# Some brief notes on the Kodaira Vanishing Theorem

## 1 Divisors and Line Bundles, according to Scott Nollet

This is a huge topic, because there is a difference between looking at an abstract variety and local equations versus the equations that define a projective variety.

Given an abstract variety $X$ and want to imbed in complex projective space, $\mathbb{P}^n$, then only the canonical line bundle $\mathcal{O}_{\mathbb{P}}(1)$ is the important object.

Let $M$ be a complex manifold of (complex) dimension $n$. A **divisor** is defined as follows. One example is: let $V \subset M$ be an analytic subvariety (locally determined by the zero set of holomorphic functions on $M$) of codimension 1. It turns out that it is locally given by a single holomorphic function equation. This is not obvious (near singular points), and this fact is not true if we generalize to codimension 2. Another way to think about this is as follows. Pick a point $p \in M$. Then near $p$, $V = \{f = 0\}$. Then the local ring $\mathcal{O}_{M,p} = \{\text{germs of holomorphic functions near } p\}$ is actually a UFD, which implies the single equation fact. Note that $\mathcal{M} = \{h : h(p) = 0\}$ is a maximal ideal in $\mathcal{O}_{M,p}$.

**Remark:**

1. If $g$ is holomorphic and $g|_V = 0$ implies that $f|g$.
2. $f$ is unique up to multiplication by functions $h$ such that $h(p) \neq 0$.

For $V$ as above except possibly singular, $V^* \setminus V_s$ ($V_s$ is singular locus) is the smooth locus. If $V_1^* \subset V^*$ is a connected component, $V_1 = \overline{V_1^*}$ is called an **irreducible analytic subvariety**. Then there are irreducible pieces

$$V = \bigcup_{j=1}^{m} V_j \ .$$

**Definition:** A **divisor** on $M$ is a formal sum

$$\sum a_i V_i, \ \ a_i \in \mathbb{Z}$$

where each $V_i$ is an irreducible analytic subvariety of dimension $n-1$. Only finitely many $a_i$ are nonvanishing. Note that *irreducible* means that the equation of definition can't be factored.

**Example:** Let $M = \mathbb{C}^2$, $V = \{x^2y + xy^2 = 0\}$ is singular at the origin $(0,0)$. $xy(x+y) = 0$. This is the union of three complex lines. $V = V_1 + V_2 + V_3$, $V_1 =$"$x$-axis".

Next, consider $xy(x+ty) = 0$ with $(x,y,t) \in \mathbb{C}^3$. As $t \to 0$, you get $x^2y = 0$. This corresponds to $V_1 + 2V_2$.

The set of all such divisors is an abelian group $\mathrm{Div}(M)$. A divisor $D$ is called **effective** if all the coefficients are nonnegative.

**Example:** Consider the equation $z^5 - z^3 = z^3 (z+1)(z-1) = 0$ on $\mathbb{C}$. The corresponding divisor is $3 \cdot V_1 + V_2 + V_3$, where $V_1 = 0$, $V_2 = -1$, $V_3 = 1$.

For $V$ an irreducible analytic subvariety, the order of vanishing of a function on $V = \{f = 0\}$ is defined as follows. If $f$ is holomorphic in a neighborhood $U$ about $p$ and $g$ is meromorphic, then the order $\mathrm{Ord}_{V,p}(g)$ is the maximal $\alpha \in \mathbb{Z}$ such that

$$g = f^\alpha \cdot h$$

with $h$ holomorphic. Thus $g$ holomorphic implies $\mathrm{Ord}_{V,p}(g) \geq 0$. It turns out that $\mathrm{Ord}_{V,p}(g)$ is independent of $p \in V$, and we call this $\mathrm{Ord}_V(g)$.

