SEMINAR ON LIE GROUPS

1. Lie Groups

Definition 1.1. *A* Lie group *G is a manifold and group for which the multiplication map $\mu : G \times G \to G$ is smooth.*

Remark 1.2. *It follows that the inverse map $i : G \to G$ defined by $i(g) = g^{-1}$ is also smooth. Proof: implicit fcn theorem + diagram*

$$\begin{array}{ccc} \mu^{-1}(e) \subseteq G \times G & \stackrel{\mu}{\to} & G \\ \pi_1 \downarrow & \nearrow & \\ G & & \end{array}$$

Example 1.3. $(\mathbb{R}^n, +)$

Example 1.4. S^1 *or* $T^n = S^1 \times ... \times S^1$

Example 1.5. $Gl(n, \mathbb{F}) \subseteq \mathbb{F}^n$*, where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$

Example 1.6. E_3 = *isometries of* $\mathbb{R}^3$ *(2 connected components) Let the orthogonal group $O_3 < E_3$ be the subgroup that fixes the origin, and let the special orthogonal group $SO(3) = SO_3 < O_3$ be the orientation-preserving elements of O_3.*

Visualizing $SO(3)$: Let u be a vector of length l in $\mathbb{R}^3$, corresponding to a rotation of angle l around the axis u. Redundancy: if $l = |u| = \pi$, u gives the same rotation as $-u$, so $SO(3)$ is the ball of radius π with antipodal points identified $= \mathbb{R}P^3$.

1.1. Matrix groups.

Theorem 1.7. *If G is a Lie group and $H < G$, then H is a Lie subgroup with the subspace topology if and only if H is closed.*

Example 1.8. *Embed $\mathbb{R}$ as an irrational slope on $\mathbb{R}^2 / \mathbb{Z}^2 = T^2$; then this is a subgroup but is not a Lie subgroup.*

Note that

$$E_3 \cong \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \subseteq Gl(4, \mathbb{R}) \text{ such that } A \in O_3 \right\}$$

(b is the translation vector)

Classical Lie (sub)groups: $Sl(n, \mathbb{F})$ (det=1), $O(n)$ ($gg^t = \mathbf{1}$, orthogonal group), $SO(n)$ ($gg^t = \mathbf{1}$, det=1, special orthogonal group), $U(n)$ ($gg^* = \mathbf{1}$, unitary group), $SU(n)$ ($gg^* = \mathbf{1}$, det=1, special unitary group), $Sp(n) = \{g \in Gl(n, \mathbb{H}) : gg^* = \mathbf{1}\}$ (symplectic group).

Why study general Lie groups? Well, a standard group could be embedded in a funny way. For example, $\mathbb{R}$ can be embedded as (e^x) as matrices, or as $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ or as $\begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}$.

Also, some examples are not matrix groups. For example, consider the following quotient of the Heisenberg group N: Let

$$\begin{aligned} N &= \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\} \\ Z &= \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}, \\ \text{Let } G &= N \diagup Z \end{aligned}$$

These groups are important in quantum mechanics. Also, consider the following transformations of $L^2(\mathbb{R})$:

$$\begin{aligned} T_a(f)(x) &= f(x-a) \\ M_b(f)(x) &= e^{2\pi i b x} f(x) \\ U_c(f)(x) &= e^{2\pi i c} f(x) \end{aligned}$$

The group of operators of the form $T_a M_b U_c$ corresponds exactly to $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$. In quantum mechanics, T_a corresponds to a unitary involution of momentum, and M is the momentum, U is phase.

Note that every Lie group is locally a matrix group. (Igor claims)

Low dimensional, connected examples:

(1) Dim 1: $\mathbb{R}$, S^1
(2) Dim 2: only nonabelian example is the space of affine transformations $x \mapsto mx + b$ of $\mathbb{R}$.
(3) Dim 3: SO_3, $SL_2(\mathbb{R})$, E_2, N (only new ones up to local isomorphism: G_1 and G_2 are locally isomorphic if there exist open neighborhoods around the identities that are homeomorphic through multiplication-preserving homeo)

2. Relationships between Lie groups

Observe that $U_2 = \{g : gg^* = \mathbf{1}\}$, $SU_2 = \{g \in U_2 : \det g = 1\}$.

For every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_2$, then $g^* = g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$,

so

$$\begin{aligned} SU_2 &= \{g \in U_2 : \det g = 1\} \\ &= \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} = S^3 \\ &= \left\{ \begin{pmatrix} t+ix & y+zi \\ -y+zi & t-ix \end{pmatrix} : (t,x,y,z) \in S^3 \right\} \\ &= \left\{ q = t\mathbf{1} + x\hat{\imath} + y\hat{\jmath} + z\hat{k} \in \mathbb{H} : (t,x,y,z) \in S^3 \right\} \\ &= Sp(1) = \text{group of unit quaternions,} \end{aligned}$$

where

$$\begin{aligned} \hat{\imath} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ \hat{\jmath} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \hat{k} &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}$$

satisfy the relations

$$\begin{aligned} \hat{\imath}^2 &= \hat{\jmath}^2 = \hat{k}^2 = -\mathbf{1} \\ \hat{\imath}\hat{\jmath} &= -\hat{\jmath}\hat{\imath} = \hat{k};\ \hat{\jmath}\hat{k} = -\hat{k}\hat{\jmath} = \hat{\imath};\ \hat{k}\hat{\imath} = -\hat{\imath}\hat{k} = \hat{\jmath}. \end{aligned}$$

This group forms a double cover of SO_3 in two different ways:

(1) The map $T : Sp(1) \to SO(3)$ is defined as follows. View $\mathbb{R}^3 = \mathrm{Im}\mathbb{H} = \mathrm{span}\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$. Then $T(q)(v) = qv\bar{q}$ gives a map from $\mathbb{R}^3$ to itself; one must check that $T(q)$ preserves $\mathrm{Im}\mathbb{H}$ and that one gets an orientation-preserving isometry of $\mathbb{R}^3$. Let's check why it is a 2-1 map: Note that for $u, v \in \mathbb{H}$, $\mathrm{Re}(u\bar{v})$ = standard Euclidean inner product. Then if $u, v \in \mathrm{Im}\mathbb{H}$, u and v are orthogonal iff $\mathrm{Re}(u\bar{v}) = 0$ iff $u\bar{v} \in \mathrm{Im}\mathbb{H}$. Also, observe that $(t_j, v_j) \in \mathrm{Re}\mathbb{H} \times \mathrm{Im}\mathbb{H}$ for $j = 1, 2$, then

$$(t_1, v_1) \cdot (t_2, v_2) = (t_1 t_2 - v_1 \cdot v_2, t_1 v_2 + t_2 v_1 + v_1 \times v_2).$$

Computation: For $q \in Sp(1)$, then $q = t + xi + yj + zk = \cos\theta + u\sin\theta$ for some $\theta \in [0, \pi]$, $u \in \mathrm{Im}\mathbb{H}$, $|u| = 1$. The claim is that $T(q)$ = rotation about $u \in \mathrm{Im}\mathbb{H}$ by an angle of 2θ.

Idea of proof of claim: use new coordinates for $\mathrm{Im}\mathbb{H} = \mathbb{R}^3$: Choose u to be a pure imaginary vector of length 1 defined as above. Then choose v perpendicular to u such that $|v| = 1$. Then $w = uv \in \mathrm{Im}\mathbb{H}$ and has norm 1. So the new basis $\{u, v, w\}$ works like the quaternions (ie also $u^2 = -1$, etc). Next, observe that $T(q)$ fixes u:

$$T(q)u = (\cos\theta + u\sin\theta)u(\cos\theta - u\sin\theta) = u.$$

Next, we check that $T(q)$ rotates the vw plane by 2θ. Consider

$$\begin{aligned} T(q)(v\cos\psi + w\sin\psi) &= (\cos\theta + u\sin\theta)(v\cos\psi + w\sin\psi)(\cos\theta - u\sin\theta) \\ &= v\cos^2\theta\cos\psi + w\cos^2\theta\sin\psi - u^2 v\sin^2\theta\cos\psi - u^2 w\sin^2\theta\sin\psi \\ &= v\cos^2\theta\cos\psi + w\cos^2\theta\sin\psi + v\sin^2\theta\cos\psi + w\sin^2\theta\sin\psi \\ &= v\cos(\psi + 2\theta) + w\sin(\psi + 2\theta). \end{aligned}$$

(2) The map $L : SU(2) \to SO(3)$ is defined as follows. View $\mathbb{R}^3$ as the skew-hermitian 2×2 matrices with zero trace. Then we let $L(M)(v) = MvM^{-1}$ (matrix multiplication). One again should check that $L(M)$ leaves the traceless skew Hermitian matrices invariant, and that one obtains an orthogonal action.

(3) For $q_1, q_2 \in Sp(1)$, view $\mathbb{R}^4 = \mathbb{H}$. We get the map

$$\pi : Sp(1) \times Sp(1) \to SO_4,$$

which is a 2-1 cover defined similarly: $\pi(q_1, q_2) = q_1 v\bar{q}_2$.

(4) Further, if we have two pairs $(q_1, q_2), (q_3, q_4)$ such that each pair is in the same image of π above, then we may act on $SO(3)$ by the action $T \times T$, using either pair, and we get a double cover

$$SO_4 \to SO_3 \times SO_3$$

Another way of understanding this map is as follows. The group SO_4 acts on $\mathbb{R}^4$ by isometries, and thus induces an action on $\Lambda^2\mathbb{R}^4 \cong \mathbb{R}^6$. But this may be decomposed as self-dual and anti-self-dual forms, and the action preserves this grading. We have $\Lambda^2\mathbb{R}^4 \cong \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4 = \mathbb{R}^3 \oplus \mathbb{R}^3$, and the action induced on the two copies of $\mathbb{R}^3$ is the $SO_3 \times SO_3$ action.

(5) Extension of $SU(2) \to SO(3)$ double cover: there are double covers – from Clifford algebra stuff

$$\begin{array}{ccc} Pin(n) & \supset & Spin(n) \\ \downarrow & & \downarrow \\ O(n) & \supset & SO(n) \end{array}$$

(6) Let the Lorentz group $SO^+_{1,3}$ be defined by

$$\begin{gathered} SO^+_{1,3} = \{g \in SL_4(\mathbb{R}) : g \text{ preserves } t^2 - x^2 - y^2 - z^2 \\ \text{and doesn't switch two sheets of hyperbola } t^2 - x^2 - y^2 - z^2 = 1\} \end{gathered}$$

Then we have the double covers

$$\begin{array}{ccccc} SU(2) & \subset & SL_2(\mathbb{C}) & \supset & SL_2(\mathbb{R}) \\ \downarrow & & \downarrow \pi & & \downarrow \\ SO(3) & \subset & SO^+_{1,3} & & SO^+_{1,2} \end{array}$$

The spinor map π can be seen as follows: View $\mathbb{R}^4$ as

$$\begin{aligned} \mathbb{R}^4 &= \left\{(t,x,y,z) \leftrightarrow \begin{pmatrix} t+x & y-iz \\ y+iz & t-x \end{pmatrix}\right\} \\ &= \{2 \times 2 \text{ Hermitian matrices}\}. \end{aligned}$$

Then $g \in SL_2(\mathbb{C})$ acts by $A \mapsto gAg^{-1}$, and $\det A = t^2 - x^2 - y^2 - z^2$, so the action preserves the quadratic form, and $\ker \pi = \{\pm \mathbf{1}\}$.
Alternative description of π above: Consider the stereographic projection $\Sigma = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}}$. See that $(x,y,z) \in S^2$ maps to $\frac{x+iy}{1-z} \in \Sigma$. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$, think of this as the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$. These transformations contain the rotations, and $SU(2) \subset SL_2(\mathbb{C})$ maps to the rotations (corresponding to the spinor map). Observe that $SL_2(\mathbb{C})$ is a 2-1 cover of the set of Möbius transformations; both a matrix and its negative map to the same transformation. Thus, $PSL_2(\mathbb{C}) = SO^+_{1,3}$ can be identified as the group of Möbius transformations of $\mathbb{C}$. Thus, the Möbius transformations on Σ are the same as the Lorentz transformations of $\mathbb{R}^4$, which is the set of holomorphic bijections of $\mathbb{P}^1_{\mathbb{C}}$.

Picture of $SL_2(\mathbb{R})$: Observe first that $SL_2(\mathbb{R}) \subset GL_2(\mathbb{C})$ is the set of Möbius transformations that fix the upper half plane. Next, $SU_{1,1} = \left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1\right\}$ is conjugate to $SL_2(\mathbb{R})$ in $GL_2(\mathbb{C})$. Observe that $SU_{1,1}$ can be identified with the set of Möbius transformations that preserves the unit disk. These two groups are conjugate via $z \mapsto \frac{z-i}{z+i}$. Thus, we can draw a picture of $SU_{1,1}$ as $S^1 \times D^2$, via $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \leftrightarrow \left(\frac{a}{|a|}, \frac{b}{a}\right)$, ie an open solid torus.

The types of elements of $SU_{1,1}$ (or $SL_2(\mathbb{R})$) are

- those with trace ≥ 2 — the union of subgroups isomorphic to $\mathbb{R}$ (> 2 hyperbolic, $= 2$ parabolic)
- those with $-2 <$ trace < 2 — the union of subgroups isomorphic to S^1 if you close up this set (elliptic)
- those with trace ≤ -2 — contains no 1-parameter subgroups (< -2 hyperbolic, $= -2$ parabolic)

Another way to look at it:

- Hyperbolic ones : matrix conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$
- Parabolic ones : matrix conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- Elliptic ones : matrix conjugate to $\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$.

3. Homogeneous Spaces

Lie groups all arise as transformation groups on manifolds. For example, S^1 acts on the sphere on S^2 by rotations. This is a group action that is not a transitive. A group action of G on X is a *transitive* action such that for all $x, y \in X$, there exists $g \in G$ such that $gx = y$.

Definition 3.1. *If G is a Lie group that acts transitively on a manifold X, then X is called a **homogeneous space**.*

Definition 3.2. *For $x_0 \in X$, the isotropy subgroup of G at x_0 is $G_{x_0} = \{g \in G : gx_0 = x_0\}$.*

Note that there is a map

$$G/G_{x_0} \xrightarrow{\psi} X$$

given by

$$gG_{x_0} \mapsto gx_0.$$

If the group action is transitive, then the map ψ is onto. By the definition of isotropy group, ψ is $1-1$. A natural topology on X is given by the quotient topology on G/G_{x_0}.

Example 3.3. *Consider the action on $O(n)$ on the unit sphere S^{n-1}. If we take the north pole NP as x_0, then $O(n)_{NP} \cong O(n-1)$.*

Example 3.4. *$SL_2\mathbb{R} \simeq$ Möbius transformations; it preserves $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. Then*

$$(SL_2\mathbb{R})_i = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} = SO(2).$$

Thus

$$\mathbb{H} = SL_2\mathbb{R}/SO(2)$$

is a homogeneous space.

Example 3.5. *$GL_n\mathbb{R}$ acts on $\mathcal{P} = \{$positive definite real symmetric matrices$\}$ by*

$$(A, P) \mapsto APA^t.$$

The isotropy subgroup at the identity $\mathbf{1}$ *is*

$$GL_n\mathbb{R} = O(n).$$

Thus

$$\mathcal{P} = GL_n\mathbb{R} \diagup O(n).$$

Example 3.6. *Grassmann variety $X = Gr_k(\mathbb{R}^n) = \{k$-dim subspaces of $\mathbb{R}^n\}$. Then $GL_n\mathbb{R}$ acts on X in the obvious way. Let x_0 be the span of the first k basis vectors. Then*

$$\begin{aligned} GL_n\mathbb{R}_{x_0} &= \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} = \left\{ \begin{array}{c} echelon \\ matrices \end{array} \right\} \\ &= GL_{k,n-k} \end{aligned}$$

So

$$Gr_k(\mathbb{R}^n) = GL_n\mathbb{R} \diagup GL_{k,n-k}$$

Also $O(n)$ acts on $Gr_k(\mathbb{R}^n)$, and this time the isotropy subgroup is

$$\begin{aligned} O(n)_{x_0} &= \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} = O(k) \times O(n-k), \end{aligned}$$

so

$$Gr_k(\mathbb{R}^n) = O(n) \diagup (O(k) \times O(n-k)).$$

Example 3.7. *$\mathcal{L}$ = the space of lattices in $\mathbb{R}^2$ = subgroups of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}^2$. Let $\mathcal{L}_1$ = the set of unimodular lattices (ie such that the area of the fundamental parallelogram is 1). Then $GL_2\mathbb{R}$ acts on $\mathcal{L}$. If $x_0 = \mathbb{Z} \oplus \mathbb{Z}$, then*

$$\begin{aligned} GL_2\mathbb{R}_{x_0} &= GL_2\mathbb{Z}, \text{ so} \\ \mathcal{L} &= GL_2\mathbb{R} \diagup GL_2\mathbb{Z}. \end{aligned}$$

Similarly,

$$\mathcal{L} = SL_2\mathbb{R} \diagup SL_2\mathbb{Z}.$$

Note that $\mathcal{L}_1$ is homeomorphic to the complement of the trefoil knot in S^3 (Milnor). Also, $\mathcal{L}$ is homeomorphic to the space of unordered triples of distinct points in $\mathbb{R}^2$ with center of mass at the origin.

3.1. **Isotropy subgroups and metrics.** Note that if a compact Lie group G acts smoothly on a manifold M, the isotropy subgroups $G_x = \{g \in G : gx = x\}$ satisfy, for all $h \in G$

$$\begin{aligned} G_{hx} &= \{g \in G : ghx = hx\} \\ &= \left\{g \in G : h^{-1}ghx = x\right\} \\ &= \left\{g \in G : h^{-1}gh \in G_x\right\} \\ &= hG_xh^{-1}. \end{aligned}$$

Thus, along an orbit of G, the isotropy subgroups are all isomorphic. Also, by the results in the last section, the orbit Gx of a point $x \in M$ is diffeomorphic to $G \diagup H_x$. Given a closed subgroup H of G, let $[H]$ denote its conjugacy class, and let

$$M\left([H]\right) = \left\{x \in M : G_x \in [H]\right\}.$$

This is called a *stratum* of the G-manifold M associated to the conjugacy class $[H]$; in fact the G-manifold is a stratified space in the usual sense of the word. Note that all of the If G and M are compact and connected, then there are finitely many such strata. Further, let

$$M\left(\geq [H]\right) = \left\{x \in M : G_x \text{ is conjugate to a subgroup of } [H]\right\}$$

Let

$$M^H = \left\{x \in M : hx = x \;\forall h \in H\right\}$$

be the fixed point set of the subgroup H. Then we have the following:

Lemma 3.8. *With notation as above,*

$$GM^H = M\left(\geq [H]\right).$$

That is, the union of orbits of points in the fixed point set of H is the set of points x whose isotropy subgroup G_x is conjugate to a subgroup of H.

3.2. **Invariant integration and invariant metrics on Lie groups.** Let α be a differential form on a G-manifold M, where G is a compact Lie group. then we say that α is a *left-invariant differential form* if for every $g \in G$

$$l_g^* \alpha = \alpha,$$

where $l_g : M \to M$ is the diffeomorphism associated to left-multiplication by g. Similarly, if G acts on M on the right, one may speak of right-invariant differential forms. If M admits a G-action from the right and left, one may speak of biinvariant differential forms. Note that there are many examples of left-invariant differential forms on a Lie group G, by the following recipe. Given any $\beta_0 \in \bigwedge^k T_{\mathbf{1}}^* G$, we may define the form β on G by

$$\beta_g := l_{g^{-1}}^* \left(\beta_0\right).$$

then this differential form is well-defined and smooth, and observe that for any $h, k \in G$, and any $v \in T_k G$,

$$\left(l_h^* \left(\beta\right)\right)_k (v) =$$

Note that this implies that every Lie group is orientable (choose a nonzero $\beta_0 \in \bigwedge^{\dim G} T_{\mathbf{1}}^* G$).

Lemma 3.9. *(existence of Haar measure) Let G be a compact Lie group. Then there exists a biinvariant volume form on G, and thus up to sign there exists a unique normalized biinvariant Borel measure dg on G; that is $\int_G dg = 1$.*

Proof. If $n = \dim G$, let ω_0 be any nonzero element of $\bigwedge^n T_{\mathbf{1}}^* G$. Then we define the n-form ω by

$$\omega_g := l_{g^{-1}}^* \left(\omega_0\right).$$

By the discussion above, ω is left-invariant. Conversely, any two left-invariant volume forms differ by a constant multiple, since $\dim \bigwedge^n T_{\mathbf{1}}^* G = 1$. Next, observe that for any h in G, since right multiplication r_h commutes with left multiplication, we have that $r_h^* \omega$ is also left-invariant, so it is a constant multiple of ω. Since $\int_G \omega = \int_G r_h^* \omega$ by the change of variables formula, that constant is 1. Thus ω is biinvariant. Replacing ω by $\left(\int_G \omega\right)^{-1}\omega$, the result follows, using the Riesz Representation Theorem to get the Borel measure. $\square$

Proposition 3.10. *Any compact Lie group G has a normalized biinvariant metric.*

Proof. Choose any metric $\langle\cdot,\cdot\rangle_0$ on G; then we define the metric $\langle\cdot,\cdot\rangle$ on G by

$$\langle v_g, w_g\rangle = \int_{G\times G} \langle l_{h_1*} r_{h_2*} v_g, l_{h_1*} r_{h_2*} w_g\rangle_{0,h_1 g h_2}\, dh_1\, dh_2.$$

Then divide this biinvariant metric by a constant so that the induced Riemannian density is the Haar measure. $\square$

Proposition 3.11. *Any Lie group G has a left-invariant metric.*

Proof. Choose any inner product $(\cdot,\cdot)$ on $T_{\mathbf{1}}G$. Then define the inner product $\langle\cdot,\cdot\rangle$ on G by

$$\langle v_g, w_g\rangle = \langle l_{g^{-1}*} v_g, l_{g^{-1}*} w_g\rangle .$$

$\square$

Proposition 3.12. *Let G be a compact Lie group with biinvariant metric, and let H be a closed subgroup. Then there is a natural left-invariant metric on the homogeneous space G/H.*

Proof. For any $h \in H$, right multiplication r_h maps (through the differential) the normal space at $g_0 h_0$ to the orbit $g_0 H$ isometrically onto the normal space at $g_0 h_0 h$ to the orbit. Thus, the transverse metric to the orbits of H in G is right H-invariant, and thus $G \to G/H$ is a Riemannian submersion, and the metric downstairs is induced from the metric upstairs, by lifting vectors to their right H-invariant fields on the orbits. By the invariance upstairs, the resulting metric is left-invariant downstairs.
Alternately, one could start with any metric on G/H. Then by averaging over G by left multiplications, one produces a left-invariant metric on G/H. $\square$

Note that in general the averaging procedure works to produce an invariant metric on any G-manifold, if G is a compact Lie group. Thus we may always assume that such a group acts by isometries on the manifold. Conversely, we state without proof:

Proposition 3.13. *The set of isometries of a compact Riemannian manifold is a compact Lie group.*

3.3. **Symmetric Spaces.** A (globally) *(Riemannian) symmetric space* is a connected Riemannian manifold M such that at every point $x \in M$, there exists an isometry $\phi : M \to M$ that fixes x and reverses geodesics through x. That is, the differential $d\phi_x : T_x M \to T_x M$ satisfies $d\phi_x(v) = -v$. A *(Riemannian) locally symmetric space* has a similar definition, but the isometry ϕ need only be defined on a neighborhood of x. From now on I will assume the adjective *Riemannian*.

