MODULI SPACES OF CURVES

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Abstract. My goal is to introduce vocabulary and present examples that will help graduate students to better follow lectures at TAGS 2018. Assuming some background in modern algebraic geometry (varieties, schemes and cohomology), I will discuss linear systems on curves, Hilbert schemes, and the Mumford-Deligne moduli space of isomorphism classes of curves of fixed genus.

1. Linear systems on curves

In this section we establish the curves to be studied, the notion of linear system, and the Riemann-Roch theorem and some of its applications. We work throughout over a fixed algebraically closed field k of arbitrary characterstic.

1.1. **Curves.** We first clarify our main object of study:

Definition 1.1. An <u>embedded curve</u> $C \subset \mathbb{P}^n_k$ is a nonsingular one dimensional closed subvariety [4, I,§5]. Letting $P_C(t) = dt + a$ be the Hilbert polynomial, the <u>geometric genus</u> of C is $g = 1 - a$ and the <u>degree</u> is $d = \deg C$.

An <u>abstract curve</u> C is a one dimensional nonsingular complete variety of finite type over k [4, IV,§1]. Letting ω_C be the sheaf of differentials[1] [4, II,§8], the <u>geometric genus</u> of C is $g = \dim H^0(\omega_C) = \dim H^1(\mathcal{O}_C)$. If $k = \mathbb{C}$, then g is the number of handles on the corresponding real surface.

Remark 1.2. An embedded curve $C \subset \mathbb{P}^n$ yields $C = \operatorname{Proj} k[x_0, \ldots, x_n]/I_C$ as an abstract curve, but the embedded curve has extra data such as the ideal I_C and the degree $\deg C$.

Remark 1.3. One can also define an abstract curve as the set of all discrete valuation rings $k \subset R \subset K$, where $k \subset K$ has transcendence degree one [4, I,§6].

Example 1.4. A general hypersurface $C \subset \mathbb{P}^2$ of degree d is a curve of genus $g = \binom{d-1}{2}$.

Example 1.5. [4, III,Exercise 5.6] For $a, b > 0$, general effective divisors on the nonsingular quadric $Q \subset \mathbb{P}^3$ of type $\mathcal{O}_Q(a, b)$ are curves C with $g = (a-1)(b-1)$ and $\deg C = a+b$. Taking $a = 2, b \geq 1$ give curves of every genus $g \geq 0$.

1.2. **Linear systems.** Any map $f : C \to \mathbb{P}^n$ gives rise to a line bundle $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ and sections $s_i = f^* x_i$ which generate L in the sense that $\mathcal{O}_C^{n+1} \xrightarrow{s_i} L$ is surjective.

$$\{f : C \to \mathbb{P}^n\} \longleftrightarrow \left\{ \begin{array}{c} L \in \operatorname{Pic} C \\ s_0, \ldots, s_n \in H^0(C, L) \\ s_i \text{ generate } L \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} L \in \operatorname{Pic} C \\ V = \langle s_0, \ldots, s_n \rangle \subset H^0(C, L) \end{array} \right\}$$

[1] Namely $\omega_C = \Delta^*(\mathcal{I}/\mathcal{I}^2)$, where $\mathcal{I}$ is the ideal of Δ in $C \times C$. We use K_C for the associated divisor.

Geometrically a hyperplane H with equation $\sum a_i x_i$ pulls back to the zero set of the section $s = \sum a_i s_i$, denoted $(s)_0$: dualize the injection $s : \mathcal{O}_C \to L$ to get $L^\vee \hookrightarrow \mathcal{O}_C$ as the ideal sheaf of the effective divisor $D = (s)_0 = \sum n_i p_i$ of degree $\deg D = \sum n_i$. These are determined by s up to $\lambda \in k^*$, so the set of all such is given by the projective space

$$\mathbb{P}V = V - \{0\}/\sim .$$

This is a linear system of dimension $r = \dim V - 1$ and degree $d = \deg L$, denoted g^r_d.

Example 1.6. (a) Take $C = \mathbb{P}^1, L = \mathcal{O}_{\mathbb{P}^1}(3)$ and $V = H^0(\mathbb{P}^1, L)$. For homogeneous coordinates s, t on $\mathbb{P}^1$ and $V = \langle s^3, s^2t, st^2, t^3\rangle$, the associated map $\mathbb{P}^1 \to \mathbb{P}^3$ is given by

$$(s,t) \mapsto (x,y,z,w) = (s^3, s^2t, st^2, t^3).$$

The image C has ideal $I_C = (xz - y^2, xw - yz, yw - z^2)$. Modulo $\operatorname{Aut}\mathbb{P}^3$, every curve of degree $d = 3$ and genus $g = 0$ arises from this construction [4, II,Example 7.8.5] and is called a twisted cubic curve.

