1. Introduction to Morse Theory

Let $M$ and $N$ be smooth manifolds, and let $F : M \to N$ be a smooth map. This induces $DF_x : T_x M \to T_{F(x)} N$.

**Definition 1.1.** $x \in M$ is **critical** for $F$ if $DF_x < \min \{\dim M, \dim N\}$. Otherwise, $x$ is **regular**.

$Cr_F = \text{the set of critical points of } F$.

**Definition 1.2.** $y \in N$ is a critical value if $F^{-1}(y)$ contains a critical point. Otherwise it is a **regular** value.

$\Delta_F = \text{the set of critical values of } F = \text{Discriminant set}$.

**Theorem 1.3.** (Morse-Sard-Federer Theorem)

1. $\Delta_F$ has measure 0.
2. If $F(M)$ has nonempty interior, then the set of regular values is dense in the image $F(M)$.

**Examples**

1. Standard $N = \mathbb{R}$. $x \in M$ is critical implies $df_x = 0$.
2. Torus maps to height function, 4 critical points.
3. Horizontal torus maps via height function, 2 critical points, union of two circles (top and bottom) as critical sets.
4. If $M \subset \mathbb{R}^n$, then $f : \mathbb{R}^n \to \mathbb{R}$ is critical at $p$ on $M$ if its tangent space $T_p M$ is tangent to a level set (in $\mathbb{R}^n$) of $f$.
5. If $F$ is proper analytic between analytic manifolds, then $\Delta_F$ is a union of submanifolds of $N$.

Recall if $M$ is a manifold and $X$ is a vector field on $M$ and $f$ is a smooth (real-valued) function, $X(f) = df(X)$.

**Lemma 1.4.** If $p_0$ is a critical point of $f$, and of $X, X', Y, Y'$ are vector fields such that $X(p_0) = X'(p_0), \ Y(p_0) = Y'(p_0)$,

then

$$(X(Yf))(p_0) = (X'(Y'f))(p_0) = (Y(Xf))(p_0)$$

(not usually true).

**Proof.** $[(XY - YX)f](p_0) = ([X,Y]f)(p_0) = df([X,Y])(p_0) = 0$.

Since $((X - X')f)(p_0) = 0$, etc. the result follows.\hfill \Box$

**Definition 1.5.** If $p_0$ is a critical point for $f : M \to \mathbb{R}$, define the Hessian as $H_{f,p_0} : T_{p_0} M \times T_{p_0} M \to \mathbb{R}$, where

$$H_{f,p_0}(X_0,Y_0) = (XYf)(p_0)$$

such that $X(p_0) = X_0, \ Y(p_0) = Y_0$. This is well-defined and symmetric, by the Lemma.
If $X = \sum X^j \partial_j$, $Y = \sum Y^i \partial_i$ in local coordinates at $p_0$,

$$H_{f,p_0} (X_0,Y_0) = \sum_{i,j} h_{ij} X^i Y^j$$

$$h_{ij} = \partial_i \partial_j f (p_0).$$

**Definition 1.6.** $p_0$ is **nondegenerate** if $H$ is nondegenerate, i.e., $H (X,Y) = 0$ for all $Y$ implies $X = 0$.

If so, $f (x) = f (p_0) + \frac{1}{2} H_{p_0} (x,x) + O (|x|^3)$.  

**Definition 1.7.** $f$ is called a **Morse function** if all critical points are nondegenerate.

Note that $H$ is nondegenerate iff $\det (h_{ij}) \neq 0$. For example, $f (x) = x^3$ from $\mathbb{R}$ to $\mathbb{R}$ has $H = 0$ at $x = 0$. The “lying down” torus has degenerate critical points ($H$ rank 1 instead of 2).