Proposition 3.14. *A connected Riemannian manifold M is locally symmetric if and only if the sectional curvature is invariant under all parallel translations, which is true if and only if the Riemann curvature tensor is covariantly constant.*

Proposition 3.15. *If a connected Riemannian manifold M is locally symmetric, simply connected, and complete, then it is globally symmetric. In particular, the universal cover of a closed locally symmetric space is globally symmetric.*

Proposition 3.16. *Every globally symmetric space is complete and locally symmetric and homogeneous.*

Proof. Note that it is easy to see that the isometry group acts transitively. Fixing x and y in the symmetric space, choose a geodesic from x to y, and then find the midpoint. Then there is an isometry that fixes the midpoint and maps x to y. □

Example 3.17. *A Riemann surface of genus > 1 and constant (negative) curvature is a locally symmetric space but not a globally symmetric space.*

Symmetric Spaces are classified since Lie groups are classified.

3.4. Fixed point sets of isometries and isotropy groups.

Proposition 3.18. *Let S be a set of isometries of a Riemannian manifold M. Then the fixed point set*

$$M^S = \{x \in M : \phi(x) = x \text{ for all } \phi \in S\}$$

is a closed, totally geodesic submanifold of M.

Proof. First of all, the set M^S is closed, because it is the intersection of inverse images of the form ϕ^{-1} (diagonal in $M \times M$). Next, it is a submanifold, because given a point $x \in M^S$, we have

$$\exp_x\left(\left(\bigcap_{\phi\in S} \ker(d\phi_x)\right) \cap \text{small ball}\right)$$

is a local trivialization of M^S (we use the fact that isometries fix geodesics), and thus M^S is a submanifold. Further, it follows that by considering all points $x \in M^S$ that M^S is totally geodesic. □

In what follows, we say that a group action on a space is *effective* if the only element of the group that fixes the entire space is the identity.

Lemma 3.19. *Let G be a compact Lie group that acts by isometries on a connected, complete Riemannian manifold M. Then for every $g \in G$, and for any $x \in M$, the action of G on M is determined by gx and by the differential dg_x.*

Proof. Note that the exponential map $\exp_x : T_xM \to M$ is onto on any such manifold. Given any $y \in M$, let $v_x \in T_xM$ be the initial velocity of a geodesic γ from x to y such that $\gamma(0) = x$ and $\gamma(1) = y$. Since G maps geodesics to geodesics and preserves distances, we have

$$\begin{aligned} gy &= g\exp_x(v_x) \\ &= \exp_{gx}(dg_x(v_x)). \end{aligned}$$

□

Proposition 3.20. *If a compact Lie group G acts effectively on a connected, smooth manifold M of dimension n, then every isotropy subgroup is isomorphic to a subgroup of $O(n)$.*

Proof. Choose an invariant Riemannian metric. Then if $x \in M$, $h \in H_x$ acts effectively by isometries by the differential $dh_x : T_xM \to T_xM$. Thus the transformations $\{dh_x : h \in H_x\}$ form a subgroup of $O(T_xM) \cong O(n)$. By the previous lemma, $h \mapsto dh_x$ is an isomorphism. □

Proposition 3.21. *Let H be a Lie subgroup of isometries of a Riemannian manifold M, and let M^H denote the fixed point set of H. Then for every $x \in M^H$, H acts (through dh_x) by isometries on the normal space $N_x\left(M^H\right) \subset T_xM$, and this action has no fixed points other than the origin.*

Proof. The differential dh_x fixes vectors tangent to the orbit Hx of x, so since H preserves angles, it follows that dh_x maps $N_x\left(M^H\right)$ to itself. If there is a vector $v_x \in T_xM \setminus \{0\}$ fixed by all elements of H, then the geodesic $t \mapsto \exp(tv_x)$ is also fixed by H, so $v_x \in N_x\left(M^H\right)$. □

Proposition 3.22. *If a compact Lie group G acts effectively on a connected G-manifold M of dimension n, then*

$$\dim G \leq \frac{n(n+1)}{2}.$$

Proof. We proceed by induction on the dimension of the manifold. The result is trivial if the manifold has dimension 0. Next, suppose that the result has been shown for all such manifolds of dimensions $< n$. Consider a point x of a connected G-manifold of dimension n. The orbit of x is diffeomorphic to G/G_x, so the dimension of the orbit is $\dim G - \dim G_x \leq n$. On the other hand, the isotropy subgroup G_x acts effectively on the unit sphere of T_xM. By the induction hypothesis, $\dim G_x$ has maximum dimension $\frac{(n-1)n}{2}$. Thus

$$\begin{aligned} \dim G &\leq \dim G_x + n \\ &\leq \frac{(n-1)n}{2} + n = \frac{n(n+1)}{2}. \end{aligned}$$

□

Note that the result is sharp in all dimensions, considering the action of $SO(n+1)$ on the n-sphere.

4. Maximal Tori

A *maximal torus* T of a compact Lie group G is a maximal connected abelian subgroup. The word *maximal* means that if $T < T' < G$, where T and T' are both connected and abelian, then $T = T'$.

Proposition 4.1. *The closure of a connected abelian subgroup of a Lie group is a closed abelian Lie subgroup.*

Proof. First, the closure of a subgroup of a Lie group is a closed Lie subgroup, because of the continuity of the group operations. Similarly, since $ab = ba$ for all a, b in the subgroup, that property continues to be true for the closure. □

Proposition 4.2. *If the dimension of a Lie group is at least 1, then there exists a closed, connected abelian Lie subgroup of positive dimension. In particular, if the Lie group is compact this means that there is a maximal torus of positive dimension.*

Proof. Let $v \in T_{\mathbf{1}}G$; then $\{\exp_{\mathbf{1}}(tv) : t \in \mathbb{R}\}$ is a one-parameter subgroup that is abelian, since $\exp_{\mathbf{1}}(tv)\exp_{\mathbf{1}}(sv) = \exp_{\mathbf{1}}((s+t)v)$. Thus, its closure is a connected Lie subgroup of positive dimension. Then in the compact case the connected component of the centralizer of this subgroup is a maximal torus. □

Lemma 4.3. *Any connected abelian subgroup of a Lie group contains a maximal torus.*

Proof. Take the connected component of the centralizer of the closure of the subgroup. $\square$

Lemma 4.4. *Maximal tori in compact Lie groups are closed Lie subgroups.*

Proof. The closure of the subgroup is also connected and abelian. $\square$

Proposition 4.5. *Every closed connected abelian subgroup of a compact Lie group is isomorphic to $T^n = (S^1)^n$.*

Proof. The subgroup is itself a Lie group, and consider the exponential map at the identity. The exponential map is onto, and by commutativity $\exp(v)\exp(w) = \exp(w)\exp(v) = \exp(v+w)$, so the exponential map is a group homomorphism. It is a local diffeomorphism, so that it is a covering map. Thus the abelian subgroup is a quotient of $\mathbb{R}^n$, so by compactness, it is isomorphic to T^n. $\square$

Theorem 4.6. *(Maximal torus theorem) Let G be a compact, connected Lie group. Then*

(1) *All maximal tori are conjugate.*
(2) *Fix a maximal torus T. Every element of G is contained in a maximal torus and is thus conjugate to an element of T.*
(3) *The codimension of a maximal torus T in G is always even.*

Proof. First of all, if G is trivial, so is this theorem, so assume that G has positive dimension. Given any element g of G, there is $v \in T_{\mathbf{1}}G$ such that $g = \exp(v)$. Then g is contained in the maximal torus containing the one-parameter subgroup $\{\exp(tv) : t \in \mathbb{R}\}$. To complete the proof, it suffices to show that

$$G = \bigcup_{g \in G} gTg^{-1}.$$

That is given $x \in G$, we must find $g \in G$ such that $x \in gTg^{-1}$, or that $xg \in gT$, or that $xgT = gT$. Thus, it suffices to show that the diffeomorphism

$$l_x : G\diagup T \to G\diagup T$$

must have a fixed point. To see this, we show that the Euler characteristic of $G\diagup T$ must be positive, which implies by the Lefschetz fixed point formula that l_x must have a fixed point. To show that $G\diagup T$ has positive Euler characteristic, let S^1 be a circle subgroup of T, which acts on $G\diagup T$ by left multiplication, and let V be the vector field on $G\diagup T$ induced by this action. Observe that V has at least one fixed point ($\mathbf{1}T$), so by the formula for the Euler characteristic in terms of such a vector field — ie a sum of positive numbers at each fixed point or fixed submanifold, the Euler characteristic is positive. $\square$

The dimension of a maximal torus in a compact Lie group is called the *rank* of the Lie group.

Example 4.7. *In $U(n)$, the diagonal matrices form a maximal torus (of dimension n).*

Example 4.8. *In $SO(2n)$, given a set of mutually orthogonal 2-planes, the rotations in those orthogonal 2-planes form a maximal torus. Same in $SO(2n+1)$. So the rank of $SO(2n)$ and of $SO(2n+1)$ is n.*

5. Representations of Lie groups

Let G be a compact Lie group. A representation of G is a finite dimensional complex vector space V and a Lie group homomorphism

$$\pi : G \to GL(V).$$

Note that π is smooth, but continuity automatically makes it smooth. Also, we require that π is a group action. We abuse notation, sometimes using g in place of $\pi(g)$.

An element $T \in \mathrm{Hom}(V, V')$ between representation spaces V and V' is an "intertwining operator" if for all $g \in G$, $gT = Tg$. In other words, it is a morphism in the category of G-spaces. The collection

$$\mathrm{Hom}_G(V, V') = \{\text{intertwining operators } V \to V'\}.$$

A G-isomorphism is an isomorphism in $\mathrm{Hom}_G(V, V')$.

Example 5.1. *The trivial representation ($gv = v$).*

Example 5.2. *$SU(n)$ acts on $\mathbb{C}^n$.*

Example 5.3. *$SO(2n)$ acts on $\mathbb{R}^n$.*

Example 5.4. *Spin representations.*

Example 5.5. *$SU(2)$ acts on $V_n(\mathbb{C}^2)$ = homogeneous polynomials of degree n in 2 variables. If $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, then we define $(gp)(\eta) = p(g^{-1}\eta)$.*

Example 5.6. *$O(n)$ acts on the space of harmonic polynomials of degree k.*

New representations from old representations: If G acts on V and W, there are obvious actions of G on $V \oplus W$, $V \otimes W$, $\bigwedge^k V$, $S^k V$, $\mathrm{Hom}(V, W)$. Note that in the last case, $(gT)(v) = g(T(g^{-1}v))$; also $\mathrm{Hom}(V, W) \cong V^* \otimes W \cong \mathrm{Hom}(V, \mathbb{C}) \otimes W$.

5.1. Irreducibility and Schur's lemma.

Let G be a compact Lie group acting on a finite-dimensional V. We say that a subspace $U \subset V$ is invariant if $GU \subset U$. We say that a representation V is *irreducible* if it has no invariant subspaces other than 0 and V. This is equivalent to V being spanned by $\{gv : g \in G\}$ for any nonzero $v \in V$.

Proposition 5.7. *(Schur's Lemma) If V and W are finite-dimensional irreducible representations of G (compact Lie group), then*

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \text{ and } W \text{ are equivalent} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $T \in \mathrm{Hom}_G(V, W)$, and assume $T \neq 0$. Consider $\ker T$ and $\mathrm{Im}T$. Note that $\ker T$ and $\mathrm{Im}T$ are G-invariant. Thus $\ker T = 0$ and $\mathrm{Im}T = W$, or $\ker T = V$. Next, let T_0, T be any nontrivial elements of $\mathrm{Hom}_G(V, W)$. Then consider TT_0^{-1} acts on W and must have an eigenvalue λ. So

$$\ker\left(TT_0^{-1} - \lambda\mathbf{1}\right)$$

is nontrivial, so $TT_0^{-1} = \lambda\mathbf{1}$, or $T = \lambda T_0$. □

Corollary 5.8. *If V is irreducible, then* $\mathrm{Hom}_G(V, V) = \mathbb{C}\mathbf{1}$.

Suppose that V is finite-dimensional inner product space. We say that G is *unitary* if G acts by isometries.

Proposition 5.9. *Given any representation of a compact Lie group G on V, there is an inner product $\langle \cdot, \cdot \rangle$, with respect to which G is unitary.*

Proof. Start with any inner product $(\cdot, \cdot)$; then let

$$\langle v, v' \rangle = \int_G (gv, gv')\, dg.$$

$\square$

We say that a representation G, V is *completely reducible* if we can write V as a direct sum of irreducible representations. (Scott: a *disappointing choice of language*.)

Corollary 5.10. *All finite-dimensional representations of compact Lie groups are completely reducible.*

Proof. Suppose V is reducible. There exists a nontrivial invariant $W \subset V$. Let $\langle \cdot, \cdot \rangle$ be an invariant inner product on V. Then one can check that $W^\perp \subset V$ is also invariant ($\langle gu, w \rangle = \langle u, g^{-1}w \rangle = 0$ etc...). So $V = W \oplus W^\perp$. Continue. Ask Dave how to finish the proof. $\square$

Any finite-dimensional representation can be written

$$V = \oplus n_i V_i$$

where each V_i is irreducible, where $n_i \in \mathbb{Z}_{\geq 0}$.

Corollary 5.11. *We have V is irreducible if and only if* $\mathrm{Hom}_G(V, V) = \mathbb{C}\mathbf{1}$.

Thank you, Barbara!

Note: we will not have uniqueness in the decomposition, because for example $V \oplus V$ can be decomposed several ways.

Let $[\pi] = [\pi : G \to GL(E_\pi)]$ be an isomorphism class of irreducible representations. Let

$$\widehat{G} = \{[\pi]\}.$$

Then $V_{[\pi]}$ be the largest subspace of V that is a direct sum of irreducible subspaces isomorphic to E_π. This is called the π-isotypic component of V. The multiplicity of π in V

$$\dim \mathrm{Hom}_G(E_\pi, V) = \frac{\dim V}{\dim E_\pi}.$$

Lemma 5.12. *If V_1, V_2 are subspaces of V that are direct sums of subspaces isomorphic to E_π, then $V_1 \oplus V_2$ is a direct sum of subspaces isomorphic to E_π.*

Proof. All W_i isomorphic to E_π; suppose $W_1 \subsetneq W_2 \oplus \ldots \oplus W_m$; then $W_1 \cap (W_2 \oplus \ldots \oplus W_m) = 0$,... $\square$

Lemma 5.13. *If U, W are irreducible representations in V and U is not equivalent to W, then U is orthogonal to W.*

The conclusion is:

$$\begin{aligned} V &\cong \bigoplus_{[\pi]\in\widehat{G}} V_{[\pi]} \cong \bigoplus_{[\pi]\in\widehat{G}} m_\pi E_\pi \\ &= \bigoplus_{[\pi]\in\widehat{G}} \dim\left(\mathrm{Hom}_G\left(E_\pi, V\right)\right) E_\pi . \end{aligned}$$

There is a G-isomorphism

$$V_{[\pi]} \to \mathrm{Hom}_G\left(E_\pi, V\right) \otimes E_\pi$$

defined as the inverse of $i_\pi : (T, v) \to T(v)$. Thus, there is a map

$$V \leftarrow \bigoplus_{[\pi]\in\widehat{G}} \mathrm{Hom}_G\left(E_\pi, V\right) \otimes E_\pi$$

6. Schur Orthogonality

Choose G a compact Lie group with finite dimensional representation V. Let $\{v_i\}$ an orthonormal basis with respect to a G-invariant inner product $(\cdot,\cdot)$, and let $\{v_i^*\}$ be the corresponding dual basis. We obtain matrix coefficients.

$$\left(gv_j, v_i^*\right)_V := v_i^*\left(gv_j\right)$$

Think of each of these as a function from G to $\mathbb{C}$. More generally, given $u, v \in V$, define

$$\begin{aligned} f_{u,v}^V &: G \to \mathbb{C} \text{ by} \\ f_{u,v}^V(g) &= (gu, v)_V . \end{aligned} \tag{6.1}$$

Any function of this form is called a *matrix coefficient*. The set of all such functions is $MC(G)$.

Theorem 6.1. *$MC(G)$ is a subalgebra of the algebra of smooth functions from $G \to \mathbb{C}$. If v_i^π is an orthonormal basis for E_π, then $\left\{ f^{E_\pi}_{v_i^\pi, v_j^\pi} \,\middle|\, \text{for all } [\pi] \in \widehat{G}, v_i, v_j \right\}$ spans $MC(G)$ (as a vector space). In particular, $MC(G)$ contains the constant functions.*

Note that $V \oplus V'$, so that $(\cdot,\cdot)_{V\oplus V'} = (\cdot,\cdot)_V + (\cdot,\cdot)_{V'}$

Theorem 6.2. *Let U, V be finite-dimensional irreducible unitary representations, and let $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Then*

$$\int_G (gu_1, u_2)_U \overline{(gv_1, v_2)_V} dg = \begin{cases} 0 & \text{if } U \ncong V \\ \frac{1}{\dim V}(u_1, v_1)_V \overline{(u_2, v_2)_V} & \text{if } U = V \end{cases} .$$

Proof. Take $u \in U$, $v \in V$; define

$$T_{u,v} : U \to V$$

by $T_{u,v} = (\cdot, u)_U v$. Then through the representation π_U, define

$$\widetilde{T_{u,v}} = \int_G g T_{u,v} g^{-1} dg.$$

Then $h\widetilde{T_{u,v}} = \widetilde{T_{u,v}}h$ for all $h \in G$. If U and V are irreducible, then by Schur's Lemma, if $U \ncong V$, $\widetilde{T_{u,v}} = 0$, and if $U = V$, $\widetilde{T_{u,v}} = cI$. Then

$$\begin{aligned}
\int_G (gu_1, u_2)_U \overline{(gv_1, v_2)_V} dg &= \int_G (gu_1, u_2)_U \left(g^{-1}v_2, v_1\right)_V dg \\
&= \int_G \left(\left(g^{-1}u_1, u_2\right)_U gv_2, v_1\right)_V dg \\
&= \int_G \left(gT_{u_2,v_2}g^{-1}u_1, v_1\right)_V dg \\
&= \left(\widetilde{T_{u_2,v_2}}u_1, v_1\right) = \begin{cases} 0 & \text{if } U \ncong V \\ c\,(u_1, v_1) & \text{if } U = V \end{cases}.
\end{aligned}$$

But taking traces of $\widetilde{T_{u_2,v_2}} = cI$, we obtain

$$\begin{aligned}
c \dim V &= \int_G \operatorname{tr}\left(gT_{u_2,v_2}g^{-1}\right) dg = \operatorname{tr}\left(T_{u_2,v_2}\right) \\
&= \operatorname{tr}\left(v_2\left(\cdot, u_2\right)\right) \\
&= \left(v_2, u_2\right).
\end{aligned}$$

So $c = \frac{(v_2,u_2)}{\dim V}$, as required. □

7. Characters

Let G be a compact Lie group with biinvariant metric, and let V be a finite dimensional representation.

Characters are functions $\chi_V : G \to \mathbb{C}$ defined by

$$\chi_V(g) = \operatorname{tr}(g).$$

Note this is not a homomorphism.

Theorem 7.1. *We have*

1. $\chi_V \in MC(G)$ *(see 6.1):* $\chi_V(g) = \sum_{i=1}^{\dim V} (ge_i, e_i)$
2. $\chi_V(\mathbf{1}) = \dim V$
3. *If* $V \cong U$*, then* $\chi_V = \chi_U$*.*
4. $\chi_V\left(hgh^{-1}\right) = \chi_V(g)$
5. $\chi_{U\oplus V} = \chi_U + \chi_V$
6. $\chi_{U\otimes V} = \chi_U\chi_V$
7. $\chi_{V^*}(g) = \chi_V\left(g^{-1}\right)$
8. $\chi_{\mathbb{C}}(g) = 1$

Theorem 7.2. *We have*

1. $\int_G \chi_V(g)\overline{\chi_W(g)}\, dg = \dim\left(\operatorname{Hom}_G(V, W)\right) = \dim\left(\operatorname{Hom}_G(W, V)\right)$*. In particular if they are irreducible then we get* 1 *or zero, and also* $\int_G \chi_V(g)\, dg = \dim\left(V^G\right)$*.*
2. $\chi_V = \chi_W$ *iff* $V \cong W$*.*
3. *We have* V *is irreducible iff* $\operatorname{Hom}_G(V, V) = 1$ *iff* $\int_G |\chi_V|^2 = 1$*.*

Proof. Part 1: if V and W are irreducible, then pick o.n.-bases $\{v_i\}\{w_j\}$, we get

$$\begin{aligned}\int \chi_V \overline{\chi_W} &= \sum_{i,j} \int (gv_i, v_i)_V \overline{(gw_j, w_j)_W} \\ &= \begin{cases} 0 & \text{if } V \ncong W \\ \frac{1}{\dim V} \sum_{i,j} \left|(v_i, v_j)\right|^2 = 1 & \text{if } V = W \end{cases}\end{aligned}$$

Next, write any representations as sum of irreducible ones: if $V = \bigoplus m_\pi E_\pi$, $W = \bigoplus n_{\pi'} E_{\pi'}$, then

$$\begin{aligned}\int \chi_V \overline{\chi_W} &= \sum m_\pi n_{\pi'} \int \chi_{E_\pi} \overline{\chi_{E_{\pi'}}} = \sum m_\pi n_\pi \\ &= \sum_{\pi,\pi'} \dim \operatorname{Hom}_G (E_\pi, E_{\pi'}) = \dim \operatorname{Hom}_G (V, W).\end{aligned}$$

The second part of (1) follows by $\chi_{\mathbb{C}}(g) = 1$ and $\operatorname{Hom}_G(\mathbb{C}, V) = V^G$.
Part 2: V is completely determined by $m_\pi = \dim \operatorname{Hom}_G(E_\pi, V)$, but this is

$$\int \chi_{E_\pi} \overline{\chi_V}.$$

Part 3: pretty clear ☐

Application: Let G_1 and G_2 be two compact Lie groups. Let V_1 and V_2 be respective representations of G_1 and G_2. Then there is a natural action of $G_1 \times G_2$ on $V_1 \otimes V_2$.