**Recap:** With $M$ a complex manifold of dimension $n$. Let $\mathrm{Div}(M)$ be the free abelian group generated by irreducible analytic subvarieties. Let $\mathcal{M}(M)$ be the set of meromorphic functions on $M$. For any $f \in \mathcal{M}(M)$, let

$$(f) = \sum_{\text{all } V} \mathrm{Ord}_V(f) \cdot V$$

be the corresponding divisor (note the sum is locally finite). Locally, any divisor can be expressed like this. However, there are obstructions to finding the corresponding global meromorphic function to a given divisor. There is a sheaf-theoretic interpretation of a divisor. Let $\mathcal{O}$ = the sheaf of holomorphic functions, let $\mathcal{O}^*$ be the sheaf of nonvanishing holomorphic functions, and let $\mathcal{M}^*$ be the sheaf of meromorphic functions that do not vanish identically. For example, $\mathcal{O}^*(U)$ is the set of holomorphic functions that are invertible on $U$. The claim is that

**Claim:** $\mathrm{Div}(M) \cong H^0(\mathcal{M}^* \diagup \mathcal{O}^*)$.

**Proof:** Let $D = \sum a_i V_i$. Pick an open cover $\{U_\alpha\}$, with $V_i|_{U_\alpha}$ given by $\{g_{\alpha i} = 0\}$. Then

$$D = \left(\prod g_{\alpha i}^{a_i}\right)$$

on $U_\alpha$. Let $f_\alpha = \prod g_{\alpha i}^{a_i} \in \mathcal{M}^*(U_\alpha)$ then $\overline{f_\alpha} \in \mathcal{M}^*(U_\alpha) \diagup \mathcal{O}^*(U_\alpha)$. On the overlaps $U_\alpha \cap U_\beta$ the element is well-defined up to $\mathcal{O}^*(U_\alpha \cap U_\beta)$, and so you get a well-defined element of $H^0(\mathcal{M}^* \diagup \mathcal{O}^*)$. The converse is even easier.

**Remark:** Using the exact sequence

$$\begin{aligned}
0 &\to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^* \diagup \mathcal{O}^* \to 0 \\
0 &\to \mathbb{C}^* \to H^0(\mathcal{M}^*) \to H^0(\mathcal{M}^* \diagup \mathcal{O}^*) = \mathrm{Div}(M) \to H^1(\mathcal{O}^*)
\end{aligned}$$

so only when the map $H^0(\mathcal{M}^*) \to H^0(\mathcal{M}^* \diagup \mathcal{O}^*)$ is onto is it true that every divisor $D = (f)$ for some $f \in \mathcal{M}^*(M)$. So the obstructions to this live in $H^1(\mathcal{O}^*)$.

**Relationship to line bundles:** Let $L \to M$ be a holomorphic line bundle. Then this is given by an open cover $\{U_\alpha\}$ and maps

$$L|_{U_\alpha} \overset{\varphi_\alpha}{\to} U_\alpha \times \mathbb{C}.$$

The (holomorphic) transition functions are

$$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}(\cdot, 1) : U_\alpha \cap U_\beta \to \mathbb{C}^*.$$

These satisfy the relations

$$\begin{aligned} g_{\alpha\beta}g_{\beta\alpha} &= \mathbf{1} \\ g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} &= \mathbf{1} \text{ (cocycle condition)} \end{aligned}$$

Question: when are two line bundles isomorphic over the base? $L \cong L'$ if $\varphi'_\alpha = f_\alpha \varphi_a$ for some $f_\alpha \in \mathcal{O}^* (U_\alpha)$. Thus

$$g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}$$

Then compute $H^1 (\mathcal{O}^*)$ with $(U_\alpha)$ and Cech complex:

$$\to \prod \mathcal{O}^* (U_\alpha) \xrightarrow{F} \prod \mathcal{O}^* (U_\alpha \cap U_\beta) \xrightarrow{G}$$

This chain complex yields cohomology $H^1 (\mathcal{O}^*) = \ker G \diagup im (F)$. Then $\ker G = \{g_{\alpha\beta}\}$ so that the cocycle condition holds. $im (F)$ implies that the first condition holds. Thus, $\mathrm{Pic}(M) = H^1 (\mathcal{O}^*) = H^1 (M, \mathcal{O}^*) = \{\text{Line bundles mod isomorphism}\}$. This is called the Picard group of $M$.