Two more linear systems are present. Note that $C \subset Q$ is a divisor of type $(2,1)$, so there are two projections to $\mathbb{P}^1$ via the isomorphism $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$:

(b) π_1 is an isomorphism giving a g^1_1. A curve C carries a $g^1_1 \iff C \cong \mathbb{P}^1$

(c) π_2 is a double cover giving a g^1_2.

Exercise 1.7. In Example 1.6 (c), compute $V \subset H^0(\mathcal{O}_{\mathbb{P}^1}(2))$.

Remark 1.8. A curve C carrying a g^1_2 is a double cover of $\mathbb{P}^1$ and is called hyperelliptic.

1.3. **Riemann-Roch.** The Riemann Roch theorem estimates the dimension $l(D)$ of the complete linear system $|D|$ associated to a divisor $D = \sum n_i p_i$. If $H^0(C, \mathcal{O}_C(D)) = 0$, then there are no effective divisors linearly equivalent to D, so we take $l(D) = -1$.

Theorem 1.9. *If D is a divisor on C, then $l(D) - l(K_C - D) = \deg D + 1 - g$.*

Corollary 1.10. $\deg K_C = 2g - 2$. [Take $D = K_C$ so that $l(K_C) = g - 1$ and $l(0) = 0$.]

Corollary 1.11. *If $\deg D > 2g - 2$, then $l(D) = \deg D - g$.* [Because $l(K_C - D) = -1$.]

Remark 1.12. In terms of the map, we observe the following [4, IV,Proposition 3.1].

(a) $P \in C$ is a base point for $|D|$ iff the map $|D - P| \hookrightarrow |D|$ given by $A \mapsto A + P$ is not onto, hence $|D|$ is base point free iff $l(D - P) = l(D) - 1$ for each P.

(b) Similarly $|D|$ separates points and tangent vectors iff $l(D - P - Q) = l(D) - 2$ for each $P, Q \in C$, including the case $P = Q$.

Corollary 1.13. *If $d = \deg D \geq 2g + 1$, then $|D|$ gives a closed immersion $C \hookrightarrow \mathbb{P}^{d-g}$.*

Proof. For $P, Q \in C$, $D, D - P$ and $D - P - Q$ are nonspecial, so apply Corollary 1.11. $\square$

Example 1.14. Three points on an elliptic curve C give a closed immersion $C \hookrightarrow \mathbb{P}^2$.

2. Hilbert schemes

A flat family in $\mathbb{P}^n$ consists of a closed immersion $X \hookrightarrow \mathbb{P}^n \times T$ with composite map $f : X \to T$ flat, meaning that for each $x \in X$ with $t = f(x)$, the map $\mathcal{O}_{T,t} \to f_*\mathcal{O}_{X,x}$ is a flat homomorphism of local rings. The following shows that flatness is the right notion:

Theorem 2.1. *If T is connected and $X \subset \mathbb{P}^n \times T$, then $X \to T$ is flat if and only if the fibers X_t have constant Hilbert polynomial.*

Example 2.2. If $C \subset \mathbb{P}^n$ is a curve and $V \subset H^0(C, L)$ gives the linear system $\mathbb{P}V$, then the total family $D \subset C \times \mathbb{P}V \subset \mathbb{P}^n \times \mathbb{P}V$ with fibers $D_s = (s)_0$ is a flat family.

The amazing thing is that there is a scheme that captures all such flat families at once. Grothendieck showed [1] that all flat families with fixed Hilbert polynomial $p(t) \in \mathbb{Q}[t]$ are captured by one universal flat family in the following theorem.