Proposition 7.3. *A finite dimensional representation W of $G_1 \times G_2$ is irreducible iff $W \cong V_1 \otimes V_2$ for finite-dimensional irreducible representations V_1 of G_1 and V_2 of G_2.*

Proof. Since the V_i are irreducible,

$$\int_{G_j} \left|\chi_{V_j}\right|^2 = 1.$$

To see that $V_1 \otimes V_2$ is irreducible, observe that

$$\begin{aligned}\int_{G_1 \times G_2} \left|\chi_{V_1 \otimes V_2}\right|^2 &= \int_{G_1 \times G_2} \left|\chi_{V_1}(g_1)\right|^2 \left|\chi_{V_2}(g_2)\right|^2 \, dg_1 \, dg_2 \\ &= \int_{G_1} \left|\chi_{V_1}\right|^2 \int_{G_2} \left|\chi_{V_2}\right|^2 = 1.\end{aligned}$$

Conversely, suppose that W is an irreducible representation of $G_1 \times G_2$. We identify $G_1 = G_1 \times \{e_2\}$, $G_2 = \{e_1\} \times G_2$, both subgroups of $G_1 \times G_2$. Write W as a direct sum of G_2-irreducible representations:

$$W = \bigoplus_{[\pi] \in \widehat{G_2}} \operatorname{Hom}_{G_2}(E_\pi, W) \otimes E_\pi.$$

Note $\operatorname{Hom}_{G_2}(E_\pi, W)$ is a G_1 representation via left multiplication, so $\bigoplus_{[\pi] \in \widehat{G_2}} \operatorname{Hom}_{G_2}(E_\pi, W) \otimes E_\pi$ is naturally a $G_1 \times G_2$ representation. In fact, it is isomorphic to the original representation via $T \otimes v \mapsto T(v)$:

$$(g_1, g_2)(T(v)) = g_1 T(g_2 v).$$

Since $G_1 \times G_2$ acts on each summand $\operatorname{Hom}_{G_2}(E_\pi, W) \otimes E_\pi$, there can be only one. Further, $\operatorname{Hom}_{G_2}(E_\pi, W)$ must be an irreducible G_1 representation. ☐

8. Peter - Weyl Theorems

Main point: Every irreducible representation of a compact Lie group is finite-dimensional.

Assume that G is a compact Lie group and V is a (complex) Hilbert space ($\dim V \leq \infty$, complete inner product space). Future results could be generalized to the case when G is a locally compact topological group and V is a topological vector space. Let $GL(V)$ be a space of bounded invertible $T \in \mathrm{Hom}(V, V)$. (In Hilbert space, bounded is the same as continuous - operator norm).

A representation (π, V) is a continuous homomorphism $\pi : G \to GL(V)$. If $\dim V < \infty$, then the character $\chi_\pi \in C(G)$ of the representation (π, V) defined by

$$\chi_\pi(g) = \mathrm{tr}\pi(g).$$

If $\dim V = \infty$, the character may not be defined.

Note that (π, V) is *irreducible* iff it has no proper nonzero invariant closed subspaces.

Example 8.1. $G = S^1 = \{\theta \in \mathbb{R}/2\pi\}$. $V = L^2(S^1)$, $\pi(\theta) f(t) = f(t-\theta)$. *All finite dimensional unitary irreducible representations of* S^1 *has dimension* 1 *(follows from Schur's Lemma). In fact, every such representation is of the form*

$$\pi_n(\theta) = e^{in\theta} : \mathbb{C} \to \mathbb{C},$$

Then $\chi_{\pi_n}(\theta) = e^{in\theta}$, *and* $\chi_{\pi_n}(g^{-1}) = \overline{\chi_{\pi_n}}(g)$. *Also* $\{\chi_{\pi_n}\}$ *is an orthonormal basis of* $L^2(S^1)$.

Example 8.2. *(generalization)* $G =$*compact abelian Lie group, the characters* $\widehat{G} = \{$*all irreducible represen* $\{\chi\}$ *forms an o-n basis of* $L^2(G)$. *If* $f \in L^2(G)$, *then we have the Fourier expansion*

$$f(g) = \sum_{\chi \in \widehat{G}} a_\chi \chi(g),$$

where

$$a_\chi = \int_G f(g) \overline{\chi}(g)\ dg$$

and we have the Plancherel formula

$$\int_G |f(g)|^2 dg = \sum_{\chi \in \widehat{G}} |a_\chi|^2.$$

If G *is abelian but only locally compact, the characters in* $\widehat{G}$ *vary continuously with the representation. We have the Fourier expansion formula*

$$\begin{aligned} f(g) &= \int_{\widehat{G}} \widehat{f}(\chi) \chi(g)\ d\chi \\ \widehat{f}(\chi) &= \int_G f(g) \overline{\chi}(g)\ dg. \end{aligned}$$

Example 8.3. *(second generalization)* G *is compact but not abelian. All irreducible representations are finite-dimensional.* $L^2(G)$ *has an orthonormal basis in the matrix coefficients of irreducible representations. The characters of irreducible representations form an o-n basis of the subspace of class functions in* $L^2(G)$.

Theorem 8.4. *(Peter and Weyl) Let* $\mathcal{H}$ *be a Hilbert space, and let* G *be a compact group. Let* $\pi : G \to GL(\mathcal{H})$ *be a unitary representation. Then* $\mathcal{H}$ *is a direct sum of finite-dimensional irreducible representations.*

Aside: compact operators. A linear operator $A : \mathcal{H} \to \mathcal{H}$ is compact if it maps bounded sequences to sequences with convergent subsequences. Equivalently, it sends bounded sets to sets whose closures are compact; or it sends the unit ball to a compact set. Compact operators form a two-sided ideal $\mathcal{K}$ in the ring of bounded operators.

Example 8.5. *All compact operators are operators of finite rank, limits of operators of finite rank (in the norm topology).*

Definition 8.6. *A bounded operator A is self-adjoint if $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in \mathcal{H}$.*

Theorem 8.7. *(Hilbert) Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a self-adjoint compact operator on a Hilbert space. Then*

1. *$\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of A.*
2. *All nonzero eigenvalues of A have finite multiplicity.*
3. *0 is the only possible accumulation point of the spectrum of A (0 is always an accumulation point if $\mathcal{H}$ is infinite dimensional).*

We restate:

Theorem 8.8. *(Peter and Weyl) Let $\mathcal{H}$ be a Hilbert space, and let G be a compact group. Let $\pi : G \to GL(\mathcal{H})$ be a unitary representation. Then $\mathcal{H}$ is a direct sum of finite-dimensional irreducible representations. (That is, $\mathcal{H} = \widehat{\bigoplus} V_\alpha$, where each V_α is irreducible "$\widehat{\bigoplus} W_\alpha$" means that each W_α is closed, and each element in $\mathcal{H}$ is a limit of finite sums (over α) of elts in W_α.)*

Proof. Let P be an orthogonal projection on an arbitrary finite-dimensional non-zero subspace of $\mathcal{H}$. Observe that $P = P^*$ is compact. Let $T = \int_G \pi(g) P \pi(g)^{-1} d\mu(g)$, where $d\mu$ is the Haar measure , and $\pi(g)^{-1} = \pi(g)^*$. The integral is to be understood as an operator norm limit of finite integral sums. It is clear that

$$\left\langle \pi(g) P \pi(g)^{-1} u, u \right\rangle \geq 0.$$

It follows that T is compact, self-adjoint, and positive, and invariant (ie $\pi(h) T = T\pi(h)$.) Because T is compact, it follows that there is a nonzero eigenvalue with finite-dimensional eigenspace. This eigenspace is an invariant subspace of π. Then we can decompose the subspace as a sum of irreducible representations. Let Σ be the set of all sets of orthogonal finite-dimensional irreducible invariant subspaces of $\mathcal{H}$, linearly ordered by inclusion. (if $S \in \Sigma$ and $U, V \in S$, then if $U \neq V$, then $U \perp V$). By Zorn's Lemma, Σ has a maximal element S'. Either $S' = \mathcal{H}$ or $S' \neq \mathcal{H}$. In the second case, find a larger element by applying the argument to $S'^{\perp}$. □

Theorem 8.9. *(Schur's Lemma, revisited) Let (V, π) and (W, π') be unitary (possibly infinite-dimensional) representations of a compact G on Hilbert spaces. If V and W are irreducible, then*

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } W \cong V \\ 0 & \text{otherwise} \end{cases}.$$

It follows from Schur's Lemma that

$$\mathcal{H} \cong \widehat{\bigoplus}_{[\pi_\alpha] \in \widehat{G}} \operatorname{Hom}_G(E_{\pi_\alpha}, \mathcal{H}) \widehat{\otimes} E_{\pi_\alpha}.$$

The quantity

$$\mathcal{H}_\alpha \cong \operatorname{Hom}_G(E_{\pi_\alpha}, \mathcal{H}) \widehat{\otimes} E_{\pi_\alpha}$$

is called the π-isotypical component, i.e. maximal closed subspace in $\mathcal{H}$ on which π acts by π_α.

8.1. **Matrix coefficients.** Let (π, V) be a representation of G, a compact group. A function of the form $\phi(g) = L(\pi(g)v)$, where $L \in V^* = \mathrm{Hom}(V, \mathbb{C})$ (continuous linear functionals) and $v \in V$. For V, $\langle \cdot, \cdot \rangle$ a Hilbert space, $V \cong V^*$.

Definition 8.10. *Let $l(g)$ or $r(g) : C(G) \to C(G)$ denote left and right translation. That is,*

$$l(g) f(x) = f\left(g^{-1}x\right), \; r(g) f(x) = f(xg).$$

Note that $l(g)$ and $r(g)$ are unitary representations on $L^2(G)$.

There are two interesting subrings of $C(G)$:

(1) $C_{\mathrm{alg}}(G)$ = subring of matrix coefficients. $\phi \in C_{\mathrm{alg}}(G)$ if $\phi(g) = \lambda(\pi(g)v)$, where (π, V) is a finite-dimensional representation, $v \in V$, $\lambda \in V^*$.
(2) $C_{\mathrm{fin}}(G)$ = subring of G-finite functions. $f \in C_{\mathrm{fin}}(G)$ iff $\{l(g) f : g \in G\}$ spans a finite dimensional vector space.

Theorem 8.11. $C_{\mathrm{alg}}(G) = C_{\mathrm{fin}}(G)$.

Proof. Step 1: $C_{\mathrm{alg}}(G) \subset C_{\mathrm{fin}}(G)$. Let $\phi \in C_{\mathrm{alg}}(G)$, ie there exists a representation (π, V) of G such that $\phi(g) = \lambda(\pi(g)v)$. By choosing an invariant inner product on V, we can assume $\pi(g)$ is unitary for all $g \in G$. By identifying V with V^*, we write $\phi(g) = \langle \pi(g)v, u \rangle = \lambda(\pi(g)v)$, where $\lambda(w) = \langle w, u \rangle$. Observe that

$$\begin{aligned} l(h)\phi(g) &= \phi\left(h^{-1}g\right) = \left\langle \pi\left(h^{-1}g\right) v, u \right\rangle \\ &= \langle \pi(h)^* \pi(g) v, u \rangle \\ &= \langle \pi(g) v, \pi(h) u \rangle, \end{aligned}$$

a matrix coefficient for v and $\widetilde{u} = \pi(h) u$. This shows that $l(h)\phi$ is a matrix coefficient for (π, V). Since $\dim V < \infty$, the dimension of the space of matrix coefficients is $\leq (\dim V)^2$.
Step 2: $C_{\mathrm{fin}}(G) \subset C_{\mathrm{alg}}(G)$. Let $f \neq 0 \in C_{\mathrm{fin}}(G)$; consider $W = \mathit{span}\{l(g) f : g \in G\}$, $\dim W < \infty$. Then $(l(g), W)$ is a unitary representation of G. Choose an orthonormal basis $\{f_1, ..., f_n\}$ of W, where $f_1 = f$. Let $l(g) f_i = \sum M_{ij}(g) f_j$. The matrix $M = (M_{ij}(g))$ is unitary. Note

$$\begin{aligned} f(g) &= l\left(g^{-1}\right) f(1) = \sum M_{ij}\left(g^{-1}\right) f_j(1) \\ &= \sum M_{ji}(g) f_j(1), \end{aligned}$$

a matrix coefficient as a linear combination. $\square$

Theorem 8.12. *(Peter and Weyl) Let G be a compact group. Then $C_{\mathrm{alg}}(G)$ is dense in $L^2(G)$.*

Proof. Enough to show that $C_{\mathrm{alg}}(G)$ is dense in $C(G)$. We will apply the Stone-Weierstrass Theorem. (If K is a compact Hausdorff space and S is a subset of $C(K)$ that separates points. Then any complex algebra with unit closed under conjugations and containing S is dense in $C(K)$.) In our case $K = G$, and $S = C_{\mathrm{alg}}(G)$. By Greg, $C_{\mathrm{alg}}(G)$ is an algebra with unit. To show that $C_{\mathrm{alg}}(G)$ separates points, we show merely that $g_0 \neq e$ can be separated from e. Consider $l(g_0) : L^2(G) \to L^2(G)$. Clearly $l(g_0) \neq I$. Apply Peter-Weyl 1st Theorem to the representation $l(g)$:

$$L^2(G) \cong \widehat{\bigoplus}_{\alpha} V_\alpha,$$

where each V_α is a finite-dimensional irreducible component. Thus there exists V_{α_0} s.t. $l(g_0)$ restricted to V_{α_0} is not the identity. This means that

$$\langle l(g_0)\, u, v\rangle \neq \langle u, v\rangle$$

for some $u, v \in V_{\alpha_0}$. Define $\phi(g) = \langle l(g)\, u, v\rangle$; this element of $C_{\mathrm{alg}}(G)$ separates e and g_0. □

Theorem 8.13. *A compact Lie group G possesses a faithful representation, i.e. there exists finite-dimensional representation (π, V) of G for which π is injective.*

Proof. Let G_0 denote the identity component in G. Suppose first that $G = G_0$. Choose any $g_1 \neq e$ in G_0. There exists a finite-dimensional representation (π_1, V_1) of G such that $\pi_1(g_1) \neq e$. Let $G_1 = \ker \pi_1$. Since G_1 is a closed subgroup of G, it is itself a compact Lie group. Moreover, G_1 does not contain a neighborhood of the identity; otherwise G_1 contains G_0 (by the exponential map). Thus the dimension of G_1 is strictly less than the dimension of G. Now, if $G_1 \neq \{e\}$, choose $g_2 \in (G_1)_0$ and repeat the procedure; find a representation (π_2, V_2) of G with $\pi_2(g_2) \neq \pi_2(e)$, etc. etc. To finish the proof for G not connected, imbed $G \diagup G_0$ in a permutation group. □

Apply the Peter-Weyl Theorem to the representation $(l(g), L^2(G))$:

$$L^2(G) \cong \widehat{\bigoplus}_{[\pi]\in\widehat{G}} \mathrm{Hom}_G\left(E_\pi, L^2(G)\right) \otimes E_\pi$$

Since $C_{\mathrm{fin}}(G)$ is an invariant (dense) subspace,

$$C_{\mathrm{fin}}(G) \cong \bigoplus_{[\pi]\in\widehat{G}} \mathrm{Hom}_G\left(E_\pi, C_{\mathrm{fin}}(G)\right) \otimes E_\pi$$

Observe that there is an isomorphism

$$\tau : \mathrm{Hom}_G\left(E_\pi, C_{\mathrm{fin}}(G)\right) \to (E_\pi)^*$$

defined by $\tau(T) = \lambda_T$, where $\lambda_T(v) = T(v)(1)$. The map τ is an G-isomorphism, since there is an inverse $\tau^{-1}(\lambda) = T_\lambda$, where

$$(T_\lambda(v))(h) = \lambda\left(\pi\left(h^{-1}\right) v\right).$$

To show that τ commutes with the action of G, we need to check that

$$g\lambda_T = \lambda_g T,$$

where $g\cdot$ is the action of G on E_π^* and λ_g is the right action on $C_{\mathrm{fin}}(G)$.

$$\begin{aligned} (g\lambda_T)(v) &= \lambda_T\left(g^{-1}v\right), \text{ and} \\ \lambda_{gT}(v) &= r(g)(T(v))(e) = T(v)(g) \\ &= \lambda\left(g^{-1}v\right). \end{aligned}$$

We can write

$$C_{\mathrm{fin}}(G) \cong \bigoplus_{[\pi]\in\widehat{G}} E_\pi^* \otimes E_\pi,$$

where the isomorphism is given by

$$\lambda \otimes v \mapsto f,$$

where $f(g) = \lambda(\pi(g^{-1})v)$. Then

$$r(g) f_{\lambda\otimes v}(h) = f_{g\lambda\otimes v}(h),$$

and

$$l(g) f_{\lambda\otimes v}(h) = f_{\lambda\otimes gv}(h).$$

Definition 8.14. *Let G be a compact group. A function $f \in C(G)$ is called a continuous class function if $f(ghg^{-1}) = f(h)$ for any $g, h \in G$. (We can extend the definition to $L^2(G)$ using the adjective a.e.)*

Theorem 8.15. *Let G be a compact group, and let $\{\chi_\pi\}$ be the set of all irreducible characters. Then*

1. *$span\{\chi_\pi\}_{\pi\in\widehat{G}} = C^{\text{class}}_{\text{fin}}(G)$*
2. *If $f \in L^2(G)$ and is a class function, then*

$$f = \sum_{[\pi]\in\widehat{G}} \langle f, \chi_\pi\rangle_{L^2} \chi_\pi.$$

Proof. Consider the diagonal embedding

$$G \hookrightarrow G \times G$$

given by $g \mapsto (g, g)$. Then G acts on $L^2(G)$ (or $C_{\text{fin}}(G)$) by

$$f(h) \overset{g\cdot}{\mapsto} f\left(g^{-1}hg\right)$$

so that f is a class function iff $g \cdot f = f$. Then

$$C_{\text{fin}}(G) \cong \bigoplus_{[\pi]\in\widehat{G}} \text{Hom}(E_\pi, E_\pi),$$

so

$$\begin{aligned} C^{\text{class}}_{\text{fin}}(G) &\cong \bigoplus_{[\pi]\in\widehat{G}} \text{Hom}_G(E_\pi, E_\pi) \\ &= \bigoplus_{[\pi]\in\widehat{G}} \mathbb{C} I_{E_\pi}. \end{aligned}$$

Under this isomorphism,

$$\chi_\pi \leftrightarrow I_{E_\pi}.$$

The theorem follows. □

9. Lie algebras

9.1. **The Lie algebra of a Lie group. Differential Geometry version:** Let G be a (compact) Lie group, let l_g denote left multiplication by g ; $l_g(x) = gx$. This a diffeomorphism, and

$$dl_g : T_xG \to T_{gx}G,$$

and the Lie algebra $\mathcal{L}(G)=\mathfrak{g}$ is by definition

$$\begin{aligned}\mathfrak{g} &= \{\text{left-invariant vector fields}\} \\ &= \{X\in\Gamma(TG): dl_g(X_x)=X_{gx}\}.\end{aligned}$$

Why is this an algebra? Note that $\mathfrak{g}$ is clearly a real vector space, and the Lie bracket

$$[X,Y]=XY-YX$$

actually preserves $\mathfrak{g}$. There is a bijection

$$\{\text{left-invariant vector fields}\}\longleftrightarrow\{T_eG\}$$

given by

$$X\mapsto X_e$$

and

$$v\in T_eG\mapsto\{dl_gv:g\in G\}\in\Gamma(TG).$$

Note that if G is a compact Lie group, we have an embedding

$$G\subset GL(n,\mathbb{C})\subset\mathbb{C}^{n^2}\cong\mathbb{R}^{2n^2},$$

and thus, T_eG is a subspace of this vector space.

Note that a tangent vector v in T_eG can be regarded as an equivalence class of curves γ with $\gamma(0)=e$, and given any $f\in C^\infty(G)$, $vf(e):=\frac{d}{dt}\gamma(f(t))\big|_{t=0}$. Using this idea, the official definition of Lie algebra is as follows.

Definition 9.1. *The Lie algebra $\mathfrak{g}$ of $G\subset GL(n,\mathbb{C})$ is defined as*

$$\mathfrak{g}=\{\gamma'(0):\gamma(0)=I,\gamma(-\varepsilon,\varepsilon)\subset G\}.$$

This is an algebra, and in fact

$$[X,Y]=XY-YX,$$

an expression that may be evaluated as matrix multiplication. Note that

$$\mathcal{L}(GL(n,\mathbb{C}))=\mathfrak{gl}(n,\mathbb{C})=M_n(\mathbb{C}).$$

Theorem 9.2. *We have*

(1) $\mathfrak{g}$ *is a real vector space.*
(2) *We have*
  (a) $[X,Y]=-[Y,X]$
  (b) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$ *(Jacobi Identity)*
(3) $\mathfrak{g}$ *is closed under* $[\cdot,\cdot]$.

Proof. We have

(1) Given $X,Y\in\mathfrak{g}$ corresponding to curves γ_1,γ_2. Let $\gamma(t)=\gamma_1(ct)\gamma_2(t)$; then

$$\left.\frac{d}{dt}\gamma(t)\right|_{t=0}=cX+Y.$$

(2) Matrix computation

(3) As in (1), let $\sigma_s(t) = \gamma_1(s)\gamma_2(t)\gamma_1(s)^{-1}$. Then for each fixed s, we have a different curves. Then $\sigma_s'(0)$ are tangent vectors; which makes a curve in the tangent space, so its derivative wrt s at $s = 0$ lives in $\mathfrak{g}$. Using the chain rule, we get

$$\begin{aligned} \sigma_s'(0) &= \gamma_1(s)\,Y\gamma_1(s)^{-1}, \text{ and} \\ \left.\frac{d}{ds}\sigma_s'(0)\right|_{s=0} &= [X,Y], \end{aligned}$$

which must be in $\mathfrak{g}$ by the argument above.

□

9.2. The exponential map.

Given $G \subset GL(n,\mathbb{C})$, then $X \in \mathfrak{g}$ corresponds to

$$\widetilde{X}(g) = dl_g X.$$

By elementary differential geometry (ODEs), there exists a unique curve γ_x such that

$$\begin{aligned} \gamma_X(0) &= I \text{ and} \\ \gamma_X'(0) &= X. \end{aligned}$$

This curve will be complete.

Theorem 9.3. *(The exponential map is the exponential map.) We have*

$$\begin{aligned} \gamma_X(t) &= \exp(tX) = e^{tX} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n, \end{aligned}$$

and exp *is a group homomorphism (in t).*

Proof. First look in $GL(n,\mathbb{C})$. Then extend $\widetilde{X}$ to all of $GL(n,\mathbb{C})$. Consider $\alpha_X(t) = e^{tX}$. Then $\alpha_X(0) = I$ and $\alpha_X'(0) = X$, so it is an integral curve of the vector field $\widetilde{X}$. Why is α_X (locally) a curve in G? Well, there is a locally unique integral curve of X on G, so α_X must locally restrict to that curve. Since $e^{mtX} = \left(e^{tX}\right)^m$, so in fact, α_X globally restricts to a curve γ_X on G. The homomorphism part is elementary. □

Example 9.4. *If $G = S^1$, $\mathfrak{g} = i\mathbb{R}$, and $\exp(t(ix)) = e^{ixt}$.*

Theorem 9.5. *If $G \subset GL(n,\mathbb{C})$,*

(1) $\mathfrak{g} = \left\{X \in \mathfrak{gl}(n,\mathbb{C}) : e^{tX} \in G \text{ for all } t \in \mathbb{R}\right\}$.
(2) exp *is a local diffeomorphism.*
(3) *If G is connected,* $\exp(\mathfrak{g}) = G$.

