator norm) limit of finite rank operators.

(Think: \( F = D(1 + D^2)^{-1/2} \), \( D \) Dirac operator, \( A = C^\infty(M) \)). If \([F, \pi(a)]\) is trace class, we say \((\mathcal{H}, \pi, F)\) is 1-summable. If this condition only holds for a dense subalgebra \( \mathcal{A} \) of \( A \), we say that this module is essentially 1-summable.

The character of an essentially 1-summable Fredholm module over \( A \) is

\[
\rho(a) = \frac{1}{2} \text{Trace} (\varepsilon F \{F, \pi(a)\}).
\]

This \( \rho \) determines an element of \( H^1_\lambda(A) \). Important commutative diagram:

\[
\begin{array}{ccc}
Fred(A) & \times & K_* (A) \\
\downarrow \text{ch} & & \downarrow \text{ch} \\
H_*^\lambda(A) & \times & H_*^\lambda(A) \to \mathbb{C}
\end{array}
\]

Picking a Fredholm module is akin to choosing a Riemannian structure.

Application: Let \( \Gamma \) be a discrete group, and let \( \mathbb{C} \Gamma \) be the complex group algebra. Let \( \mathbb{C} \Gamma \subseteq B(\ell^2(\Gamma)) \) be the left regular representation. Then

\[
\mathbb{C} \Gamma = \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma : a_\gamma \in \mathbb{C} \right\}.
\]

Then

\[
a_\gamma \gamma : \ell^2(\Gamma) \to \ell^2(\Gamma)
\]

is defined by

\[
a_\gamma \gamma (\delta_\alpha) = a_\gamma \delta_{\gamma \alpha}
\]

The norm closure of \( \mathbb{C} \Gamma \) in \( B(\ell^2(\Gamma)) \) is called the reduced \( C^* \)-algebra \( C^*_r(\Gamma) \) of \( \Gamma \).

Noncommutative connectivity conjecture:

**Conjecture 4.1.** (Bass Idempotent Conjecture): If \( \Gamma \) is torsion-free, then \( \mathbb{C} \Gamma \) has no nontrivial idempotents (i.e. \( e \neq 0,1 \)).

**Conjecture 4.2.** (Kadison Conjecture): If \( \Gamma \) is torsion-free, then \( C^*_r(\Gamma) \) has no nontrivial idempotents.

(Note Baum-Connes Conjecture implies both of these and the Borel Conjecture and ... )
5. A proof of Kadison’s conjecture for $F_2$

Let $F_2$ be the free group on two generators.
Let $\mathbb{C}F_2 \subseteq C^*_r(F_2) \subset \mathcal{B}(\ell^2(F_2))$ - reduced group $C^*$-algebra.

Here,
\[
\sum_{\gamma \in F_2} a_{\gamma} \gamma : \ell^2(F_2) \to \ell^2(F_2)
\]
\[
\left( \sum_{\gamma \in F_2} a_{\gamma} \right)(\delta_{\alpha}) = \sum_{\gamma \in F_2} a_{\gamma} \gamma \alpha,
\]
where
\[
\langle \delta_{\alpha}, \delta_{\beta} \rangle = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}
\]

**Theorem 5.1.** (Kadison Conjecture): $C^*_r(F_2)$ contains no nontrivial idempotents.

**Definition 5.2.** Let $\tau : A \to \mathbb{C}$ be a trace on a $C^*$-algebra $A$ ($\tau(ab) = \tau(ba)$). We say $\tau$ is
- **positive** if $\tau(a^*a) \geq 0$ for all $a \in A$.
- **faithful** if $\tau(a^*a) = 0$ iff $a = 0$.

**Example 5.3.** The function $\tau : \mathbb{C}F_2 \to \mathbb{C}$ defined by
\[
\tau \left( \sum_{\gamma \in F_2} a_{\gamma} \right) = a_1
\]
extends to a positive faithful trace on $C^*_r(F_2)$.

**Theorem 5.4.** Let $A$ be a $C^*$-algebra that admits a positive faithful trace $\tau$ such that $\tau(1) = 1$. Let $(\mathcal{H}, \pi, F)$ be an essentially 1-summable Fredholm module on $A$. Let
\[
\mathcal{A} = \{ a \in A : F\pi(a) - \pi(a) F \in L^1(\mathcal{H}) \}.
\]
(Then $\mathcal{A}$ is a dense subalgebra of $A$.) Suppose the character $\rho$ on $(\mathcal{H}, \pi, F)$ agrees with $\tau$ on $\mathcal{A}$. Then there is no nontrivial idempotent on $\mathcal{A}$.

Note that a character $\rho : \mathcal{A} \to \mathbb{C}$ is $\rho(a) = \frac{1}{2} \text{Trace}(\varepsilon F(F\pi(a) - \pi(a) F))$ (Hilbert space trace).

(Sketch) The inclusion $\mathcal{A} \hookrightarrow A$ induces an isomorphism:
\[
K_0(\mathcal{A}) \to K_0(A).
\]
(reason: $\mathcal{A}$ is closed under the holomorphic functional calculus, i.e. if $a \in \mathcal{A}$ and $f$ is holomorphic in an open domain containing the spectrum of $A$, then
\[
f(a) := \int_C \frac{f(z)}{a - z} dz \in \mathcal{A}.
\]
Therefore, we may assume an idempotent $e$ in $A$ actually lives in $\mathcal{A}$. By $K$-theory nonsense, we may assume also that $e^* = e$.

From our commutative diagram,
\[
\begin{array}{ccc}
Fred(\mathcal{A}) & \times & K_*(\mathcal{A}) \\
\downarrow & & \downarrow \text{index} \\
H^*_\lambda(\mathcal{A}) & \times & H^\lambda(\mathcal{A}) \\
\end{array} \to \mathbb{C}
\]
By hypothesis, we see that \( \tau(e) = \rho(e) \in \mathbb{Z} \). We also know that
\[
\tau(e) = \tau(e^*e) \geq 0
\]
because \( \tau \) is positive. But \( 1 - e \) is also a self-adjoint idempotent,
\[
\tau(1 - e) \geq 0, \\
1 - \tau(e) \geq 0
\]
so \( \tau(e) \leq 1 \). If \( \tau(e) = 0 \), then \( \tau(e^*e) = 0 \) so \( \tau(e) = 0 \) by faithfulness.
If \( \tau(e) = 1 \), then \( \tau((1 - e^*)(1 - e)) = 0 \), and \( 1 - e = 0, e = 1 \).