Theorem 2.3. *There is a scheme $\mathrm{Hilb}^n_{p(m)}$ and flat family (1) $X \subset \mathbb{P}^n \times \mathrm{Hilb}^n_{p(m)}$ such that for each flat family (2) $C \subset \mathbb{P}^n \times T \to T$ with fibers having Hilbert polynomial $p(t)$, (2) arises from (1) as a pullback from a unique map $T \to \mathrm{Hilb}^n_{p(m)}$.*

Remark 2.4. The Hilbert schemes $\mathrm{Hilb}^n_{p(t)}$ have been both objects of study and a useful tool since since their arrival in 1960. A few comments about their nature:

(a) Grothendieck proved that $\mathrm{Hilb}^n_{p(t)}$ is projective over $\operatorname{Spec} \mathbb{Z}$, so every flat family is encoded by a projective scheme with integer coefficients.
(b) Over a field k, Hartshorne showed [3] that $\mathrm{Hilb}^n_{p(t)}$ is connected for each $p(t)$, but examples show it can have many irreducible components.
(c) The universal property implies that $\mathrm{Hilb}^n_{p(t)}$ is unique up to unique isomorphism.
(d) Taking $T = \operatorname{Spec} k$ shows that there is a bijection

$$\{\text{Points of } \mathrm{Hilb}^n_{p(t)}\} \leftrightarrow \{\text{Closed subschemes } X \subset \mathbb{P}^n \text{ with Hilbert polynomial } p(t)\}$$

Exercise 2.5. Prove that $\mathrm{Hilb}^n_1 = \mathbb{P}^n$ with universal family $X = \Delta(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{P}^n$.

Notation 2.6. For degree d and genus g, we write $\mathrm{Hilb}^n_{d,g} = \mathrm{Hilb}^n_{dm+1-g}$.

Example 2.7. The twisted cubic curve $C \subset \mathbb{P}^3$ has ideal $I = (y^2 - xz, xw - yz, yw - z^2)$ generated by the maximal minors of $M = \begin{pmatrix} z & y & x \\ w & z & y \end{pmatrix}$. The Hilbert-Burch theorem gives a free resolution $0 \to S(-3)^2 \xrightarrow{M} S(-2)^3 \to I_C \to 0$. We vary entries of M to give flat families of curves parametrized by $t \in \mathbb{A}^1$ and evaluate the limiting curve as $t \to 0$.

(a) $M_t = \begin{pmatrix} z & y & x \\ w & zt & y \end{pmatrix} \Rightarrow I_{C_0} = (wy, xw - yz, y^2)$, so C_0 is the union of $x = y = 0$ and doubling of $w = y = 0$ on Q.
(b) $M_t = \begin{pmatrix} z & y & xt \\ w & z & y \end{pmatrix} \Rightarrow I_{C_0} = (wy - z^2, zy, y^2)$, a triple line on the quadric cone.
(c) $M_t = \begin{pmatrix} z & y & xt \\ wt & z & y \end{pmatrix} \Rightarrow I_{C_0} = (z^2, zy, y^2) = (x, y)^2$, the thick triple line.
(d) Projection from a point gives a nodal plane cubic with a spatial embedded point at the node [4, III,Ex. 9.8.4] (see also [7]).

Example 2.8. Work of Piene and Schlessinger [7] describes $\mathrm{Hilb}^3_{3,0}$. It has two irreducible components: (a) the 12-dimensional closure of the family of twisted cubic curves and (b) the 15-dimensional family whose general member is the union of a plane cubic and an isolated point. They meet transversely in an irreducible subvariety of dimension 11.

3. The moduli space M_g of curves of genus $g \geq 2$

We construct a scheme M_g whose closed points are in bijective correspondence with isomorphism classes of curves of genus g.

Example 3.1. If $g = 0$, then applying Corollary 1.13 to $D = P \in C$ gives a closed immersion $C \to \mathbb{P}^1$ which must be an isomorphism, so M_0 is a point.

Example 3.2. The j-invariant shows that $M_1 = \mathbb{A}^1$.

In view of Examples 3.1 and 3.2 we assume $g \geq 2$. We need a way to produce all the curves and recognize isomorphism classes. This is provided by the following:

Observations 3.3. Let C be a curve of genus $g \geq 2$.

(a) Fix $n \geq 3$. Then the n-canonical complete linear system $|nK_C|$ gives a closed embedding $C \hookrightarrow \mathbb{P}^N$ by Corollary 1.13, with $N = n(2g-2)-g = (2n-1)(g-1)-1$. The image is an <u>$n$-canonical curve</u> of degree $d = n(2g-2)$ and genus g.

(b) If $j : C \hookrightarrow \mathbb{P}^N$ is an n-canonical embedding and $\Psi \in \operatorname{Aut}\mathbb{P}^N$, then $\Psi \circ j$ is also an n-canonical embedding.