Proof. (2) $d(\exp)X = X$, so the inverse function theorem tells you that it is a local diffeomorphism.(3) onto a nbhd of identity is sufficient. □

9.3. Classical Lie Algebras.

Note that

$$\mathcal{L}(GL(n,\mathbb{C})) = \mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C}).$$

(A nbhd of the identity in $M_n(\mathbb{C})$ is contained in $GL(n,\mathbb{C})$, so the Lie algebra must be all of $M_n(\mathbb{C})$.)

Next, $SL(n,\mathbb{C}) = \{X \in GL(n,\mathbb{C}) : \det(X) = 1\}$, so that

$$\begin{aligned}
\mathfrak{sl}(n,\mathbb{C}) &= \left\{Y \in M_n(\mathbb{C}) : \det\left(e^{tY}\right) = 1 \text{ for all } t\right\} \\
&= \left\{ATA^{-1} \in M_n(\mathbb{C}) : T \text{ is upper triangular and } \det\left(e^{tY}\right) = 1 \text{ for all } t\right\} \\
&= \left\{ATA^{-1} \in M_n(\mathbb{C}) : T \text{ is upper triangular and } \det\left(e^{tT}\right) = 1 \text{ for all } t\right\} \\
&= \left\{ATA^{-1} \in M_n(\mathbb{C}) : T \text{ is upper triangular and } e^{t\cdot \operatorname{tr}(T)} = 1 \text{ for all } t\right\} \\
&= \{Y \in M_n(\mathbb{C}) : \operatorname{tr}(Y) = 0\}.
\end{aligned}$$

Next, $U(n,\mathbb{C}) = \{X \in GL(n,\mathbb{C}) : XX^* = I\}$. Then

$$\begin{aligned}
\mathfrak{u}(n,\mathbb{C}) &= \left\{Y \in M_n(\mathbb{C}) : \left(e^{tY}\right)^{-1} = e^{-tY} = e^{tY^*} \text{ for all } t\right\} \\
&= \{Y \in M_n(\mathbb{C}) : Y + Y^* = 0\}.
\end{aligned}$$

Similarly, we see

$$\mathfrak{o}(n,\mathbb{R}) = \left\{Y \in M_n(\mathbb{R}) : Y + Y^t = 0\right\}.$$

One may also obtain these equations by taking derivatives of the defining group equations.

9.4. **Lie algebra homomorphisms.** Given two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, we call

$$\phi : \mathfrak{h} \to \mathfrak{g}$$

a *Lie algebra homomorphism* if it is a linear transformation such that

$$[\phi x, \phi y] = \phi[x,y]$$

for all $x, y \in \mathfrak{h}$. Given a Lie group homomorphism

$$\psi : H \to G,$$

the map

$$d\psi := d\psi_I : \mathfrak{h} \to \mathfrak{g}$$

is a Lie algebra homomorphism. Note that

$$d\psi(X) = \frac{d}{dt}\,\psi\left(e^{tX}\right)\Big|_{t=0}.$$

Theorem 9.6. *We have*

(1) $d\psi[X,Y] = [d\psi(X), d\psi(Y)]$.
(2) $\psi \circ \exp = \exp \circ d\psi$
(3) *If ψ_1 and ψ_2 are two Lie group homomorphisms from H to G, if H is connected and $d\psi_1 = d\psi_2$, then $\psi_1 = \psi_2$.*

Corollary 9.7. *If G_1, G_2 are isomorphic Lie groups, then $\mathfrak{g}_1, \mathfrak{g}_2$ are isomorphic as Lie algebras.*

As a consequence, the Lie algebra structure from $G \subset GL(n,\mathbb{C})$ is independent of the embedding.

One of the standard homomorphisms is the conjugation homomorphism (for $g \in G$)

$$c_g : G \to G$$

given by $c_g(h) = ghg^{-1}$, which takes the identity to itself. We define the adjoint representation Ad of G on $\mathfrak{g}$ as

$$Ad_g := dc_g : \mathfrak{g} \to \mathfrak{g},$$

but we can also think of this as

$$Ad : G \to GL(\mathfrak{g}).$$

This is actually a Lie group homomorphism, which may be calculated explicitly. Note that (with $G \subset GL(n, \mathbb{C})$)

$$\begin{aligned} Ad(g)(X) &= dc_g(X) = \frac{d}{dt} c_g\left(e^{tX}\right)\Big|_{t=0} \\ &= gXg^{-1}. \end{aligned}$$

Note that $Ad(I) = I$.

Then we define $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by

$$ad := dAd : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g}).$$

Note that (with $G \subset GL(n, \mathbb{C})$)

$$\begin{aligned} ad(X)Y &= \frac{d}{dt} Ad\left(e^{tX}\right)Y\Big|_{t=0} \\ &= \frac{d}{dt} e^{tX} Y e^{-tX}\Big|_{t=0} \\ &= [X, Y]. \end{aligned}$$

9.5. **Lie subgroups and subalgebras.** Let G be a compact complex Lie group. We often assume $G \subset M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$. Why is this nice? Recall that

$$\mathfrak{g} = T_e G.$$

Through the isomorphism,

$$[A, B] = AB - BA.$$

Recall that if $\phi : H \to G$ be a Lie group homomorphism, then $d\phi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism.

Theorem 9.8. *With $G \subset GL(n, \mathbb{C})$, there is a bijection between the set of connected Lie subgroups of G and the set of Lie subalgebras of $\mathfrak{g}$. If H is a connected Lie subgroup of G, then the corresponding subalgebra is a Lie subalgebra for H.*

Proof. Hard part is to start off with Lie subalgebra and exponentiate to get to the group. It is not obvious that it does not generate a larger subgroup. Uses Frobenius theorem; need to show that the span the left-invariant vector fields form an involutive distribution, exactly the Lie subalgebra condition. □

9.6. **Covering homomorphisms.**

Theorem 9.9. *Let H and G be connected Lie subgroups of $GL(n, \mathbb{C})$, and let $\phi : H \to G$ be a Lie group homomorphism. Then ϕ is a covering iff $d\phi$ is an isomorphism.*

Proof. If ϕ is a covering, it is clear that $d\phi$ is an isomorphism.

On the other hand, if $d\phi$ is an isomorphism. Then ϕ is a local diffeomorphism by the inverse function theorem. So there is a neighborhood U_0 of $e \in H$ that gets mapped diffeomorphically to a neighborhood V_0 of $e \in G$. Find smaller neighborhoods such that the neighborhoods are connected and such that $VV^{-1} \subset V_0$, $U = \phi^{-1}(V) \cap U_0$. It is easy to show that ϕ is surjective, because the image $\phi(U)$ contains a neighborhood of e. Next, note that $\phi^{-1}(V) = U \ker \phi$. Need to see that $\phi^{-1}(V)$ is a disjoint union of sets isomorphic to U. Suppose that

$u_1\gamma_1 = u_2\gamma_2$ with $u_i \in U$, $\gamma_i \in \ker\phi$. This implies $\gamma_2\gamma_1^{-1} = u_2^{-1}u_1 \in U_0 \cap \ker\phi = \{e\}$. Thus $\gamma_1 = \gamma_2$, $u_1 = u_2$. So ϕ is a covering over e. For $g \in G$, $g = \phi(h)$. Consider gV and $\phi^{-1}(g)V = hU\ker\phi$. the connected components are $hU\gamma$, which are disjoint since h acts as a diffeo on H. □

Here is an application.

Theorem 9.10. *Let H and G be connected Lie subgroups of $GL(n,\mathbb{C})$, and suppose H is simply connected. Further, suppose that $\psi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism. Then there is a unique homomorphism of Lie groups $\phi : H \to G$ such that $d\phi = \psi$.*

Remark 9.11. *Simple connectivity is important, because for example there is no Lie group homomorphism $\phi : S^1 \to \mathbb{R}$ that induces the identity on Lie algebras.*

Proof. Consider $H \times G \subset GL(2n,\mathbb{C})$. The Lie algebra of $H \times G$ is $\mathfrak{h} \oplus \mathfrak{g}$. Let $\mathfrak{a} = \{X + \psi X : X \in \mathfrak{h}\} \subset \mathfrak{h} \oplus \mathfrak{g}$. This is a Lie algebra. We check

$$\begin{aligned}[X+\psi X, Y+\psi Y] &= [X,Y] + [\psi X, \psi Y] \\ &= [X,Y] + \psi[X,Y].\end{aligned}$$

Now, $\mathfrak{a} \subset \mathfrak{h} \oplus \mathfrak{g}$, so it generates a Lie subgroup A of $H \times G$. Consider the projections $\pi_G, \pi_H : H \times G \to G$ or H. Then

$$\begin{aligned}d\pi_H(X+\psi X) &= X, \\ d\pi_G(X+\psi X) &= \psi X.\end{aligned}$$

Then $d\pi_H$ is an isomorphism, it gives us a covering homomorphism $\pi_H : A \to H$. Since H is simply connected, π_H is an isomorphism. So $\pi_G \circ \pi_H^{-1}$ induces the desired Lie algebra isomorphism. □

10. Abelian Lie Subgroups and Structure Theorems

10.1. **Overview.**

- In the unitary group $U(n)$, each element g is conjugate to a diagonal matrix.
- The diagonal matrices in $U(n)$ form a torus $T \cong \mathbb{T}^n$, which is a maximal abelian subgroup in $U(n)$.
- In fact, any abelian subgroup of $U(n)$ is conjugate to a subgroup of T. (Commuting matrices can be simultaneously diagonalized.)
- No such statements can be made about $Gl(n,\mathbb{C})$ (without making a certain Russian professor very angry).
- However, similar statements are true for any compact Lie group G. In particular, one can always choose a maximal torus $T \cong \mathbb{T}^n$ in G, and then the following theorem is true.

Theorem 10.1. *If G is a connected compact Lie group, then*

1. *every element of G is conjugate to an element of T, and*
2. *any connected abelian subgroup of G is conjugate to a subgroup of T. In particular, any two maximal tori are conjugate.*

Remark 10.2. *The word "connected" cannot be omitted. Not every maximal abelian subgroup is a torus, as the following example shows. Consider $SO(3)$ with maximal torus $SO(2)$. But the subgroup of diagonal matrices form a subgroup of order 4 that is not isomorphic to any subgroup of $SO(2)$.*

Proof. Conceptual proof of theorem:
Note that (2) follows from (1), because any compact connected abelian group A contains an element g whose powers are dense in A. Then $x^{-1}gx \in T \Rightarrow x^{-1}Ax \subset T$.
To prove (1) by algebraic topology: to find $x \in G$ such that $x^{-1}gx \in T$ is equivalent to finding the fixed point of the map $f_g : G/T \to G/T$ defined by $f_g(xT) = gxT$. Note that f_g depends continuously on g, $g \in G$, so f_g is homotopic to f_e. Theorem from topology: If X is a compact space with nonzero Euler number, then every continuous map $f : X \to X$ which is homotopic to the identity has a fixed point. (proof: Lefschetz number of f is nonzero) Note that the Euler characteristic of G/T is nonzero. Note that $\chi(G/T)$ = order of the Weyl group of G, which is $N(T)/T$. □

10.2. **Lie algebra approach.**

Theorem 10.3. *Let G be a Lie subgroup of $Gl(n, \mathbb{C})$ (true for any Lie group)*

1. *For $X, Y \in \mathfrak{g}$, $[X, Y] = 0$ iff e^{tX} and e^{sY} commute. In this case, $e^{X+Y} = e^X e^Y$.*
2. *If A is a connected Lie subgroup of G, then A is abelian iff $\mathfrak{a}$ is abelian.*

Proof. (1) implies (2).
Proof of (1). It is enough to show $e^{t(X+Y)} = e^{tX}e^{tY}$ for all t. Take the derivative: LHS= $(X+Y)e^{t(X+Y)}$, RHS= $Xe^{tX}e^{tY} + e^{tX}Ye^{tY}$=$(X+Y)e^{tX}e^{tY}$. By existence-uniqueness theorem for ODEs for linear systems, the equation holds. □

11. Weyl Group

Let G be a compact, connected Lie group (real or complex). We want to

- classify such G
- Find irreps for G and calculate their characters

The idea is to relate the irreps of G to the irreps of its maximal torus $T^m \cong S^1 \times ... \times S^1$, which are all one-dimensional and labelled by m-tuples of integers. That is, $\chi_{(n_1,...,n_m)}(\theta) = e^{i(\sum n_j \theta_j)}$.

It is necessary to study the coset space G/T, which is a homogeneous space with transitive G-action. It is actually a Kähler manifold and a projective algebraic variety.

Example 11.1. *Suppose G is the unitary group $U(n)$. Then $T = \left\{diag\left[e^{i\theta_1}, ..., e^{i\theta_n}\right]\right\}$, and G/T is a flag manifold. What is a flag manifold? Let $G = GL(n, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) acting on $V = \mathbb{K}^n$. A flag f in V is a collection of subspaces*

$$f = (0 = U_0 \subset U_1 \subset ... \subset U_n = V).$$

One can characterize f by a choice of basis

$$u \in U_1, u_2 \in U_2 \diagdown U_1, ..., u_i \in U_i \diagdown U_{i-1}, ...$$

Now it is clear that $GL(n, \mathbb{K})$ acts transitively on the flag manifold F. Let f_0 be a specific flag determined by an orthonormal basis $e_1, ..., e_n$. The subgroup of $GL(n, \mathbb{K})$ which fixes f_0 is B =group of upper triangular matrices. Thus the flag is isomorphic to $F = GL(n, \mathbb{K}) \diagup B(n, \mathbb{K})$. Next, by applying Gramm-Schmidt to columns of $A \in GL(n, \mathbb{C})$, we can write A as $A = UB$, where U is unitary and B is upper triangular. Moreover, observe that $U(n) \cap B(n, \mathbb{C}) = T$. Thus, we have the homeomorphism

$$F \cong GL(n, \mathbb{K}) \diagup B(n, \mathbb{K}) \cong U(n) \diagup T.$$

Why is F a projective algebraic variety? Let $u_1, ..., u_n$ be a basis of V defining a flag f. Notice that $u_1 \wedge ... \wedge u_i$ depends (up to a nonzero scalar multiple) only on U_i, and hence the tensor product

$$\begin{aligned} u_1 \otimes (u_1 \wedge u_2) \otimes ... \otimes (u_1 \wedge ... \wedge u_n) &\in V \otimes \Lambda^2 V \otimes ... \otimes \Lambda^n V \\ &= E \end{aligned}$$

where V has the standard basis $e_1, ..., e_n$. The expression above depends only on the flag F. Hence if $P(E)$ is the projective space of E, then we have a mapping

$$\phi : F \to P(E)$$

given by $f \mapsto$image of the map above in $P(E)$. Note that ϕ is injective and $X = \phi(F)$ is a closed subvariety of $P(E)$.

A maximal torus $T \subset G$ comes with the action of a Weyl group on it.

Definition 11.2. *Given a maximal torus T in a compact, connected Lie group G, the normalizer $N(T)$ of T is the subgroup of G defined by*

$$N(T) = \left\{ g \in G : gTg^{-1} = T \right\}.$$

The subgroup T is a normal subgroup of $N(T)$, and the quotient

$$W(G,T) = N(T) \diagup T$$

is called the Weyl group.

Remark 11.3. *There are many different maximal tori but are all conjugate, so different choices of T lead to isomorphic $W(G,T)$.*

Remark 11.4. *$W(G,T)$ acts on T by conjugation.*

Remark 11.5. *In addition, for every $t \in T$, we can consider $N(t) = \{g \in G : gtg^{-1} = t\}$. If $\dim N(t) = \dim T$, then t is called regular. If $\dim N(t) > \dim T$, t is said to be singular.*

Example 11.6. *Let $G = U(n)$, $T =$diag$\left(e^{i\theta_1}, ..., e^{i\theta_n}\right)$. Then $W(G,T) \cong S_n$, acting by permuting $(\theta_1, ..., \theta_n)$. Regular elements correspond to those where the θ_j's are distinct; the other elements of T are singular.*

12. Representation Ring

Let $\pi : G \to GL(V, \mathbb{C})$ be a representation. Note that $[\pi]$ is characterized up to isomorphism by its character $\chi_\pi(g) = \mathrm{Tr}(\pi(g))$. Note that χ is constant on conjugacy classes, and

$$\begin{aligned} \chi_{\pi_1 \oplus \pi_2} &= \chi_{\pi_1} + \chi_{\pi_2}, \\ \chi_{\pi_1 \otimes \pi_2} &= \chi_{\pi_1} \cdot \chi_{\pi_2}. \end{aligned}$$

The character ring $R(G)$ is a free abelian group generated by irreducible characters of complex representations of G (with multiplication). Elements of $R(G)$ are virtual characters (formal differences of characters).

13. Weyl Integration Formula

Let G be a compact, connected Lie group. Recall that G has a unique (up to a constant multiple) regular Borel measure dg that is invariant under left translations and such that it is nonstupid (ie $\mu_G(U) > 0$ for open nonempty sets U). This dg is called a Haar measure and is automatically right-invariant.

We would like to compute

$$\int_G f(g)\ dg$$

for integrable f. If f is a class function, i.e. $f(hgh^{-1}) = f(g)$ for all $g, h \in G$, then one can express this integral as an integral over a maximal torus T in G, since every element of g is conjugate to an element of T. This formula is called the Weyl integration formula.

Theorem 13.1. *(Weyl Integration Formula) If f is a class function and if dg and dt are normalized Haar measures on G and T, then*

$$\int_G f(g)\ dg = \frac{1}{|W|}\int_T f(t)\det\left(\left[Ad\left(t^{-1}\right) - I\right]\big|_{\mathfrak{p}}\right)\ dt.$$

Here W is the Weyl group of G, that is

$$W = N(T)\diagup T,$$

a finite group, and $|W|$ is the order of the Weyl group. The adjoint representation Ad is defined as follows. Recall that G acts on itself by conjugation. For every $h \in G$,

$$c_h(g) = hgh^{-1}$$

is an isomorphism. The adjoint representation of G on its Lie algebra $\mathfrak{g}$ is

$$\begin{aligned} Ad(h)\,Y \quad &: \quad = dc_h(Y) = \left.\frac{d}{d\tau}\right|_{\tau=0} he^{\tau Y}h^{-1} \\ &= \quad hYh^{-1} \text{ if } G \subset GL(n,\mathbb{C}). \end{aligned}$$

Note that since G is compact, there is an Ad-invariant inner product on $\mathfrak{g}$, so this may be made to be a unitary representation. Now, let $\mathfrak{t} \subset \mathfrak{g}$ denote the Lie algebra of T, and let $\mathfrak{c} = \mathfrak{t}^{\perp}$, so that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ is an orthogonal direct sum. Since $Ad(t)$ maps $\mathfrak{t}$ to $\mathfrak{t}$, the unitary-ness means that $Ad(t)$ maps $\mathfrak{p}$ to $\mathfrak{p}$.

Example 13.2. $G = U(n)$. *Then* $T = \{\operatorname{diag}(t_1, ..., t_n)\}$. *Then*

$$\int_{U(n)} f\ dg = \frac{1}{n!}\int_0^{2\pi}...\int_0^{2\pi} f(t_1,...,t_n)\prod_{i<j}|t_i - t_j|^2\,\frac{d\theta_1}{2\pi}...\frac{d\theta_n}{2\pi},$$

where $t_j = e^{i\theta_j}$. *Note that*

$$\mathfrak{u}(n) = \left\{X \in M(n,\mathbb{C}) : X + \overline{X^T} = 0\right\}.$$

Note that

$$\mathfrak{t} = \mathfrak{u}(n) \cap \{\text{diagonal matrices}\}.$$

An Ad-invariant inner product

$$\langle X, Y\rangle = \operatorname{tr}\left(X\overline{Y^T}\right).$$

Trick: expand computations to $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{p}\subset\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{u}(n)$*. Now* $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{u}(n)=M(n,\mathbb{C})$*, since every matrix* A *may be written*

$$A=\frac{A-\overline{A^T}}{2}+i\frac{A+\overline{A^T}}{2i},$$

and both parts are skew Hermitian. Let E_{ij} *be the matrix with* 1 *in the* ij *place and* 0 *otherwise. Then*

$$\begin{aligned}\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{p} &= (\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{t})^{\perp}\\ &= \operatorname{span}\{E_{ij}: i\neq j\}.\end{aligned}$$

Note that E_{ij} *are eigenvectors of* Ad*. Note that*

$$\begin{aligned}Ad\left(t^{-1}\right)E_{ij} &= t^{-1}E_{ij}t\\ &= t_i^{-1}t_jE_{ij}.\end{aligned}$$

The determinant is

$$\begin{aligned}\det\left(\left[Ad\left(t^{-1}\right)-I\right]\Big|_{\mathfrak{p}}\right) &= \prod_{i\neq j}\left(t_i^{-1}t_j-1\right)\\ &= \prod_{i<j}\left(t_i^{-1}t_j-1\right)\left(t_j^{-1}t_i-1\right)\\ &= \prod_{i<j}\left(t_i-t_j\right)\left(t_i^{-1}-t_j^{-1}\right)\\ &= \prod_{i<j}\left|t_i-t_j\right|^2.\end{aligned}$$

We need the following for the proof of the integration formula.