Given a symmetric bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, there is a basis with respect to which it’s diagonal. Nondegenerate implies the matrix is full rank. And the number of negative eigenvalues is called the **index** of the critical point. This is also the largest dimension of a subspace on which $H$ is negative definite. Let

$$\lambda (f,p_0) = \text{index of } f \text{ at } p_0.$$  

Define the **Morse polynomial**

$$P_F (t) = \sum_{\mu \in \mathbb{C}_r F} t^{\lambda (f,p)} = \sum_{\lambda \in \mathbb{Z}} \mu_f (\lambda) t^\lambda$$

where $\mu_f (\lambda)$ is the number of critical points with index $\lambda$.

**Theorem 1.8.** ("Morse lemma") If $f$ is Morse and $p$ is a nondegenerate critical point, there exists a neighborhood $U$ of $p$ and coordinates such that $x^i (p) = 0$ and such that $f (x) = f (p_0) + \frac{1}{2} H_{f,p} (x)$.

**Corollary 1.9.** There exist coordinates such that

$$f (x) = f (p_0) - \sum_{j=1}^{\lambda} (x_j)^2 + \sum_{j=\lambda+1}^{m} (x_j)^2$$

These are called coordinates adapted to $f$.

## 2. Existence of Morse Functions

Assume $M$ is imbedded in $E = \mathbb{R}^{2m+1}$, with some metric. Let $\Lambda$ be a smooth manifold. Suppose $F : \Lambda \times E \to \mathbb{R}$ is smooth, which restricts to $f : \Lambda \times M \to \mathbb{R}$. Let $f_\lambda : \Lambda \times M \to \mathbb{R}$ be the family of functions. There is a natural surjection by restriction:

$$p_x : E^* \to T^*_x M.$$  

So $df_\lambda (x) = Pf_\lambda (x)$. Let

$$\partial^* f : \Lambda \to T^*_x M$$

$$\lambda \mapsto df_\lambda (x).$$

**Definition 2.1.** We say $F : \Lambda \times E \to \mathbb{R}$ is **sufficiently large relative to $M$** if $\dim \Lambda \geq \dim M$ and $\forall x \in M$, the point $0 \in T^*_x M$ is a regular value for $\partial^* f$.  


Theorem 2.2. If $F$ is sufficiently large, then there exists a subset $\Lambda_\infty \subset \Lambda$ of measure zero such that $f_\lambda$ is Morse for all $\lambda \in \Lambda - \Lambda_\infty$.

Example: let $\Lambda = E^*$, let $H = E^* \times E \to \mathbb{R}$. (evaluation). Claim: this is sufficiently large. This is the height function.

Example: Let $\Lambda = E$, let $F = R : E \times E \to \mathbb{R}$, $R(\lambda, x) = \frac{1}{2} |x - \lambda|^2$. This $R$ is sufficiently large. The resulting Morse functions are exhaustive, i.e, the sublevel sets \( \{ x \in M : f(x) \leq s \} \) are compact (assuming $M$ is properly embedded in $E$, equivalent to $M$ being embedded in $E$ as a closed subspace).

Example: Let $\Lambda$ be the set of positive definite matrices on $E$. Let $F : \Lambda \to E$ be $(A, x) \mapsto \frac{1}{2} (Ax, x)$. This $F$ is sufficiently large and exhaustive.

The point: there exist Morse functions.

Definition 2.3. A Morse function is called resonant if two critical points have the same critical value.

Theorem 2.4. Any resonant Morse function on a compact manifold can be approximated arbitrarily closely in the $C^2$ topology by a nonresonant Morse function.

3. The topology of Morse functions

3.1. Surgery, handle attachment, cobordism. Let $D^k$ be the closed $k$-disk. Let $\partial D^k = S^{k-1}$ be the boundary.

Given $X$ and a map from $\partial D^k \to X$, we can do a handle attachment (cell attachment). A good way to build CW complexes. But this bad in the manifold world.

Instead, we will do surgery (attaching handles one dimension up). Let $M$ be a smooth manifold, given an embedding of a sphere into $M$ and a trivial normal bundle that extends the embedding. (more next time)