(c) If $\varphi : C' \xrightarrow{\sim} C$ is an isomorphism, then the composite $C' \to C \to \mathbb{P}^N$ is also an n-canonical embedding because $\varphi^* K_C = K_{C'}$. It follows that each isomorphism class of genus g curves appears a single $\operatorname{Aut}\mathbb{P}^N$-orbit.

Taking $G = \operatorname{Aut}\mathbb{P}^N = \mathrm{PGL}(N+1)$, this leads to the following.

Construction 3.4. Let $U \subset \mathrm{Hilb}^N_{n(2g-2),g}$ be the open subset of smooth connected curves and $K \subset U$ the subset of n-canonical curves. Then G acts on K and the set K/G of G-orbits is in bijective correspondence with isomorphism classes of genus g curves.

3.1. **Geometric Invariant theory.** Given a group G acting on a variety X, geometric invariant theory [5, 6] establishes conditions under which the orbit space X/G exists as the points of an algebraic variety. Here are a summary of some results from this area. Below we take G to be a *reductive* group, of which $G = \mathrm{PGL}(N+1)$ is an example.

3.2. **Affine case.** If $X = \operatorname{Spec} A$ is affine, things work out well: the action of G on X induces an action $G \times A \to A$, when we can define $A^G = \{a \in A : ga = a \forall g \in G\}$ and the map $\phi : \operatorname{Spec} A \to Y = \operatorname{Spec} A^G$ does the job. In particular, for any open $U \subset Y$ for which the G-orbits in $\phi^{-1}(U)$ are closed, then $U = \phi^{-1}(U)/G$ is the space of orbits.

3.3. **Projective case.** If $X = \operatorname{Proj} R$ where R is graded, things are more difficult.

3.3.1. *GIT Semistability.* Assuming that G acts on X *linearly* , meaning through the action of the general linear group on R_1, one can again form the G-invariant subring $R^G \subset R$ and the map $\pi : X \to Y = \operatorname{Proj} R^G = X//G$, but typically π will not be defined for all $x \in X$, so we define the semistable locus $X^{ss} \subset X$ to be the open set where it is defined. Concretely $x \in X$ is <u>semistable</u> if there is some $f \in R^G$ with $x \in X_f$, otherwise x is nonstable. Then we get a morphism $X^{ss} \to X//G$.

3.3.2. *GIT stability.* Unfortunately when we use semistable points as above, the morphism might collapse distinct orbits together in $X//G$. To remedy this situation, we define $x \in X^{ss}$ to be <u>stable</u> if each orbit Gx is closed in X^{ss} and $\dim Gx = \dim G$ (in other words, $|\mathrm{Stab}(x)| < \infty$). If $X^s \subset X^{ss}$ is the subset of stable points, then $\pi : X^s \to X//G$ separates the orbits and $X^s \to \pi(X^s)$ is the orbit space we are seeking.

With the notions above, we can almost state the main result here. We need one more definition to make it all work. The condition that $\mathrm{Stab}(x)$ be finite says that the corresponding curve C should have a finite automorphism group. For C a (smooth connected) curve of genus $g \geq 2$, this is always true [4, IV, Exercise 2.5]. For an elliptic curve C, $\operatorname{Aut} C$ is infinite, but the automorphisms fixing any point $P \in C$ is finite [4, IV, Corollary 4.7] and for $C = \mathbb{P}^1$ every automorphism is uniquely determined by the images of three distinct points [4, I, Exercise 6.6]. With this in mind, we describe two types of nodal curves that also have finite automorphism groups, the Deligne-Mumford stable curves:

(a) the curves C such that every smooth rational component meets the other components in at least 3 points.
(b) the curve C such that every rational component of the normalization of C has at least 3 points lying over the singular points of C.

Finally we can state the main result. We have defined $K \subset U \subset \mathrm{Hilb}$ above. Enlarge K to $\tilde{K}$ by adding the curves of types (a) and (b) shown above and let $\tilde{K}^{ss} = \tilde{K} \cap \mathrm{Hilb}^{ss}$.

Theorem 3.5. *Assume $n \geq 5$ in the construction above.*

(1) *$\tilde{K}^{ss}$ is a smooth closed subscheme of* Hilb*.*
(2) *$\tilde{K}^{ss} = \tilde{K}^s$: these curves are already GIT stable.*
(3) *$\tilde{K}^{ss} = \tilde{K}$.*

In particular, $\tilde{K}/G$ is a projective variety containing K/G as a dense open subset.

References

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