Proposition 13.3. *$N(T)$ is a closed subgroup of G. The connected component $N(T)_0$ in $N(T)$ is T, and the Weyl group $W=N(T)\diagup T$ is finite.*

Proof. $N(T)$ is closed since T is closed. We have a well-defined homomorphism $c: N(T)\to \operatorname{Aut}(T)$ defined by

$$c(g)\,t=gtg^{-1}$$

If we think of $T\cong\mathbb{R}^n\diagup\mathbb{Z}^n$, then $\operatorname{Aut}(T)=GL(n,\mathbb{Z})$. Thus, any connected group of automorphisms of T must act trivially. Thus, if $h\in N(T)^0$, then h commutes with each element of T. If $N(T)^0\neq T$, it must contain a one-parameter subgroup $n(t)$. The closure of the group generated by T and $n(t)$ properly contains T and is abelian, a contradiction. Thus, $N(T)^0=T$. The quotient $W=N(T)\diagup T$ is a Lie group, the quotient of a compact Lie group by its connected component of the identity, so it is finite. □

Proposition 13.4. *The centralizer $C(T)$ of a maximal torus T is T.*

Proof. Since $C(T)\subseteq N(T)$, T is of finite index in $C(T)$ by the last proposition. Thus, if $x\in C(T)$. Thus, if $x\in C(T)$, we have $x^n\in T$ for some n. Let t_0 be a generator T. Since the n^{th} power map $T\to T$ is surjective, there exists $t\in T$ such that $x^nt^n=(xt)^n=t_0$. Now, xt is in some maximal torus T' which contains t_0. By maximality, $T=T'$, so $x\in T$. □

Proposition 13.5. *There exists a dense open subset $\Omega\subset T$ such that the $|W|$ elements wtw^{-1} for $w\in W$ are all distinct for each $t\in\Omega$.*

Proof. For any $w \in W$, define $\Omega_w = \{t \in T : wtw^{-1} \neq t\}$. Clearly, Ω_w is open. If $w \neq \mathbf{1}$ and t is a generator of T, then $t \in \Omega_w$. Thus Ω_w contains all generators of the torus, ie the set

$$\{(t_1, ..., t_n) \in T : \{t_1, ..., t_n\} \text{ lin indep over } \mathbb{Q}\}.$$

Kronecker's theorem implies that the set of generators is dense in T. Thus, Ω_w is dense and open, so $\Omega = \bigcap_{\mathbf{1} \neq w \in W} \Omega_w$ is dense and open. □

Proof. (of Weyl Integration Formula)
Let $X = G/T$, which is a manifold. Now consider the map

$$\phi : X \times T \to G$$

defined by $\phi(xT, t) = xtx^{-1}$, which is well-defined. We will use this as a "change of coordinates". Observe that $X \times T$ and G are orientable manifolds of the same dimension. We choose volume elements on $\mathfrak{g}$ and on $\mathfrak{t}$ so that the Jacobians of the exponential maps $\mathfrak{g} \to G$ and $\mathfrak{t} \to T$ are 1. We now compute the Jacobian J_ϕ. Parametrize a neighborhood of xT in X by a chart based on the neighborhood of the origin of $\mathfrak{p}$.

$$\mathfrak{p} \supset U \mapsto xe^U T.$$

Let $t \in T$. We parametrize a neighborhood of $t \in T$ by

$$\mathfrak{t} \supset V \mapsto te^V.$$

We parametrize any element of G by

$$\begin{array}{ccc} (U, V) & \to & \left(xe^U T, te^V\right) \in X \times T \\ & & \downarrow \\ & & xe^U te^V e^{-U} x^{-1} \end{array}$$

We translate on the left by $t^{-1}x^{-1}$ and on the right by x:

$$\begin{array}{ccc} (U, V) & \to & \left(xe^U T, te^V\right) \in X \times T \\ & & \downarrow^{\phi} \\ & & t^{-1}e^U te^V e^{-U} = e^{Ad\left(t^{-1}\right)U} te^V e^{-U} \end{array}$$

The differential $d\phi$ of this map is

$$U \times V \mapsto \left(Ad\left(t^{-1}\right) - I\right)_{\mathfrak{p}} U \times V.$$

The theorem follows after realizing that ϕ is a $|W|$-fold cover on the nice part of T. □

14. Weight and Root Systems

Let G be a compact, connected Lie group of dimension d. Let $T \subset G$ be a maximal torus. Let $k = \dim T =: \operatorname{rank}(G)$. Recall that

(1) Each $g \in G$ is conjugate to some element of T.
(2) If $\phi : G \to GL(V)$ is a complex representation, the corresponding character $\chi_\phi = \operatorname{tr}(\phi) : G \to \mathbb{C}$, a class function.
(3) Two representations are equivalent ($\tilde{}$) iff the corresponding characters are equal.

Lemma 14.1. *If ϕ, ψ are representations on V, then $\phi \tilde{} \psi$ iff $\phi|_T \tilde{} \psi|_T$.*

Proof. $\phi \tilde{} \psi$ iff $\chi_\phi = \chi_\psi$ iff $\chi_\phi|_T = \chi_\psi|_T$ iff $\phi|_T \tilde{} \psi|_T$. □

Since T is abelian, irreducible representations are 1-dimensional, so $\phi|_T = \phi_1 \oplus \phi_2 ... \oplus \phi_n$, with $n = \dim V$. This leads to a bookkeeping device.

Given $\phi : G \to GL(V)$ complex representation, $\phi|_T = \phi_1 \oplus \phi_2 ... \oplus \phi_n$. Let $\mathfrak{t} \cong \mathbb{R}^k =$ the Lie algebra of T. Thus we have

$$\begin{array}{ccc} \mathbb{Z}^k & \to & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathfrak{t} \cong \mathbb{R}^k & \xrightarrow{d\phi_j} & \mathbb{R} \\ \downarrow^{\exp} & & \downarrow^{\exp} \\ T & \xrightarrow{\phi_j} & U(1) \subset \mathbb{C} \end{array}$$

Definition 14.2. *The weight system $\Omega(\phi)$ of a representation ϕ is the set $\left\{ d\phi_j : \mathbb{R}^k \to \mathbb{R} \right\} \in \mathfrak{t}^*$. In fact, $d\phi_j\left(\mathbb{Z}^k\right) \subset \mathbb{Z}$.*

Notation 14.3. $\Omega(\phi)$ *=weight system. For $\omega \in \Omega(\phi)$, $m(\omega, \phi) =$ multiplicity of ω in ϕ.*

Note that the weight system of a real representation is the corresponding weight system of its complexification.

Application to adjoint representation:

$$\begin{array}{rcl} c & : & G \times G \to G \\ (g, x) & \mapsto & gxg^{-1} = c(g)\, x \end{array}$$

For fixed $g \in G$, $c(g, \cdot) : G \to G$ maps the identity to itself, and so it induces a map on the Lie algebra, so one obtains the differential $Ad(g) : \mathfrak{g} \to \mathfrak{g}$, or

$$Ad : G \to Aut(\mathfrak{g}) \subset GL(\mathfrak{g}) = GL\left(\mathbb{R}^d\right) \hookrightarrow GL\left(\mathbb{C}^d\right).$$

Definition 14.4. *The root system of G is the system of nonzero weights of the (complexified) adjoint representation, or the nonzero weights of $Ad \otimes \mathbb{C}$, notated $\Delta(G)$.*

Facts:

- multiplicity of 0 is $k = \dim T$
- Thm coming up later implies that nonzero weights have multiplicity 1.

Explicitly, if $T = \left(S^1\right)^k$, $\exp(x_1, ..., x_k) = \left(e^{2\pi i x_1}, ..., e^{2\pi i x_k}\right)$, we have the weights are

$$\begin{aligned} d\phi_j(x_1, ..., x_k) &= \sum_{i=1}^k n_i x_i,\ n_i \in \mathbb{Z}, \\ \phi_j\left(e^{2\pi i x_1}, ..., e^{2\pi i x_k}\right) &= \prod_{i=1}^k e^{2\pi i n_i x_i} \end{aligned}$$

Remark 14.5. *The weight system $\Omega(\phi)$ is a complete invariant of the complex representation ϕ, as is χ_ϕ.*

Remark 14.6. *These invariants are related by the following. For $t \in \mathfrak{t}$, $\chi_{\phi_j(\exp(t))} = e^{2\pi i d\phi_j(t)}$. Thus,*

$$\chi_{\phi(\exp(t))} = \sum_{w \in \Omega(\phi)} e^{2\pi i w(t)} \text{ (with multiplicity)}$$

Remark 14.7. *Facts about characters:*

(1) $\chi_{\phi\oplus\psi} = \chi_\phi + \chi_\psi \Longrightarrow \Omega(\phi\oplus\psi) = \Omega(\phi)\cup\Omega(\psi)$
(2) $\chi_{\phi\otimes\psi} = \chi_\phi\chi_\psi \Longrightarrow \Omega(\phi\otimes\psi) = \Omega(\phi)+\Omega(\psi)$
(3) $\chi_{\phi^*} = \overline{\chi_\phi} \Longrightarrow \Omega(\phi^*) = -\Omega(\phi)$

Remark 14.8. *The zero weight has multiplicity k in $\Omega(Ad\otimes\mathbb{C})$. Note that Ad is trivial on the torus, so the multiplicity must be at least k.*

$c: G\times G\to G$ conjugation

$Ad: G\times\mathfrak{g}\to\mathfrak{g}$ Adjoint representation

$Ad|_T : T\times\mathfrak{g}\to\mathfrak{g}$ Adjoint representation, restricted to T. Let $\mathfrak{t}$ denote Lie algebra of T in $\mathfrak{g}$, which is contained in the fixed point set of $Ad|_T$ in $\mathfrak{g}$. Next, take the orthogonal complement $\mathfrak{w}$ with respect to an Ad-invariant metric on $\mathfrak{g}$, so that $\mathfrak{g}=\mathfrak{t}\oplus\mathfrak{w}$. The claim is that T fixes no nonzero w in $\mathfrak{w}$. If T fixes $\langle w\rangle\subset\mathfrak{w}$. Applie exp*, so that $T\times G\to G$ has a T-fixed 1-parameter subgroup $\phi:\mathbb{R}\to G$. Since T fixes all of the subgroup, and $T\subset\langle T,\phi(1)\rangle$, which implies $\phi(t)\in T$ for all t, which implies by maximality that $\phi'(0)=w\in\mathfrak{t}$, a contradiction. Next, when we tensor with $\mathbb{C}$, the adjoint action fixed $\mathfrak{w}$ and $i\mathfrak{w}$, and the same proof applies.*

Thus $\Delta(G)$ consists of $d-k$ nonzero weights.

Example 14.9. *Consider $U(n)$ acting on $\mathbb{C}^n$. $\mu: U(n)\to GL(\mathbb{C}^n)$, $d=n^2$, $k=n$,*

$$T = diag\left(e^{2\pi i\theta_1},...,e^{2\pi i\theta_n}\right),$$

so $\mu|_T = \phi_1\oplus\phi_2\oplus...\oplus\phi_n$, with ϕ_j acting on the j^{th} component of $\mathbb{R}^n$, so $d\phi_j(\theta)=\theta_j$. So

$$\begin{aligned}\Omega(\mu) &= \{\theta_j : 1\le j\le n\}\\ \chi_\mu|_T &= \sum_{j=1}^n e^{2\pi i\theta_j}.\end{aligned}$$

Then

$$\begin{aligned}\Omega(\mu^*) &= \{-\theta_j : 1\le j\le n\}\\ \chi_{\mu^*}|_T &= \sum_{j=1}^n e^{-2\pi i\theta_j}\end{aligned}$$

Finally,

$$Ad_{U(n)}\otimes\mathbb{C} = \mu\otimes\mu^*.$$

Root system:

$$\begin{aligned}\chi_{Ad_{U(n)}\otimes\mathbb{C}}\Big|_T &= \left(\sum e^{2\pi i\theta_j}\right)\left(\sum e^{-2\pi i\theta_k}\right) = n+\sum_{j\ne k} e^{2\pi i(\theta_j-\theta_k)},\\ \text{so }\Delta(U(n)) &= \{\theta_j-\theta_k : 1\le j\ne k\le n\}.\end{aligned}$$

Example 14.10. *Let $G=S^3$, which double covers $SO(3)$ as follows. $S^3=\{unit\ quaternions\}$.*

$$S^3\times\mathbb{H}\to\mathbb{H}$$

via $(g,h)\mapsto ghg^{-1}$. $\mathbb{R}\subset\mathbb{H}$ is invariant, and the action decomposes as $\mathbb{H}=\mathbb{R}\oplus\mathrm{Im}\mathbb{H}$. The action

$$S^3\times\mathrm{Im}\mathbb{H}\to\mathrm{Im}\mathbb{H}$$

can be described as follows. For $g=\cos\theta+u\sin\theta$, with $u\in\{x\in\mathrm{Im}\mathbb{H} : |x|=1\}$, the action is rotation about u with angle 2θ. Consequence: the adjoint action $S^3\times S^3\to S^3$ is the *restriction of this action. Choose the maximal torus $T = \mathbb{C} \subset \mathbb{H}$. For $e^{i\theta} \in T$, the orbit is $\left\{ g e^{i\theta} g^{-1} \right\}$ is a 2-sphere of radius $|\sin\theta|$. Thus the conjugacy class of $e^{i\theta}$ is a 2-sphere of radius $|\sin\theta|$ with the real part $\cos\theta$.*

Now S^3 acts on homogeneous polynomials $P_k = \mathbb{C}[z_1, z_2]_k$ of degree k. The action: for $a, b \in \mathbb{C}$, $a + bj \in \mathbb{H}$.

$$\phi_1(a + jb)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$$

gives the action on P_1, and it naturally extends to P_k: $\phi_k : S^3 \to GL(P_k)$, dimension $k+1$. The weight system: $\phi_k|_T$ acts by $\phi_k\left(e^{i\theta}\right) z_1 = e^{i\theta} z_1$, $\phi_k\left(e^{i\theta}\right) z_2 = e^{-i\theta} z_2$. Thus the monomials of degree k are eigenvectors for the action : z_1^k, $z_1^{k-1} z_2$, ..., z_2^k corresponding to eigenvalues $e^{ik\theta}$, $e^{i(k-2)\theta}, ..., e^{-ik\theta}$. Thus

$$\chi_{\phi_k}\left(e^{i\theta}\right) = \sum_{j=0}^{k} e^{i(k-2j)\theta} = \frac{\sin((k+1)\theta)}{\sin(\theta)}.$$

The weight system: $\Omega(\phi_k) = \{k, k-2, ..., -k\}$.

***Claim:** $\{\phi_k\}$ is $\widehat{G} = \widehat{S^3}$. We first check irreducibility:*

$$\int_{S^3} \chi_{\phi_k}(g)\,\overline{\chi_{\phi_k}}(g)\ dg = 1?$$

Note that $\mathrm{Vol}(S^3) = 2\pi^2$, so if dV is standard volume and dg is the Haar measure, then $dg = \frac{1}{2\pi^2} dV$. So

$$\begin{aligned} \int_{S^3} \chi_{\phi_k}(g)\,\overline{\chi_{\phi_k}}(g)\ dg &= \frac{1}{2\pi^2}\int_{S^3} \chi_{\phi_k}(g)\,\overline{\chi_{\phi_k}}(g)\ dV \\ &= \frac{1}{2\pi^2}\int_0^{\pi} \chi_{\phi_k}\left(e^{i\theta}\right)\overline{\chi_{\phi_k}}\left(e^{i\theta}\right)\ 4\pi(\sin\theta)^2\, d\theta \end{aligned}$$

since each conjugacy class of $e^{i\theta}$is a sphere of radius $\sin\theta$. So

$$\begin{aligned} \int_{S^3} \chi_{\phi_k}(g)\,\overline{\chi_{\phi_k}}(g)\ dg &= \frac{1}{2\pi^2}\int_0^{\pi} \frac{\sin^2((k+1)\theta)}{\sin^2(\theta)}\ 4\pi(\sin\theta)^2\, d\theta \\ &= 1. \end{aligned}$$

Thus each ϕ_k is irreducible. Now we check completeness: If ψ is an irreducible complex representation of dimension $k+1$, want to show $\psi\tilde{}\phi_k$. Suppose not; then

$$\int_{S^3} \chi_{\phi_k}(g)\,\overline{\chi_{\psi}}(g)\ dg = 0.$$

But then

$$\begin{aligned} \int_{S^3} \chi_{\phi_k}(g)\,\overline{\chi_{\psi}}(g)\ dg &= \frac{1}{2\pi^2}\int_0^{\pi} \chi_{\phi_k}\left(e^{i\theta}\right)\overline{\chi_{\psi}}\left(e^{i\theta}\right)\ 4\pi(\sin\theta)^2\, d\theta \\ &= \frac{2}{\pi}\int_0^{\pi} \sin((k+1)\theta)\,\overline{\chi_{\psi}}\left(e^{i\theta}\right)\ (\sin\theta)\, d\theta = 0 \end{aligned}$$

for every k. Since $\overline{\chi_{\psi}}\left(e^{i\theta}\right)\sin\theta$ is constant on conjugacy classes, it is an even function of θ. Thus, not all Fourier coefficients could be zero. Contradiction. So $\widehat{G} = \{\phi_k\}$.

In particular, $Ad \otimes \mathbb{C}$ is some multiple of ϕ_2. (In fact, it is the same as ϕ_2.) Thus,

$$\Omega(Ad \otimes \mathbb{C}) = \{-2, 0, 2\},$$

so $\Delta\left(S^3\right)=\{\pm 2\}$.

Example 14.11. *(Consequence of previous example). We can obtain* $\widehat{SO(3)}$ *by using the double cover* $S^3 \to SO(3)$.

$$\{\pm 1\} \to S^3 \xrightarrow{\pi} SO(3) \to GL(V)$$

So any irreducible representation of $SO(3)$ *induces an irreducible representation of* S^3 *that* ± 1 *maps to the trivial representation. (This is a 1-1 correspondence.) So the irreducible representations of* $SO(3)$ *are the same as those* $\{\phi_k : \phi_k(-\mathbf{1}) = \mathbf{1}\}$. *So this is all the* ϕ_k *with* k *even (ie only odd dimensional representations). To compute the weights (* k *even),*

$$\left\{\text{unit complex numbers } e^{it}\right\} = S^1 = T \subset S^3 \to \overline{T} \subset SO(3),$$

can be realized as

$$e^{it} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix}.$$

The maps on the Lie algebra $\mathbb{R}$ *is multiplication by* $k, k-2, ..., -k$, *so we will get weights (since* $d\phi_k = d\pi \circ d\psi_k = 2d\psi_k$*).*

$$\Omega(\psi_k) = \left\{\frac{k}{2}, \frac{k-2}{2}, ..., -\frac{k}{2}\right\}.$$

Thus, the root system of $SO(3)$ *is*

$$\Delta(SO(3)) = \{\pm 1\}.$$

14.1. **Classification of Rank 1 Compact, Connected Lie Groups.** Let G be such a group. We will show that $G = S^1$ or $SO(3)$ or S^3. Let $S^1 \cong T \subset G$ be a maximal torus.

$$\begin{aligned} Ad|_T &= \mathbf{1} \oplus \psi_1 \oplus \psi_2 \oplus ... \\ \mathfrak{g} &= \mathfrak{t} \oplus \mathbb{R}^2(\psi_1) \oplus \mathbb{R}^2(\psi_2) \oplus ... \end{aligned}$$

The map is

$$\psi_j : e^{it} \mapsto \begin{pmatrix} \cos(n_j t) & -\sin(n_j t) \\ \sin(n_j t) & \cos(n_j t) \end{pmatrix}.$$

Put the integers in increasing order

$$n_1 \le n_2 \le ...$$

We may assume $n_j \ge 0$ (up to conjugation – so up to equivalent representations). Examples: if $G = S^3$, $Ad|_T = \mathbf{1} \oplus \psi_1$, $n_1 = 2$. If $G = SO(3)$, $Ad|_T = \mathbf{1} \oplus \psi_1$, $n_1 = 1$.

Claim: $\mathfrak{g}_1 = \mathbb{R} \oplus \mathbb{R}^2(\psi_1) \subset \mathfrak{g}$ is a Lie subalgebra.

Proof: Assume $n_1 > 0$. Choose $H\tilde{}1 \in \mathbb{R}$, $X, Y \in \mathbb{R}^2(\psi_1)$ such that

$$\begin{aligned} Ad(\mathrm{Exp}(tH))X &= \cos(n_1 t)X + \sin(n_1 t)Y \\ Ad(\mathrm{Exp}(tH))Y &= -\sin(n_1 t)X + \cos(n_1 t)Y \end{aligned}$$

We differentiate at $t = 0$ to obtain

$$\begin{aligned} [H, X] &= n_1 Y \\ [H, Y] &= -n_1 X. \end{aligned}$$

Further, by using the Jacobi identity

$$[H,[X,Y]] = [[H,X],Y] + [X,[H,Y]] = 0,$$

so $H = c\,[X,Y]$ (otherwise there would be a 2-dim torus). Thus $\mathfrak{g}_1$ is a Lie subalgebra.

Claim: $c > 0$. $Ad\,(\mathrm{Exp}\,(tX))$ preserves angles and is orthogonal, so

$$\langle Ad\,(\mathrm{Exp}\,(tX))\,Y, Ad\,(\mathrm{Exp}\,(tX))\,H\rangle = \langle Y, H\rangle\,.$$

Differentiating,

$$\langle [X,Y]\,, H\rangle + \langle Y, [X,H]\rangle = 0,$$

so

$$c\,|H|^2 = -\,\langle Y,[X,H]\rangle = n_1\,|Y|^2\,,$$

so $c > 0$.

Claim: $\mathfrak{g}_1$ is isomorphic to the Lie algebra of $SO\,(3)$.

Claim:only 3 possibilities.

Case1: if G is commutative, then $G = S^1$.

We now assume G is not commutative. Use $\mathfrak{g}_1 \subset \mathfrak{g}$ (isomorphic to S^3, simply connected). Lie's theorem implies that there exists a Lie group homomorphism

$$\Phi : S^3 \to G$$

with

$$d\Phi_e : \mathfrak{g}_1 \hookrightarrow \mathfrak{g}.$$

What is the image of Φ?

$$\begin{array}{ccc} S^3 & & \\ \downarrow^{\pi} & \searrow & \\ \mathrm{Im}\Phi & \subset & G \end{array}$$

Note $\pi =$ Lie group mod a cover map. What is $\ker\pi$? It must be a discrete normal subgroup, which implies $\ker\pi \subset Z\,(S^3)$. (Since $gNg^{-1} = N$ is true for all g, therefore, since g can be chosen sufficiently small, it maps $gng^{-1} = n$, so n commutes with a neighborhood of the identity). Since $Z\,(S^3) = \{\pm 1\}$, $S^3 \diagup \ker\pi = S^3$ or $SO\,(3)$. So we just need to show Φ is onto. If $\Phi\,(S^3) = G_1 \neq G$, look at $V = \bigoplus\limits_{j>1} \mathbb{R}^2\,(\psi_j) = \mathfrak{g}_1^{\perp}$. Then $S^1 = T$ acts on V (by ϕ). Look at the weight system $\Omega\,(\phi \otimes \mathbb{C}) = \Omega\,(\phi|_T \otimes \mathbb{C}) = \bigcup\limits_{j>1} \Omega\,(\psi_j \otimes \mathbb{C})$.

Two cases:

- $G_1 = SO\,(3) \implies$ any irred representation contains the zero weight, but the zero weight does not appear in $\bigcup\limits_{j>1} \Omega\,(\psi_j \otimes \mathbb{C})$.
- $G_1 = S^3 \implies n_1 = 2$ ($\Delta = \{\pm 2\}$), so $n_j \geq 2$ for all j. Then $\bigcup\limits_{j>1} \Omega\,(\psi_j \otimes \mathbb{C}) = \bigcup\limits_{j>1} \{\pm n_j\}$, no zero or ± 1 appears. But for any irreducible representation of S^3, $\Omega = \{k, k-2, ..., -k\}$ must contain 0 or ± 1.

Thus the only rank 1 connected compact Lie groups are S^1, $SO\,(3)$, S^3.

14.2. **Multiplicity Theorem.** The group G acts on $\mathfrak{g}$ via the adjoint representation.

Theorem 14.12. *The multiplicity of each nonzero weight in* $\Omega\left(Ad_G \otimes \mathbb{C}\right)$ *is one. Moreover, if* $0 \neq \alpha \in \Delta(G)$, *then* $k\alpha \in \Delta(G)$ *iff* $k = \pm 1$. *(The multiplicity of the zero weight is the rank of* G.*)*

Proof. Let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t}$.

$$\begin{aligned} Ad|_T &= \mathbf{1} \oplus \psi_1 \oplus \psi_2 \oplus ... \\ \mathfrak{g} &= \mathfrak{t} \oplus \sum \mathbb{R}^2 \left(\psi_j\right)_{(\pm\alpha)} \end{aligned}$$

as a direct sum of $Ad|_T$-invariant subspaces. (Subscript denotes weight.) $\{\pm\alpha\}$ runs through all pairs of nonzero weights in $\Omega\left(Ad_G \otimes \mathbb{C}\right)$ with multiplicity. For $H \in \mathfrak{t}$, the action of $Ad\left(\exp H\right)$ on $\mathbb{R}^2\left(\psi_j\right)_{(\pm\alpha)}$ is

$$\begin{pmatrix} \cos\left(2\pi\alpha\left(H\right)\right) & -\sin\left(2\pi\alpha\left(H\right)\right) \\ \sin\left(2\pi\alpha\left(H\right)\right) & \cos\left(2\pi\alpha\left(H\right)\right) \end{pmatrix}$$

We want each α to appear exactly once. Fix such an α. Let $\mathfrak{t}_\alpha = \ker\left(\alpha : \mathfrak{t} \to \mathbb{R}\right)$. Then let $T_\alpha \subset T$ be a codimension 1 subtorus with $\mathfrak{t}_\alpha$ as its Lie algebra. Let $G_\alpha = Z_G^0\left(T_\alpha\right) =$ the connected component of the centralizer of T_α in G. Let $\widetilde{G_\alpha} = G_\alpha \diagup T_\alpha$. (Remark: $Z_G\left(T_\alpha\right)$ is actually connected in the first place.) Let $\mathfrak{g}_\alpha$ be the Lie algebra of G_α. Note that $\mathfrak{g}_\alpha = F\left(T_\alpha, \mathfrak{g}\right)$ is the fixed point set of T_α in $\mathfrak{g}$. Then we write

$$\mathfrak{g}_\alpha = F\left(T_\alpha, \mathfrak{g}\right) = \mathfrak{t} \oplus \sum_{0 \neq \beta, \mathfrak{t}_\beta = \mathfrak{t}_\alpha} \mathbb{R}^2\left(\psi_j\right)_{(\pm\beta)}$$

Thus,

$$\widetilde{\mathfrak{g}_\alpha} = \mathfrak{g}_\alpha \diagup \mathfrak{t}_\alpha = \mathfrak{t} \diagup \mathfrak{t}_\alpha \oplus \sum_{0 \neq \beta, \mathfrak{t}_\beta = \mathfrak{t}_\alpha} \mathbb{R}^2\left(\psi_j\right)_{(\pm\beta)}.$$

Thus, $\mathfrak{t} \diagup \mathfrak{t}_\alpha \cong S^1$ is a max torus in $\widetilde{G_\alpha}$. Thus, by previous work, $\widetilde{G_\alpha} = S^1$, $SO\left(3\right)$, or S^3. So there is only one element in the summation. But since $\mathfrak{t}_\beta = \mathfrak{t}_\alpha$ iff $\beta = c\alpha$ or $c\beta = \alpha$, each weight must have multiplicity 1. $\square$

15. Lie algebras and their root systems

Some definitions:

Definition 15.1. *A Lie group is called **simple** if all its normal subgroups are discrete.*

All simple, simply connected, compact Lie groups are known. They come in four families and five exceptional groups. (See the table in any Lie group book.) We restrict ourselves to compact, connected Lie groups.

Theorem 15.2. *Every compact, connected Lie group has the form* $G = K \diagup H$, *where* K *is a finite product of* $SO\left(2\right)$*'s and the groups listed in the table, and* H *is a discrete subgroup of* $Z\left(K\right)$ *(the center of* K*).*

Recall:

Theorem 15.3. *We have*

(1) *For any real Lie algebra* $\mathfrak{g}$, *there exists a Lie group* G *such that* $\mathcal{L}\left(G\right) = \mathfrak{g}$, *and this group is unique up to a local isomorphism.*

(2) *Among all connected Lie groups G such that $\mathcal{L}(G) = \mathfrak{g}$, there is exactly one that is simply connected Lie group $\widetilde{G}$.*
(3) *Every other group is $\widetilde{G}/H$, where H is in the center of $\widetilde{G}$.*

Definition 15.4. *A Lie group is called **semisimple** if the corresponding Lie algebra is semisimple.*

Remark 15.5. *A compact, connected G is semisimple if and only if $Z(G)$ is finite.*

15.1. Types of Lie algebras.

Definition 15.6. *A Lie algebra $\mathfrak{g}$ is an **extension** of a Lie algebra $\mathfrak{g}_1$ by a Lie algebra $\mathfrak{g}_2$ if there exists the following short exact sequence of Lie algebra homomorphisms:*

$$0 \to \mathfrak{g}_2 \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{g}_1 \to 0$$

That is, $\mathfrak{g}$ contains an ideal isomorphic to $\mathfrak{g}_2$, and $\mathfrak{g}_1 \cong \mathfrak{g}/\mathfrak{g}_2$.

Definition 15.7. *An extension as above is called **central** if $i(\mathfrak{g}_2) \subseteq Z(\mathfrak{g})$, **trivial** if the map p admits a section (i.e. a homomorphism $s : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $p \circ s = \mathbf{1}$. Then $\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ (semidirect product).*

The types of Lie algebras:

(1) **commutative**, ie $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.
(2) **solvable**, ie minimal collection of Lie algebras containing all abelian Lie algebras and is closed under extensions.
(3) **nilpotent**, i.e. minimal collection of Lie algebras containing all abelian Lie algebras and is closed under central extensions.
(4) **semisimple**, i.e. minimal collection of Lie algebras containing all nonabelian **simple** Lie algebras and is closed under extensions. Simple means it has no proper ideals.

Theorem 15.8. *(Levi) Any Lie algebra $\mathfrak{g}$ has a unique maximal solvable ideal $\mathfrak{t}$, and the quotient Lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{t}$ is semisimple, and $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{t}$.*

Theorem 15.9. *(Cartan) Any semisimple Lie algebra is isomorphic to a direct sum of simple Lie algebras.*

Thus, in order to classify Lie algebras, we need to

(1) describe all simple Lie algebras
(2) describe all solvable Lie algebras
(3) describe all semidirect products of $\mathfrak{g}_1 \ltimes \mathfrak{g}_2$, where $\mathfrak{g}_1$ is semisimple and $\mathfrak{g}_2$ is solvable.

According to Kirillov, (1) above is known (Dynkin, etc.), but (2) and (3) are hopeless. We thus restrict to semisimple Lie algebras.

15.2. Abstract root systems.

Definition 15.10. *A finite set $R \subset \mathbb{R}^n$ is called a **root system** if it satisfies*

(1) *(R1) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha,\beta \in R$.*
(2) *(R2) $S_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in R$ for all $\alpha,\beta \in R$.*

Example 15.11. *The root system for $SO(n)$ is*

$$\{e_i - e_j, i \neq j, 1 \leq i, j \leq n\}.$$

Geometrically, (R1) means that the angles between vectors α and β can only be $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{4}, \frac{2\pi}{3}$ (exercise). Moreover, for each angle between α and β, the ratio $\frac{|\alpha|}{|\beta|}$ must be one of $\left\{1, 2, 3, \frac{1}{2}, \frac{1}{3}\right\}$. The second condition (R2) implies the following. Let M_α denote the hyperplane in $\mathbb{R}^n$ orthogonal to α. Let S_α denote the reflection wrt M_α. Then $\beta \in R$ implies $S_\alpha(\beta)$ is in R for all $\alpha, \beta \in R$.

Similar root systems are those that are equivalent under orthogonal transformations and dilations.

There are some special root systems which satisfy additional axioms. A root system could be

(1) **nondegenerate** if R spans $\mathbb{R}^n$.
(2) **indecomposable** if R cannot be written as $R = R_1 \sqcup R_2$ with $R_1 \perp R_2$.
(3) **reduced** if $\alpha \in R$ implies $2\alpha \notin R$.
(4) **simply-laced** if all $\alpha \in R$ have the same length.

Note that any root system may be made non-degenerate by replacing $\mathbb{R}^n$ by the span of R. The *rank* of a root system is the defined to be the dimension of the span of R.

Example 15.12. *For all $n \in \mathbb{Z}_+$, there is only one indecomposable, non-reduced, non-simply laced root system of rank n. This root system is called BC_n.*

$$BC_n = \{\pm e_i \pm e_j, \pm e_k, \pm 2e_k\}_{i \neq j, k}$$

(simply laced systems are called ADE systems)

Note that each reflection in $\mathbb{R}^n$ is an orthogonal transformation. The subgroup W of $O(n)$ generated by reflections $S_\alpha, \alpha \in R$ is called the *Weyl group* of R. Then $\mathbb{R}^n \setminus \{M_\alpha : \alpha \in R\} = \coprod C_i^\circ$ is a disjoint union of the connected components, called the open Weyl chambers. The Weyl chambers are $C_i = \overline{C_i^\circ}$, the closures of the open Weyl chambers.

We say that the vector $v \in \mathbb{R}^n$ is *regular* if $v \in C_i^\circ$ for some i and *singular* if $v \in M_\alpha$ for some α.

Linear orders on $\mathbb{R}^n$: (eg lexicographical order)

Lemma 15.13. *Any linear order on $\mathbb{R}^n$ is a lexicographic order induced by the choice of basis in $\mathbb{R}^n$.*

Choose a linear order on $\mathbb{R}^n$; this induces a corresponding order on R. This divides the roots into positive and negative roots. Let R_+ (resp. R_-) be the set of positive (negative) roots. It turns out that R has finitely many orders.

Proposition 15.14. *For any linear order on $\mathbb{R}^n$, the convex cone generated by R_+ is exactly the dual cone to one of the Weyl chambers.*

Given a $K \subset \mathbb{R}^n$ is a cone, its *dual cone* K^* is $\{v : (v, \alpha) \geq 0, \text{ for all } \alpha \in K\}$. Then observe that $(K^*)^* = K$. We can define the *positive Weyl chamber* to be

$$C_+ := \{v \in \mathbb{R}^n : (v, \alpha) \geq 0, \text{ for all } \alpha \in R_+\}.$$

Definition 15.15. *$\alpha \in R_+$ is* ***decomposable*** *if $\alpha = \beta + \gamma$ for some $\beta, \gamma \in R_+$. Otherwise, the root is called indecomposable or a* ***simple root****.*

Lemma 15.16. *Every root $\alpha \in R$ has a unique decomposition $\alpha = \sum c_k \alpha_k$, where each α_k is simple, and the c_k are either all positive integers or all negative integers.*

Proof. First, it is obvious that every positive root is the sum of simple roots. Similarly true for negative roots. Uniqueness follows from the fact that if α_i and α_j are simple, then $(\alpha_i, \alpha_j) < 0$. (Proof: assume $(\alpha_i, \alpha_j) > 0$ for some i, j. Then $k = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}_+$. If $(\alpha_i, \alpha_i) > (\alpha_j, \alpha_j)$, then $\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \le 2\frac{\|\alpha_i\|\|\alpha_j\|}{(\alpha_i,\alpha_i)} < 2$, so $k = 1$. Then $S_{\alpha_i}(\alpha_j) = \alpha_j - \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}\alpha_i = \alpha_j - \alpha_i$. If $S_{\alpha_i}(\alpha_j)$ is simple, it is either positive or negative, but then a simple root would be decomposable. This is a contradiction.) Next, suppose $\sum c_k\alpha_k = \sum d_k\alpha_k$. Then $\sum (c_k - d_k)\alpha_k = 0$, so there is some relation between simple roots, so $v = \sum_{i\in I} b_i\alpha_i = \sum_{j\in J} e_j\alpha_j$ with all b_i, e_j positive. but then $0 < (v, v) = \sum b_i e_j (\alpha_i, \alpha_j) < 0$. Contradiction. $\square$

Let Π be the set $\{\alpha_1, ..., \alpha_n\}$ of simple roots (we assume R spans $\mathbb{R}^n$, ie R is nondegenerate). The root system R can be reconstructed from Π (if nondecomposable)

Proposition 15.17. *The group W acts simply transitively on the set of Weyl chambers. (simply transitive: transitive and free)*

Recall that

$$C_+ = \{\lambda \in \mathbb{R}^n : (\lambda, \alpha) \ge 0 \text{ for all } \alpha \in R_+\}$$

is the positive Weyl chamber.

Lemma 15.18. *Let $\lambda \in C_+^\circ$, and let $\mu \in \mathbb{R}^n$ be an arbitrary vector. Consider the orbit $W(\mu)$ of μ under W. Then $W(\mu)$ has a unique common point with C_+ which is the nearest point to λ of $W(\mu)$.*

Corollary 15.19. *W acts simply transitively on the set of all linear order relations on R so that the number of order relations on R is the same as the number of Weyl chambers, which is the order of W.*

Corollary 15.20. *The stabilizer in W of any vector λ in $\mathbb{R}^n$ is generated by reflections in the mirrors that contain λ.*

We say that $\lambda \in \mathbb{R}^n$ is **regular** iff $\text{stab}(\lambda) = 0$.

Let $\Pi = \{\alpha_1 \le ... \le \alpha_n\}$ be the system of simple roots. The **Cartan matrix** $A = (A_{ij}) \in M_n(\mathbb{Z})$ is defined by

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, 1 \le i, j \le n$$

Dynkin proved that A determines Π and vice versa. Note that W acts on R by orthogonal transformations. This implies that A does not depend on the choice of linear order, So, W does not change A but permutes orders.

Dynkin diagrams: The information about our root system that is encoded in A can be represented as a **Dynkin graph** Γ_A. By definition, Γ_A is defined by:

(1) Vertices are labeled by simple roots (or the numbers $1, ..., n$).
(2) Two different vertices i, j are joined by $n_{ij} = A_{ij} \cdot A_{ji}$ edges.
(3) If $|\alpha_i| > |\alpha_j|$, we add arrow directed from i to j.

The **Dynkin diagram** is defined to be the corresponding undirected graph.

The following properties can be used to show that A can be reconstructed from the graph Γ_A.

Proposition 15.21. *We have*

(1) $A_{ii} = 2$.

(2) $A_{ij} \leq 0$ *and* $A_{ij} = 0$ *iff* $A_{ji} = 0$.
(3) *All principal minors of* A *are positive. In particular,* $n_{ij} = A_{ij} \cdot A_{ji} \in \{0, 1, 2, 3\}$.

Example 15.22. *List of all Cartan matrices for* $n = 2$*: (*Γ_A *and* R*)*

(1) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
(2) $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.
$A_{ji} = A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$, *so* $|\alpha_1| = |\alpha_2| = 1$.
(3) $\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$.
$\sqrt{2}\,|\alpha_1| = |\alpha_2|$
(4) $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ *or* $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$
$\sqrt{3}\,|\alpha_1| = |\alpha_2|$, *angle between* α_1 *and* α_2 *is* $150°$.

See http://en.wikipedia.org/wiki/Dynkin_diagram#Rank_1_and_rank_2_examples for the corresponding root systems and diagrams.

16. Examples of Root Systems

Recall that studying root systems, Dynkin diagrams, and Cartan matrices are equivalent.

Example 16.1. *The system* A_n*. Let* $\{e_i\}_{0 \leq i \leq n}$ *be the standard basis in* $\mathbb{R}^{n+1}$*. Let*

$$R = \{\alpha_{ij} := e_i - e_j \mid 0 \leq i \neq j \leq n\}$$

This is a root system for $SU(n+1)$*. This is degenerate of tank* n*, indecomposable, reduced (*$\alpha \in R \Rightarrow 2\alpha \notin R$*), and simply laced (all* α_{ij} *have same length).* $|R| = n^2 - n$*. Let* $x^0, ..., x^n$ *be the standard coordinates in* $\mathbb{R}^{n+1}$*. The mirrors* M_{ij} *are given by* $x^i = x^j$*. They split* $\mathbb{R}^{n+1}$ *into* $(n+1)!$ *Weyl chambers. With the standard lexicographic order on* $\mathbb{R}^{n+1}$*,*

$$R_+ = \{\alpha_{ij} : i < j\}.$$

The positive Weyl chamber C_+ *is*

$$C_+ = \left\{\lambda \in \mathbb{R}^{n+1} : (\lambda, \alpha) \geq 0, \alpha \in R_+\right\}.$$

Note that

$$\#\,(\text{Weyl chambers}) = |W|.$$

Note that C_+ *is defined by* $x^0 \geq x^1 \geq ... \geq x^n$*. Also,* W *is generated by permutations* $x^i \leftrightarrow x^j$*, so* $W \cong S_{n+1}$*. The system of simple roots :*

Lemma 16.2. *The system of simple roots for* A_n *is* $\Pi = \{\alpha_k := \alpha_{k-1,k} \mid 1 \leq k \leq n\}$.

Proof. If $j > i+1$, then $\alpha_{ij} = \alpha_{i,i+1} + \alpha_{i+1,j}$. By independence, Π must be simple roots. □

Example 16.3. *(Root system* D_n*) Roots system for* $SO(2n, \mathbb{R})$*. Let* $R \subset \mathbb{R}^n$*,* $n \geq 2$*, be the set* $\{\pm e_i \pm e_j\}_{1 \leq i \neq j \leq n}$*. So* $|R| = 2n(n-1)$*. This is a nondegenerate system of rank* n*, indecomposable, reduced, simply laced, and this system contains* A_{n-1} *as a subset. Nevertheless, this is not a sum of two root systems. In this example, the mirrors are* $x^i = \pm x^j$*. The Weyl group is*

$$W = S_n \ltimes \mathbb{Z}_2^{n-1}.$$

This is the semidirect product (G, $N \triangleleft G$ is normal, $H < G$; $G = H \ltimes N$ if one of the following holds

1. $G = NH$ *and* $N \cap H = \{e\}$
2. $G = HN$ *and* $N \cap H = \{e\}$
3. *every* $g \in G$ *s.t.* $g = nh$ *(or* hn*) uniquely*
4. *There is a homomorphism* $G \to H$ *which is the identity on* H *and has kernel* N.

). The positive Weyl chamber is

$$C_+ = \left\{x : x^1 \geq ... \geq x^{n-1} \geq |x^n|\right\}$$

The system of simple roots for D_n *is* $\Pi = \{e_k - e_{k-1} \mid 1 \leq k \leq n-1\} \cup \{e_{n-1} + e_n\}$. *Note that* $e_1 + e_2 = (e_1 - e_2) + ... + (e_{n-1} - e_n) + (e_n + e_2)$, *so this is right.*

Example 16.4. *(Root system* E_8*) The root system* $R \subset \mathbb{R}^8$ *is the set of* $4\begin{pmatrix} 8 \\ 2 \end{pmatrix} + 2^7 = 112 + 128 = 240$, *given by*

$$\{\pm e_i \pm e_j \mid i \neq j\} \cup \left\{\sum_{i=1}^{8} \varepsilon_i e_i \mid \varepsilon_i = \pm 1 \text{ and } \bigcap_{i=1}^{8} \varepsilon_i = 1\right\}.$$

Note that $D_8 \subset E_8$. *The simple roots are*

$$\begin{aligned} \alpha_k &= e_{k+1} - e_{k+2}, 1 \leq k \leq 6, \\ \alpha_7 &= e_7 + e_8, \alpha_8 = \frac{1}{2}(e_1 - ... - e_7 + e_8) \end{aligned}$$

Theorem 16.5. *Connected Dynkin graphs corresponding to reduced root systems form 4 infinite series and 5 isolated examples:*

$$\begin{aligned} A_n &: \circ - ... - \circ, n \geq 1 \\ B_n &: \circ - ... - \circ \Rightarrow \circ, n \geq 2 \\ C_n &: \circ - ... - \circ \Leftarrow \circ, n \geq 3 \\ D_n &: \circ - ... - \circ \begin{matrix} -\circ \\ -\circ \end{matrix}, n \geq 4 \\ E_n &: \circ - ... - \circ \begin{matrix} -\circ \\ -\circ - \circ \end{matrix}, n = 6, 7, 8 \\ F_4 &: \circ - \circ \Rightarrow \circ - \circ \\ G_2 &: \circ \Rrightarrow \circ \end{aligned}$$

Note $A_1 \cong B_1 \cong C_1, B_2 \cong C_2, A_3 \cong D_3, E_5 \cong D_5, E_4 \cong A_4, E_3 \cong A_2 + A_1$.

17. Weyl character formula and more

17.1. **The Main Results.** Statements of results to follow. Let K be a simple compact Lie group.

Theorem 17.1. *We have:*

1. *Any unitary irreducible representation* π *of* K *is finite-dimensional and can be uniquely extended to a holomorphic irreducible representation of the simply connected complex Lie group* G *such that* $\mathfrak{g} = Lie\,(K)_{\mathbb{C}}$.

(2) *A unitary irreducible representation π is characterized up to equivalence by its highest weight λ, which can be any dominant weight. (A **dominant weight** is a linear combination of fundamental weights with nonnegative integer coefficients.)*

(3) *(Weyl character formula) The character of a unitary irreducible representation π_λ with heighest weight λ is given by the Weyl formula*

$$\chi_\lambda(t) = \frac{\sum\limits_{w\in W} (-1)^{l(w)} e^{w\cdot(\lambda+\rho)}(t)}{\sum\limits_{w\in W} (-1)^{l(w)} e^{w\cdot(\rho)}(t)},$$

where $\rho = \frac{1}{2}\sum \alpha_k$, with the sum over all simple positive roots.

(4) *The dimension of π_λ is given by*

$$d_\lambda = \prod_{\alpha\in R_+} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}.$$

(5) *The infinitessimal character of π_λ takes the value*

$$I_\lambda(\Delta_2) = (\lambda+2\rho,\lambda) = |\lambda+\rho|^2 - |\rho|^2$$

on the quadratic Casimir element Δ_2.

(6) *The multiplicity of the weight μ in the unitary irreducible representation π_λ is*

$$m_\lambda(\mu) = \sum_{w\in W} (-1)^{l(w)} P(w\cdot(\lambda+\rho) - (\mu+\rho)),$$

where P is the Kostant partition function on the root lattice.

17.2. **Killing Form.** Let $\mathfrak{g}$ be any (real or complex) finite dimensional Lie algebra. Recall that it has a distinguished representation ad which acts on itself:

$$ad(X) : [X,\cdot] \in End(\mathfrak{g}).$$

There is an invariant bilinear form on $\mathfrak{g}$ called the Killing form. Note that this is the differential of $Ad : G \to End(\mathfrak{g})$ at e, where Ad is defined by

$$Ad(g)X = gXg^{-1}.$$

Definition 17.2. *The Killing form on $\mathfrak{g}$ is defined to be*

$$K(X,Y) = tr(ad(X)\circ ad(Y)).$$

Theorem 17.3. *We have*

(1) $K(X,Y) = K(Y,X)$.
(2) $K(Ad(g)X, Ad(g)Y) = K(X,Y)$
(3) $K(ad(Z)X,Y) = -K(X,ad(Z)Y)$

Proof. (1) is obvious. (2) is a consequence of the formula $ad(Ad(g)X) = Ad(g)\,ad(X)\,Ad(g)^{-1}$. (3): differentiate (2) with respect to a family $g(t)$ with $g'(0) = Z$:

$$0 = \frac{d}{dt} K(Ad(g)X, Ad(g)Y)|_{t=0} = K(ad(Z)X,Y) + K(X,ad(Z)Y).$$

□

Remark 17.4. *For semisimple compact groups G, the Killing form on $\mathfrak{g}$ is negative definite. If G is compact, then $\mathfrak{g}$ has a unique normalized, biinvariant inner product. Moreover, $ad(X)$ is skew adjoint for this inner product (same proof as above). Then*

$$K(X,X) = tr(ad(X)\,ad(X)) = -tr\left(ad(X)(ad(X))^*\right),$$

which must be a negative operator. The definiteness comes from the definition of semisimpleness.

17.3. **Noli me necare, cape omnias pecunias meas! (No! We don't have a recession!) (Roots and weights, redefined).** Let G be a compact, connected Lie group. Let π be an irreducible representation of G on an n-dimensional complex vector space V. We may choose a Hermitian inner product so that π is unitary. This representation is completely determined by its character $\chi_\pi : G \to \mathbb{C}$. Let T be a maximal torus. All complex irreducible representations of T are 1-dimensional.

Definition 17.5. *A* ***weight*** *is an irreducible representation of T. For any representation (π, V) of G, the* ***weight space*** *corresponding to a given weight is the subspace of V on which T acts by a given weight.*

There are several ways to label weights, i.e. either in terms of T or of the Lie algebra $\mathfrak{t}$. Now, $T = (S^1)^k = (\mathbb{R}/\mathbb{Z})^k$. Then the weight is a choice of k integers $\bar{n} = (n_1, ..., n_k)$, equivalent to $\theta_{\bar{n}} : ([x_1], ..., [x_k]) \mapsto e^{2\pi i(n_1x_1+...+n_kx_k)}$. Also, the corresponding map on Lie algebra is $(\theta_{\bar{n}})_* : (x_1, ..., x_k) \mapsto n_1x_1 + ... + n_kx_k$ (this is really $\frac{1}{2\pi i} d\theta_{\bar{n}}$). Note that

$$(\theta_{\bar{n}})_* : \mathfrak{t} \to \mathbb{C},$$

so it is an element of $\mathfrak{t}^*$ (dual space). Thus, the weights are elements of $\mathfrak{t}^*$ that take integer values on the integer lattice inside $\mathfrak{t}$ (integer lattice $= \exp^{-1}(e)$, $\exp : \mathfrak{t} \to T$)

Now we will consider representations of T on the real vector space. In this case irreducible representations are homomorphisms

$$\theta : T \to SO(2).$$

Two types of irreducible representations:

(1) The one-dimensional trivial representation on $\mathbb{R}$.
(2) Nontrivial representations on $\mathbb{R}^2$.

We let

$$\theta_{\bar{n}}([x_1], ..., [x_k]) = \begin{pmatrix} \cos(2\pi(\theta_{\bar{n}})_*(x_1, ..., x_k)) & -\sin(2\pi(\theta_{\bar{n}})_*(x_1, ..., x_k)) \\ \sin(2\pi(\theta_{\bar{n}})_*(x_1, ..., x_k)) & \cos(2\pi(\theta_{\bar{n}})_*(x_1, ..., x_k)) \end{pmatrix}.$$

Note that $\theta_{\bar{n}}$ and $-\theta_{\bar{n}}$ are equivalent representations (conjugate by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.)

Definition 17.6. *A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called an* ***ideal*** *if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.*

If $H \subset G$ is a Lie subgroup, then $\mathfrak{h} \subset \mathfrak{g}$ is an ideal iff H is normal.

Definition 17.7. *A Lie algebra $\mathfrak{g}$ is called* ***simple*** *iff it has no nontrivial proper ideals and it is not 1 dimensional. A Lie group is simple if its Lie algebra is simple.*

Definition 17.8. *A Lie algebra $\mathfrak{g}$ is called* ***semisimple*** *if it has no nontrivial abelian ideals. A Lie group is semisimple if its Lie algebra is.*

For example, $U(n)$ is not semisimple, and $SU(n)$ is semisimple. In $U(n)$, the subgroup of diagonal matrices forms an abelian normal subgroup.

We mostly are interested in real Lie groups.Then $\mathfrak{g}$ is a real vector space. It turns out that the classification of Lie algebras corresponding to compact semisimple real Lie groups is equivalent to the classification of compact semisimple complex Lie groups (via complexification). Each complex semisimple Lie algebra has a unique associated real Lie algebra of a compact Lie group, so that the complex Lie algebra is the complexification of the real one.

17.4. **Complexification.** For a real Lie algebra $(\mathfrak{g}, [\cdot\ ,\ \cdot])$, we define its complexification $(\mathfrak{g}_{\mathbb{C}}, [\cdot\ ,\ \cdot]_{\mathbb{C}})$ as a complex vector space

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$$

with bilinear operation $[\cdot\ ,\ \cdot]_{\mathbb{C}}$ extended from $[\cdot\ ,\ \cdot]$ by linearity. That is, every finite dimensional Lie algebra has a basis

$$[e_i, e_j] = \sum_k C_{ij}^k e_k,$$

where C_{ij}^k are the structure constants. Now, we take complex linear combinations and use the same structure constants to get the complexification.

Proposition 17.9. *(Cartan Criterion) A Lie algebra is semisimple iff the Killing form is nondegenerate.*

If addition, if the group is compact, the Killing form is negative definite. Therefore, the negative of the Killing form is a positive definite inner product on $\mathfrak{g}$.

17.5. **Roots.** Last time, we defined

- weights - complex irreducible representations of a maximal torus.
- complex weights can be labelled by elements of $\mathfrak{t}^*$ (also called weights). If $T = (S^1)^k$, then $(\theta_{\vec{n}})_* \in \mathfrak{t}^*$ is $\theta_{\vec{n}}(x_1, ..., x_k) = n_1 x_1 + ... + n_k x_k$.

Let G be a compact Lie group. A **Cartan subalgebra** of $\mathfrak{g}$ is a Lie subalgebra that is the Lie algebra of a maximal torus T. Equivalently, Cartan subalgebra = maximal abelian (Lie algebra zero) subalgebra.

We wish to consider the adjoint representation $Ad : G \to GL(\mathfrak{g})$ defined by $g \mapsto gAg^{-1}$. We can study the weights of Ad. The nonzero weights of this representation are called roots. The maximal torus T acts trivially on $\mathfrak{t}$ via this representation, so the trivial weight will appear with multiplicity $rank(G) = \dim(T)$. Scott showed that

$$\begin{aligned} \mathfrak{g} &= \mathfrak{t} \oplus \mathfrak{g}/\mathfrak{t} \cong \mathfrak{t} \oplus \mathfrak{t}^{\perp}, \\ Ad(g) &: \mathfrak{t}^{\perp} \to \mathfrak{t}^{\perp}, \end{aligned}$$

$\mathfrak{g}$ has an invariant inner product.

Definition 17.10. *The **roots** of G are the non-trivial weights of the adjoint representation on the real vector space $\mathfrak{g}$. More explicitly, the roots are some nonzero elemetns $\alpha \in \mathfrak{t}^*$, taking integer values in the integer lattice $\exp^{-1}(e)$, where $\exp : \mathfrak{t} \to T$ is the exponential map.*

Thus, $\mathfrak{g}/\mathfrak{t} = \bigoplus \mathfrak{g}_{\alpha_i}$, where $\mathfrak{g}_{\alpha_i}$ is a 2-dimensional vector space called root space of α_i. Scott proved that $\mathfrak{g}_{\alpha_i}$ all have multiplicity one.

Corollary 17.11. *The manifold G/T is even dimensional.*

Now we complexify $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{t}_{\mathbb{C}} \oplus V_{\mathbb{C}}$, where $V = \mathfrak{g} \diagup \mathfrak{t}$. Choosing a complex structure on $V_{\mathbb{C}}$ is the same as choosing a symmetric matrix J with $J^2 = -I$. Then $V_{\mathbb{C}} = V^{0,1} \oplus V^{1,0}$ ($\pm i$ eigenspaces of J). This choice allows us to write

$$V_{\mathbb{C}} = \bigoplus_{\text{positive roots } \alpha_i} \left(\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i}\right)$$

by declaring $\alpha_i \in V^{0,1}$ to be "positive". Choosing a complex structure is like choosing a basis $(e_1, ..., ie_1, ...)$.

Here is another confusing part. How does $\mathfrak{t}$ sit inside $\mathfrak{t}_{\mathbb{C}}$?

- as a real part of $\mathfrak{t}_{\mathbb{C}}$
- as $2\pi i\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$

This leads to definitions of *real roots* and *complex roots*.

Definition 17.12. *The* ***real roots*** *are labelled by maps* $\alpha : \mathfrak{t} \to \mathbb{R}$*, integral on* $\exp^{-1}(e) \subset T$*. On* $\mathfrak{g}_\alpha$*,* T *acts by*

$$Ad\left(\left[x_1\right], ..., \left[x_k\right]\right)\big|_{\mathfrak{g}_\alpha} = \begin{pmatrix} \cos\left(2\pi\alpha\left(x_1, ..., x_n\right)\right) & -\sin\left(2\pi\alpha\left(x_1, ..., x_n\right)\right) \\ \sin\left(2\pi\alpha\left(x_1, ..., x_n\right)\right) & \cos\left(2\pi\alpha\left(x_1, ..., x_n\right)\right) \end{pmatrix}.$$

The adjoint action ad of the Lie algebra $\mathfrak{t}$ *on* $\mathfrak{g}$ *is given by*

$$ad\left(x_1, ..., x_k\right)\big|_{\mathfrak{g}_\alpha} = \begin{pmatrix} 0 & -2\pi\alpha\left(x_1, ..., x_n\right) \\ 2\pi\alpha\left(x_1, ..., x_n\right) & 0 \end{pmatrix}.$$

Notice that the eigenvalues of this matrix are $\pm 2\pi i\alpha$*, and complexification allows us to use root spaces that are eigenvectors.*

Definition 17.13. *The* ***complex roots*** *are labelled by complex-linear maps* $\alpha_{\mathbb{C}} : \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$*. Also* $\mathfrak{g}_\alpha \otimes \mathbb{C} = \mathfrak{g}_{\alpha_{\mathbb{C}}} \otimes \mathfrak{g}_{-\alpha_{\mathbb{C}}}$*. The root space is* $\mathfrak{g}_{\alpha_{\mathbb{C}}}$*. Then* T *acts by*

$$Ad\left(\left[u_1\right], ..., \left[u_k\right]\right)\big|_{\mathfrak{g}_{\alpha_{\mathbb{C}}}} = e^{\alpha_{\mathbb{C}}(u_1, ..., u_k)}.$$

The adjoint action ad of the Lie algebra $\mathfrak{t}$ *on* $\mathfrak{g}$ *is given by*

$$\begin{aligned} ad\left(u_1, ..., u_k\right)\big|_{\mathfrak{g}_{\alpha_{\mathbb{C}}}} &= \alpha_{\mathbb{C}}\left(u_1, ..., u_k\right) \\ &= 2\pi i n_1 u_1 + ... + 2\pi i n_k u_k \end{aligned}$$

Notice that the eigenvalues of this matrix are $\pm 2\pi i\alpha$*, and complexification allows us to use root spaces that are eigenvectors.*

One can also think of roots as eigenvalues of the adjoint action. For all $X, Y \in \mathfrak{t}_{\mathbb{C}}$, we have $[X, Y] = 0$. Consider

$$ad\left(X\right) \circ ad\left(Y\right) - ad\left(Y\right) \circ ad\left(X\right) = ad\left(\left[X, Y\right]\right) = 0.$$

Thus the operators $ad\left(X\right)$ commute for $X \in \mathfrak{t}_{\mathbb{C}}$. Secondly, the operators $ad\left(H\right) : V_{\mathbb{C}} \to V_{\mathbb{C}}$ are also skew adjoint (with respect to the invariant inner product) and commuting. Thus, they are simultaneously diagonalizable. Let $X \in \mathfrak{g}_{\alpha_{\mathbb{C}}}$. Then

$$[H, X] = ad\left(H\right) X = \alpha_{\mathbb{C}}\left(H\right) X,$$

and so $\alpha_{\mathbb{C}}\left(H\right)$ is the eigenvalue of $ad\left(H\right)$ on $\mathfrak{g}_{\alpha_{\mathbb{C}}}$. Thus, root spaces are simultaneous eigenspaces, and the roots are eigenvalues.

Lemma 17.14. $\left[\mathfrak{g}_{\alpha_{\mathbb{C}}}, \mathfrak{g}_{\beta_{\mathbb{C}}}\right] \subset \mathfrak{g}_{\alpha_{\mathbb{C}} + \beta_{\mathbb{C}}}$.

Proof. If $H \in \mathfrak{t}_{\mathbb{C}}$, $X_\alpha \in \mathfrak{g}_{\alpha_{\mathbb{C}}}$, $X_{\beta_{\mathbb{C}}} \in \mathfrak{g}_{\beta_{\mathbb{C}}}$, then

$$[H, [X_\alpha, X_\beta]] = -[X_\alpha, [X_\beta, H]] - [X_\beta, [H, X_\alpha]]$$

by the Jacobi identity. Then

$$\begin{aligned} [H, [X_\alpha, X_\beta]] &= [X_\alpha, \beta_{\mathbb{C}}(H) X_\beta] - [X_\beta, \alpha_{\mathbb{C}}(H) X_\alpha] \\ &= (\alpha_{\mathbb{C}}(H) + \beta_{\mathbb{C}}(H)) [X_\alpha, X_\beta]. \end{aligned}$$

□

17.6. **The War on Terrorism via Lie Groups (or $SU(n)$, Weyl chambers, and diagrams of a group).**

17.6.1. ***Example***. $SU(n) = \{A \in GL(n, \mathbb{C}) : A^* A = I \text{ and } \det A = 1\}$. Note $A^* = \overline{A^T}$.
$\mathfrak{g} = \{A \in M(n, \mathbb{C}) : A + A^* = 0, tr(A) = 0\}$ (real vector space). We complexify to get $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C}) = \{A \in M(n, \mathbb{C}) : tr(A) = 0\}$. Idea: every matrix is the sum of a hemitian and skew-hermitian matrix, and B is hermitian iff iB is skew hermitian.

Maximal torus $T = \{\text{diagonal matrices}\} \subset SU(n)$

$$[D_{jj}] = \begin{bmatrix} D_{11} & & 0 \\ & \ddots & \\ 0 & & D_{nn} \end{bmatrix},$$

where $D_{jj} = e^{i\theta_j}$, $\prod e^{i\theta_j} = 1$. The corresponding Cartan subalgebra is the skew-hermitian diagonal matrices

$$H_\lambda = diag[\lambda_1, ..., \lambda_n],$$

where $\sum \lambda_j = 0$ and λ_j pure imaginary. Note that $H_\lambda : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ via $ad(H_\lambda)(B) = [H_\lambda, B]$. The root space of this action for the root α_{jk} is $span\{E_{jk}\}$ (1 in jk entry and 0 otherwise). Then

$$[H_\lambda, E_{jk}] = (\lambda_j - \lambda_k) E_{jk} = \alpha_{jk}(H_\lambda) E_{jk}.$$

Thus, $\alpha_{jk}(H_\lambda) = (\lambda_j - \lambda_k)$ (secretly, this is $\alpha_{jk} = e^j - e^k$, with $\{e^i\}$ a dual basis in $\mathfrak{g}^*$.)

For example, $G = SU(3)$ has six roots: $\alpha_{12}, \alpha_{23}, \alpha_{13}$ and their negatives. We declare $\alpha_{12}, \alpha_{23}, \alpha_{13}$ to be the positive roots, and since $\alpha_{13} = \alpha_{12} + \alpha_{23}$, and we have two simple roots α_{12}, α_{23}. The complex roots are $\mathfrak{t}$ = real part of complexified $\mathfrak{t}_{\mathbb{C}}$. The real roots are $\mathfrak{t} = 2\pi i \mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$.

17.6.2. ***Weyl Chambers***. Let's recall that the Weyl group $W(G, T) = N(T) \diagup T$, and we think of $N(T)$ is the set of all $g \in G$ such that $gTg^{-1} \subset T$, and in fact $W(G, T)$ is the group of outer automorphisms of T. Note that $W(G, T)$ acts on the set of roots as follows. A root θ_α is a homomorphism $\theta_\alpha : T \to \mathbb{C}$, and an element $w \in W(G, T)$ corresponds to a homomorphism $w : T \to T$ via $w(t) = xtx^{-1}$. then $w(\theta_\alpha) = \theta_\alpha \circ w = \theta_{w_\alpha}$.

Recall that $t \in T$ is called *singular* if $\dim N(t) = \dim \{g \in G : gtg^{-1} = t\} > \dim T$. The element t is regular if $\dim N(t) = \dim T$. If $G = SU(n)$, then regular elements correspond to diagonal matrices with distinct eigenvalues, because $W(G, T) \cong S_n$ (permutes the diagonal elements).We want to understand the structure of the set of singular points in T.

Definition 17.15. *For each root θ_α, there is a codimension 1 subgroup U_α of the torus such that*

$$U_\alpha = \ker(\theta_\alpha) = \ker(\theta_{-\alpha}).$$

Claim: for any $t \in U_\alpha$, t is a singular point of T. Why? Consider a complex, 1-dimensional root space $\mathfrak{g}_\alpha$. This is generated by a 1-dimensional subgroup of G such that $v \in \mathfrak{g}_\alpha$, so $\exp(\tau v) \in G$. Note that $\exp(\tau v) \notin T$, because its action on the torus by the identity. Also, this subgroup commutes with T. Thus $\exp(\tau v)$ gives an element of $N(t)$ not in T, so t is singular.

Consider $\bigcap U_\alpha = Z(G) =$ center of G.

The **Stieffel diagram of a group** is the set $\{\exp^{-1}(U_\alpha)\} \subset \mathfrak{t}$. This is an infinite set of parallel hyperplanes. The **infinitessimal diagram of** G consists of the hyperplanes $Lie(U_\alpha) \subset Lie(G)$. On the Lie algebra level, $\theta_\alpha(\exp(\tau v)) = 0$ is one way to think of it. Othewise $\theta_\alpha(v) = 0$. For $SU(n)$, $\theta_\alpha(t) = e^{2\pi i \alpha(t)}$. Then $\theta_\alpha \in \mathfrak{t}^*$, and you can write $\langle \theta_\alpha, v \rangle = 0$.

So the diagram of $SU(3)$ is as follows. If we do have an invariant inner product, $\alpha_{12}, \alpha_{13}, \alpha_{23}$ and so on are the roots, and the angle is $60°$ between them in $\mathbb{R}^2$. The hyperspaces $Lie\left(U_{\alpha_{ij}}\right)$ are the planes perpendicular to the roots. The Weyl chamber is defined by

Definition 17.16. *Given a choice of positive roots, a Weyl chamber is a set of the form* $\{v \in \mathfrak{t} : \varepsilon_i \alpha_i(v) > 0$ *for all* $i\}$, *where each* ε_i *is* ± 1.

17.7. **Roots of evil and weights of sin (or, Roots and the Killing form).** Last time, "we" "studied" combinatorial structures coming from the roots and the action of the Weyl group. Recall:

- the Weyl group acts by conjugation on the set of roots. $W(G,T) = N(T) \diagup T$.
- infinitessimal diagram of G. This consists of hyperplanes $\ker \alpha$ for each root $\alpha : \mathfrak{t} \to \mathbb{C}$.
- Weyl chamber can be defined in terms of the roots as follows:
  Choose a system of positive roots (same a choosing a basis in the Lie algebra, same as choosing a complex structure on the complexification of the Lie algebra). A Weyl chamber is a set of the form $\{v \in \mathfrak{t} : \varepsilon_i \alpha_i(v) > 0 \text{ for all } i\}$, where each ε_i is ± 1, and $\{\alpha_i\}$ is the set of positive roots.
- A system of simple roots S is a maximal linearly independent set of positive roots.
- S defines the fundamental Weyl chamber $K(S) = \{v \in \mathfrak{t} : \alpha_i(v) > 0 \text{ for all } i\}$.

Theorem 17.17. *The Weyl group acts simply transitively on the set of possible choices of* S.

Theorem 17.18. *The Weyl group is generated by reflections with respect to hyperplanes corresponding to simple roots in* S.

In order to define reflections, we need to fix an *Ad*-invariant inner product on $\mathfrak{t}$ and $\mathfrak{t}^*$. Recall that the Killing form on $\mathfrak{g}$ is $K(X,Y) = Tr(ad(X)\, ad(Y))$.

Theorem 17.19. *If* $\mathfrak{g}$ *is a Lie algebra of a semisimple Lie group, then*

(1) $K(X,Y)$ *is nondegenerate.*
(2) *if* G *is in addition compact, then* $K(X,Y)$ *is negative definite.*

Note that K can be extended linearly from $\mathfrak{g}$ to $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$. We assumed that $\mathfrak{g}$ is identified with the purely imaginary part of $\mathfrak{g}_{\mathbb{C}}$. Then K is actually positive definite on $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$. So we define our inner product on $\mathfrak{g}$ by

$$\langle \cdot\, , \cdot \rangle = K(\cdot\, , \cdot).$$

If we restrict our inner product to $\mathfrak{t}_{\mathbb{C}}$ and use the fact that all $ad\,(H) \in \mathfrak{t}_{\mathbb{C}}$ are diagonal, we can write down $\langle \cdot , \cdot \rangle$ explicitly. Note that $ad\,(H)$ is diagonal with roots $\alpha_i\,(H)$ on the diagonal.

$$ad\,(H)\,X_i = [H, X_i] = \alpha_i\,(H)\,X_i$$

for X_i in the α_i root space.

$$\langle H, H\rangle = \sum_{\alpha_i \in R} (\alpha_i\,(H))^2\,,$$

with R the set of all roots. Note that $\alpha_i\,(H)$ are real.

Example 17.20. *If $G = SU\,(n)$, then $\mathfrak{t} =$ diagonal traceless matrices. If $\vec{\lambda} = [\lambda_1, ..., \lambda_n]$,*

$$H_\lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

$\alpha_{ij}\,(H_\lambda) = \lambda_i - \lambda_j$. Then

$$\langle H_\lambda, H_\lambda\rangle = \sum_{i \neq j} (\lambda_i - \lambda_j)^2 = 2n \sum \lambda_j^2$$

since the trace is zero.

We can use $\langle \cdot , \cdot \rangle$ on $\mathfrak{t}$ to identify $\mathfrak{t}$ and $\mathfrak{t}^*$, and thus we can generate a positive definite inner product on $\mathfrak{t}^*$.

Recall that elements of $W\,(G, T)$ act on roots by permuting them. Incidentally, the Weyl group of $SU\,(n) = S_n$. It turns out that for any root α, there is a distinguished element $s_\alpha \in W\,(G,T)$ such that s_α acts on the torus and leaves invariant $U_\alpha = \ker \alpha$ $(\alpha : T \to \mathbb{C})$. On $\mathfrak{t}$, s_α leaves invariant the infinitessimal diagram of the group (fixes the Lie subalgebra of U_α). This action is an isometry in our Ad-invariant inner product, since the Weyl group acts by conjugation and $Ad\,(g)$ on the torus. If we identify $\mathfrak{t}$ with $\mathfrak{t}^*$, then s_α acts on $\mathfrak{t}^*$:

$$s_\alpha\,(X) = X - \frac{2\,\langle \alpha, X\rangle}{\langle \alpha, \alpha\rangle}\alpha,$$

so it maps α to $-\alpha$.

What element of $N\,(T)$ gives the reflection s_α? Answer: $\exp\left(\frac{\pi}{2}\,(X_\alpha - X_{-\alpha})\right)$, where $X_\alpha \in \mathfrak{g}_\alpha$, $(ad\,(H)\,X_\alpha = \alpha\,(H)\,X_\alpha)$ the root space for the root α. Recall that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \left(\bigoplus_i \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i}\right)$.

Example 17.21. *$G = SU\,(3)$, $H_\lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, $\sum \lambda_i$, $\alpha_{12}\,(H_\lambda) = \lambda_1 - \lambda_2$. Then*

$$\mathfrak{g}_{\alpha_{12}} = span \left\{ X_{\alpha_{12}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \in \mathfrak{g}_{\mathbb{C}}.$$

Note that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}\,(3, \mathbb{C})$.

$$[H_\lambda, X_{\alpha_{12}}] = (\lambda_1 - \lambda_2)\,X_{\alpha_{12}}.$$

Then $X_{-\alpha_{12}} = X_{\alpha_{12}}^T$. *Then*

$$X_{\alpha_{12}} - X_{-\alpha_{12}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$g = \exp\left(\frac{\pi}{2}\left(X_{\alpha_{12}} - X_{-\alpha_{12}}\right)\right) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can check that

$$gH_\lambda g^{-1} = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

17.8. **Pushkin™ is our everything, Putin™ is our forever (or, Review of $SU(2)$ representations).** So far we studied the adjoint representation Ad and its nonzero weights are the roots of the group. Let G be a compact Lie group. We want to study arbitrary complex unitary finite-dimensional representations (π, V). We have

$$V = \bigoplus V_\alpha,$$

with $\alpha \in \mathfrak{t}^*$ are weights of our representation. The V_α is the α-weight space. Thus for every vector $v \in V_\alpha$ and $H \in \mathfrak{t}$, then

$$Hv = \alpha(H)\, v.$$

$H|_{V_\alpha}$ is diagonal with $\alpha(H)$ on the diagonal.

We want to solve the following problem: For a compact Lie group G, identify the irreducible representations, compute their weights and multiplicities.

Lemma 17.22. *(Relation between roots and arbitrary weights) If X is in $\mathfrak{g}_\beta$ (β rootspace), then $X : V_\alpha \to V_{\alpha+\beta}$. That is, if α is a weight and β is a root, then $\alpha + \beta$ is again a weight.*

Proof. If $v \in V_\alpha$, $H \in \mathfrak{t}$, then

$$\begin{aligned} HXv &= XHv + [H, X]\, v \\ &= X(\alpha(H)\, v) + \beta(H)\, Xv \\ &= (\alpha(H) + \beta(H))\, v. \end{aligned}$$

□

Now, we restrict to $SU(2)$. There is a simple explicit construction of all complex irreducible representations of $SU(2)$. Consider the space $V_2^n =$ all homogeneous polynomials of degree n in $z = (z_1, z_2)$. A typical element is

$$f(z_1, z_2) = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \ldots + a_n z_2^n.$$

Then $SU(2)$ acts by

$$\pi(U)\, f(z) = f\left(U^{-1} z\right).$$

Note V_2^n has complex dimension $n+1$, and this representation is irreducible with characters

$$\chi_n(\theta) = e^{-in\theta} + e^{-i(n-2)\theta} + \ldots + e^{in\theta} = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

The Lie algebra picture of this representation:

$$Lie\left(SU\left(2\right)\right)=\left\{A:A+A^{*}=0,\ Tr\left(A\right)=0\right\}.$$

The complexification is $\mathfrak{sl}\left(2,\mathbb{C}\right)=$ traceless 2×2 matrices. The basis consisting of

$$\begin{aligned}
X^{+} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ X^{-}=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\mathfrak{t} &= \operatorname{span}\left\{H\right\} \\
\left[\lambda H,X^{+}\right] &= 2\lambda X^{+}=\alpha\left(\lambda H\right)X^{+} \\
\left[\lambda H,X^{-}\right] &= -2\lambda X^{-}=\alpha\left(\lambda H\right)X^{-}
\end{aligned}$$

so the roots are α and $-\alpha$, where

$$\alpha\left(\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}\right)=\lambda.$$

So

$$\begin{aligned}
V_{\alpha} &= \operatorname{span}\left\{X^{+}\right\}, \\
V_{-\alpha} &= \operatorname{span}\left\{X^{-}\right\}.
\end{aligned}$$

The induced representation π_{*} of $\mathfrak{sl}\left(2,\mathbb{C}\right)$ on V_{2}^{n}: if $X\in Lie\left(SU\left(2\right)\right)$, $\pi_{*}\left(X\right)f=\left.\frac{d}{dt}\right|_{t\to 0}\pi\left(\exp\left(2\pi tX\right)\right)f$. Because X^{+}, X^{-}, H are not in the group, you can write:

$$X^{+}=\frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}+\frac{1}{i}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\right)$$

You do something similar with X^{-}, H. It turns out that

$$\begin{aligned}
\pi_{*}\left(H\right)f &= -z_{1}\frac{\partial f}{\partial z_{1}}+z_{2}\frac{\partial f}{\partial z_{2}} \\
\pi_{*}\left(X^{+}\right)f &= -z_{2}\frac{\partial f}{\partial z_{1}} \\
\pi_{*}\left(X^{-}\right)f &= -z_{1}\frac{\partial f}{\partial z_{2}}
\end{aligned}$$

On monomials in V_{2}^{n},

$$\begin{aligned}
\pi_{*}\left(H\right)\left(z_{1}^{j}z_{2}^{k}\right) &= \left(-j+k\right)\left(z_{1}^{j}z_{2}^{k}\right) \\
\pi_{*}\left(X^{+}\right)\left(z_{1}^{j}z_{2}^{k}\right) &= \left(-j\right)\left(z_{1}^{j-1}z_{2}^{k+1}\right) \\
\pi_{*}\left(X^{-}\right)\left(z_{1}^{j}z_{2}^{k}\right) &= \left(-k\right)\left(z_{1}^{j+1}z_{2}^{k-1}\right)
\end{aligned}$$

So the weights are $\left(-n,-n+2,...,n-2,n\right)$. The weight spaces are spans of those monomials. The roots are ± 2. Note that $\pi_{*}\left(H\right)\left(cz_{1}^{n}\right)=\beta\left(cz_{1}^{n}\right)z_{1}^{n}=-ncz_{1}^{n}$, so $\beta\left(cz_{1}^{n}\right)=-nc$.

Now, $\pi_{*}\left(X^{+}\right)$ is the "raising operator" that increases the eigenvalue by two. That is

$$\pi_{*}\left(X^{+}\right):\beta\text{-eigenspace}\to\left(\beta+2\right)\text{-eigenspace}.$$

Similarly, $\pi_{*}\left(X^{-}\right)$ is the "lowering operator".

Example 17.23. *Representations on V_{2}^{3}. The weights are $-3,-1,1,3$, and X^{+} maps for example the weight space of -1 to that of 1. The weight space corresponding to the highest weight $n=3$ is the kernel of the raising operator, and the rest of the weight* *spaces can be constructed by applying the lowering operator to the highest weight space. If* $\ker\left(\pi_*\left(X^+\right)\right)$ =*span*(v)*, then* $v = z_2^3$ *is called a highest weight vector.*

17.9. **Yes, I have freedom of speech (provided it's not prime time TV), or Fundamental Representations and Highest Weight Theory.** We will study arbitrary representations of higher rank compact Lie groups. We will talk about the purely Lie-algebraic aspect of the picture: good for computations. The weakness of this approach: lack of explicit constructions of irreducible representations. One can take quotients of infinite dimensional modules of the enveloping algebra (Verma modules). There is also the geometric Borel-Weil construction, but it is on the level of the group (not Lie algebra).

Let G be a compact Lie group, and recall that G has adjoint representation $Ad(g)$ on $\mathfrak{g}$. Also, we have ad : representation of $\mathfrak{g}$ on itself. If we restrict Ad and ad to the torus T or $\mathfrak{t}$, then we have a decomposition of the complexified Lie algebra into

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha \text{ simple rt}} \left(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\right).$$

Coroots and Fundamental Weights

We understand $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$. Here is what we want to do: let $\pi : G \to U(V)$ be some complex representation. Then $d\pi : \mathfrak{g}_{\mathbb{C}} \to \operatorname{End}(V)$ can be decomposed: for each simple root α, we would like to identify a copy of $\mathfrak{sl}_2(\mathbb{C}) \subset \mathfrak{g}_{\mathbb{C}} \diagdown \mathfrak{t}$, which we will call $\mathfrak{sl}_2(\mathbb{C})_\alpha$. The weights of (π, V) will be classified by how they behave under the maximal abelian subalgebras of each $\mathfrak{sl}_2(\mathbb{C})_\alpha$. Recall, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, and in particular, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{t}$. Each $\mathfrak{g}_\alpha$ is one-dimensional. Now, for each α, choose the set of three generators $X_\alpha^+ \in \mathfrak{g}_\alpha$, $X_\alpha^- \in \mathfrak{g}_{-\alpha}$, $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$, satisfying $\mathfrak{sl}_2(\mathbb{C})$ relations:

$$\begin{aligned}
\left[H_\alpha, X_\alpha^+\right] &= \alpha\left(H_\alpha\right) X_\alpha^+ = 2X_\alpha^+ \\
\left[H_\alpha, X_\alpha^-\right] &= -\alpha\left(H_\alpha\right) X_\alpha^- = -2X_\alpha^- \\
\left[X_\alpha^+, X_\alpha^-\right] &= H_\alpha.
\end{aligned}$$

Moreover, we can choose X_α^- to be the adjoint of X_α^+. The span of $\{X_\alpha^+, X_\alpha^-, H_\alpha\} = \mathfrak{sl}_2(\mathbb{C})_\alpha$. Then

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \text{ simple}} \mathfrak{sl}_2(\mathbb{C})_\alpha$$

The element $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ is canonically associated with a root α and is called a **coroot** of α ($=\breve{\alpha}$)

Definition 17.24. *The* ***coroot*** H_α *is the unique element in* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-a}]$ *satisfying* $\alpha(H_\alpha) = 2$.

Recall that representations of $\mathfrak{sl}_2(\mathbb{C})$ decompose into weight spaces V_β (V_2^n). On each V_β, $H \in \mathfrak{t}$ acts with integral eigenvalue $\beta(H)$. Let α be a root, and choose H_α. Then $H_\alpha V_\beta = \beta(H_\alpha) V_\beta$, and $\beta(H_\alpha)$ must be an integer.

This motivates the definition of the weight lattice. Note that a lattice Λ in a vector space V is a discrete, additive subgroup such that the weights of Λ spans V.

The **weight lattice** is the lattice of weights $\Lambda_w \in \mathfrak{t}^*$, the set of $\beta \in \mathfrak{t}^*$ such that $\beta(H_\alpha) \in \mathbb{Z}$ for all simple roots α. Notice that we also can form the root lattice Λ_R (defined by simple roots), and $\Lambda_R \subset \Lambda_W$.

For simply connected G, all elements in Λ_W are weights of the representations. For nonsimply connected groups (eg $G = SO(3)$), only a sublattice corresponds to weights.

Now we want to use the Weyl group $W(G,T)$ to understand the relations between roots and weights. The relation: $W(G,T)$ is defined in terms of roots, and it acts on weights. Recall that $W(G,T)$ is generated by reflections $S_\alpha : \mathfrak{t}^* \to \mathfrak{t}^*$. That is,

$$S_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha, \ \beta \in \mathfrak{t}^*.$$

(only need simple roots for generators). Let's now relate this to coroots. Recall that coroot H_α has the property that $\alpha(H_\alpha) = 2$ and $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. What is the dual of H_α with respect to the invariant inner product?

$$(H_\alpha)^* = \frac{2\alpha}{\langle \alpha, \alpha\rangle} \in \mathfrak{t}^*$$

To see this,

$$\frac{2\alpha}{\langle \alpha, \alpha\rangle}(H_\alpha) = 2\frac{\langle \alpha, \alpha\rangle}{\langle \alpha, \alpha\rangle} = 2.$$

The S_α equation becomes

$$S_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha,$$

if β is a weight. Now, if α_i and α_j are simple roots,

$$\begin{aligned} S_{\alpha_i}(\alpha_j) &= \alpha_j - 2\frac{\langle \alpha_j, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\alpha_i \\ &= \alpha_j - A_{ji}\alpha_i, \end{aligned}$$

where $A_{ji} = \alpha_j(H_{\alpha_i})$ is the Cartan matrix.

Definition 17.25. *The dual basis to the basis of coroots is called the* ***basis of fundamental weights***

$$\{\omega_{\alpha_i} = \omega_i\}$$

satisfying $\omega_{\alpha_i}\left(H_{\alpha_j}\right) = \delta_{ij}$.

For arbitrary weights,

$$\beta = \sum_i \beta(H_\alpha)\omega_i,$$

and the $\beta(H_\alpha)$ are integers. Then if $\beta = \alpha_j$, then

$$\begin{aligned} \alpha_j &= \sum_i \alpha_j(H_{\alpha_i})\omega_i \\ &= \sum_i A_{ji}\omega_i \end{aligned}$$

This equation tells you how to get a basis of simple roots from the basis of fundamental weights, where $A_{ji} = 2\frac{\langle \alpha_j, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}$ is the Cartan matrix.

Example 17.26. $G = SU(n)$, $\mathfrak{t} = diag(\lambda_1, ..., \lambda_n) = H_\lambda$, *where* $\lambda_1 + ... + \lambda_n = 0$. *The roots are*

$$\alpha_{ij}(H_\lambda) = \lambda_i - \lambda_j.$$

We write $\alpha_{ij} = e^i - e^j$. *The simple roots are*

$$\{\alpha_{i,i+1}\}_{i=1}^{n-1}.$$

The coroots are

$$H_{\alpha_{i,i+1}} = E_{ii} - E_{i+1,i+1}$$
$$\mathfrak{g}_{\alpha_{ij}} = span\left\{E_{ij}\right\}.$$

The fundamental weights are

$$\begin{aligned} \omega_{i,i+1}\left(H_\lambda\right) &= \omega_i\left(H_\lambda\right) = \lambda_1 + \ldots + \lambda_i \\ &= e^1 + \ldots + e^i \end{aligned}$$

More concretely, if $G = SU\left(3\right)$, $\alpha_1 = \alpha_{12} = e^1 - e^2$ and $\alpha_2 = \alpha_{23} = e^2 - e^3$ are the simple roots. (Note $e^1 + e^2 + e^3 = 0$) The Cartan matrix is

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Note that $\alpha_1 = 2\omega_1 - w_2$, $\alpha_2 = -\omega_1 + 2\omega_2$, where $\omega_1 = e^1$, $\omega_2 = e^1 + e^2$. We now draw a diagram. The vectors α_{12}, 0 and α_{23}, and the angle between α_{12} and α_{23} is $120°$. Then ω_1 makes an angle of $\pi/6$ with α_{12}, and ω_2 makes an angle of $-\pi/6$ with α_{23}.

$$0 - - -^{\sqrt{2}} - - - - - - - \alpha_{12}$$

17.10. **Highest Weight Theorem.**

Definition 17.27. *A weight β is called **dominant if** $\langle\beta, \alpha_i\rangle \geq 0$ for all simple roots α_i, i.e. it is in the closure of the fundamental Weyl chamber.*

Definition 17.28. *For a given representation of G on V, a vector $v \in V$ is called a **highest weight vector** if for all positive roots α, $X_\alpha \in \mathfrak{g}_\alpha$ implies that $X_\alpha v = 0$.*

If the highest weight vector $v \in V_\beta$ (weight space of β in the decomposition of V), then β is called the highest weight of the representation.

Here is the Weyl recipe for constructing irreducible representations. Find the roots, then find the weight lattice.

(1) Pick a dominant weight λ.
(2) Assume one has a representation with this highest weight, and pick a highest weight vector.
(3) Take your weight vector, and apply to it all possible negative roots, ie apply all possible elements of negative root spaces ($\mathfrak{g}_{-\alpha}$) to this vector to generate a full representation.

For example, look at the representation of SU(2).

Theorem 17.29. *(Highest Weight Theorem) For any dominant weight $\lambda \in \Lambda_\omega$, there exists a unique irreducible representation V_λ of G with highest weight λ.*

(We will prove later.)

Remark 17.30. *Weights of an irreducible representation must lie inside a convex hull of the figure one gets by acting on the highest weight by elements of the Weyl group.*

Recall that the fundamental weights form a basis for the weight lattice. Recall that the dominant weight satisfies $\langle\beta, \alpha_i\rangle \geq 0$. But

$$\begin{aligned} \langle\beta, \alpha_i\rangle &= \beta\left(H_{\alpha_i}\right) \geq 0, \\ \omega_i\left(H_{\alpha_i}\right) &= \delta_{ij} \end{aligned}$$

so dominant weights are linear combinations of fundamental weights with non-negative integer coefficients.

17.11. **Examples of Highest Weight Theorem.** If $\pi : G \to U(V)$ isa finite dimensional unitary representation, a weight β is a **highest weight** of (π, V) if there exists $0 \neq v \in V_\beta$ such that for any positive root α, $X_\alpha \in \mathfrak{g}_\alpha$ implies $X_\alpha v = 0$. Such a v is called a **highest vector**.

Remark 17.31. *If you know that (π, V) is irreducible, then $\dim_{\mathbb{C}} V_\beta = 1$. (not obvious)*

Remark 17.32. *Also for irreducible (π, V), the highest weight β is the largest weight in the lexicographic order determined by the basis of fundamental weights.*

Remarks about calculating weights of irreducible representations:

- Let β be a dominant weight, ie
$$\beta = \sum_{i=1}^{n} k_j \omega_j,$$
where $\{\omega_j\}$ is the basis of fundamental weights (dual basis to the basis of co-roots), k_j nonnegative integers. Let $H_{\alpha_i} \in \mathfrak{g}$ be the coroot corresponding to ω_i. Then
$$H_{\alpha_i}(\cdot) = 2\frac{\langle \alpha_i, \ \cdot \ \rangle}{\langle \alpha_i, \alpha_i \rangle}$$
for $\cdot \in \mathfrak{g}^*$. Applying β to both sides,
$$\begin{aligned} \beta(H_{\alpha_i}) &= \sum_{i=1}^{n} k_j \omega_j (H_{\alpha_i}) \\ &= k_i. \end{aligned}$$
Thus,
$$k_i = \beta(H_{\alpha_i}) = 2\frac{\langle \alpha_i, \ \beta \ \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$
The numbers $(k_1, ..., k_n)$ are called the Dynkin coefficients of a dominant weight β. So irreducible representations are in 1-1 correspondence with the Dynkin coefficients of its highest weight, which are in 1-1 correspondence with n-tuples of non-negative integers.

Two observations:

- Weights of an irreducible representation must lie inside the convex hull of the figure one gets by acting on the highest weight by all elements of the Weyl group.
- Roots act on weights by translation. That is, if $v \in V_\beta$ and $X \in \mathfrak{g}_\alpha$, then $Xv \in V_{\beta+\alpha}$. So each weight is embedded in some sequence
$$\beta + k\alpha, ..., \beta, ..., \beta - j\alpha.$$
Let's act on this sequence by an element of the Weyl group, $s_\alpha \in W(G, T)$. Applying s_α, we get
$$s_\alpha (\beta + p\alpha) = \beta - m\alpha$$
so there is a relation between m and p above. We get
$$s_\alpha(\beta) - p\alpha = \beta - m\alpha,$$

or

$$s_\alpha(\beta) = \beta - 2\frac{\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha = \beta + (p-m)\,\alpha,$$

so

$$m - p = 2\frac{\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}.$$

This is an important condition that must be satisfied.

Example 17.33. *Let $G = SU(3)$. Suppose that a highest weight of an irreducible representation has Dynkin coefficient $(1,0)$. We will work in the basis of generalized weights $\{\omega_1, \omega_2\}$. Then the highest weight is $\beta_1 = k_1\omega_1 + k_2\omega_2 = \omega_1$. Because of the Remark above, another weight is*

$$\beta_2 = S_{\alpha_1}(\beta_1) = \beta_1 - 2\frac{\langle \beta_1, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\alpha_1 = \beta_1 - k_1\alpha_1 = \beta_1 - \alpha_1.$$

What is the Dynkin coefficient of this weight? Recall the relationship between roots and fundamental weights is

$$\alpha_j = \sum_i A_{ji}\omega_i,$$

where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is the Cartan matrix. In particular,

$$\alpha_1 = A_{11}\omega_1 + A_{12}\omega_2,$$

so

$$\textit{Dynkin vector of } \beta_2 = \textit{Dynkin vector of } \beta_1 - \textit{first row of } A.$$

so

$$\begin{aligned} S_{\alpha_2}(\beta_1) &= \beta_1 \\ \beta_3 &= S_{\alpha_2}(\beta_2) = \textit{Dynkin vector of } \beta_2 - \textit{second row of } A \end{aligned}$$

These are the only possible weights you can get.

$$\begin{aligned} A &= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \leftrightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \\ (1,0) &\to (-1,1)\ \textit{(subtracting first row)} \\ &\to (0,-1)\ \textit{(subtracting second row)} \end{aligned}$$

Example 17.34. *What about the highest weight $(1,1)$?*

$$\begin{aligned} A &= \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \\ (1,1) &\to \textit{sub } \alpha_1\ (-1,2) \\ &\to \textit{sub } \alpha_2 \textit{ from orig } (2,-1) \\ \beta_1 - \alpha_1 - \alpha_2 &= (0,0) \\ \beta_1 - \alpha_1 - 2\alpha_2 &= (1,-2)\,,\ \beta_1 - 2\alpha_1 - \alpha_2 = (-2,1) \\ \beta_1 - \alpha_1 - 2\alpha_2 &= (-1,-1) \end{aligned}$$

This is really the Adjoint representation of $SU(3)$, of dimension 8. But there are only 7 weights, so $(0,0)$ has multiplicity 2. You get a hexagon with the center.

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