There is an exact sequence of sheaves, inducing the long exact sequence in cohomology:

$$\begin{aligned} 0 &\to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^* \diagup \mathcal{O}^* \to 0 \\ 0 &\to H^0 (\mathcal{O}^*) = \mathbb{C}^* \to H^0 (\mathcal{M}^*) \to H^0 (\mathcal{M}^* \diagup \mathcal{O}^*) = \mathrm{Div}(M) \xrightarrow{\delta} H^1 (\mathcal{O}^*) \to ... \end{aligned}$$

All maps above are pretty clear except the map $\delta$. The geometric description of $\delta$ is as follows. Given a divisor $D \in H^0 (\mathcal{M}^* \diagup \mathcal{O}^*)$ and an open cover $\{U_\alpha\}$, $f_\alpha \in \mathcal{M}^* (U_\alpha)$ (unique mod $\mathcal{O}^*$). Set $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$ on $U_\alpha \cap U_\beta$. Then

$$\begin{aligned} g_{\alpha\beta}g_{\beta\alpha} &= \mathbf{1} \\ g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} &= \mathbf{1} \text{ (cocycle condition)} \end{aligned}$$

This gives a line bundle. The fractions are only well-defined up to $\mathcal{O}^*$. If $\{f'_\alpha\}$ also give $D$, then $g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta}$ gives a different line bundle. We see that $h_\alpha = \frac{f_\alpha}{f'_\alpha} \in \mathcal{O}^* (U_\alpha)$. Then $g'_{\alpha\beta} = \frac{h_\beta}{h_\alpha} g_{\alpha\beta}$. The existence of the elements $\frac{h_\beta}{h_\alpha} \in \mathcal{O}^*$ implies that $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ give isomorphic line bundles. The map from $D$ to this line bundle is the map $\delta$. Note that the meromorphic functions map to trivial line bundles through $\beta$.

**Remarks:**

1. $\delta$ is a group homomorphism.

   $\delta (D + D') = \delta (\{f_\alpha\} + \{f'_\alpha\}) = \delta (\{f_\alpha f'_\alpha\}) = \delta (D) \otimes \delta (D')$ (transition functions are multiplied).

   $\delta (-D) = \delta (D)^*$.

2. $\delta (D) = \delta (\{f_\alpha\})$ is trivial iff there exist $h_\alpha \in \mathcal{O}^* (U_\alpha)$ such that $\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta}$, iff $\frac{f_\alpha}{h_\alpha} = \frac{f_\beta}{h_\beta}$ on $U_\alpha \cap U_\beta$, iff there exists $f \in \mathcal{M}^* (M)$ such that $f|_{U_\alpha} = \frac{f_\alpha}{h_\alpha}$, so that $D = (f)$.

3. Given $D$ corresponding to $\{f_\alpha\}$, we get more that just $\delta(D) = \mathcal{L}(D)$, more than just the transition functions. We also have $f_\alpha \in \mathcal{M}^*(U_\alpha)$ satisfying $f_\beta g_{\alpha\beta} = f_\alpha$ on $U_\alpha \cap U_\beta$. This data is exactly a global meromorphic section of $L \to M$. (If you can choose each $f_\alpha$ to be holomorphic, then we have a holomorphic section.) The divisor is effective (nonegative coefficients) iff the $f_\alpha$ are holomorphic iff the correspondinf section is holomorphic.

**Example:** on $\mathbb{P}^1$, the divisor $p - q$ corresponds to the function $\frac{x_0}{x_1}$ in homogeneous coordinates $[x_0, x_1]$.

So $Div(M) \to Pic(M)$ gives $D \mapsto (\mathcal{L} \to M, s \in H^0(\mathcal{L} \otimes \mathcal{M}))$. Conversely, if $s$ is a global meromorphic section of $\mathcal{L}$, then $s$ corresponds to $\{f_\alpha\}$ such that $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, so they have the same poles and same zeros on the intersections. So for $V$ irreducible, $\mathrm{Ord}_V(f_\alpha) = \mathrm{Ord}_V(f_\beta) = \mathrm{Ord}_V(s)$. Take $D = \sum_{irred\ V} \mathrm{Ord}_V(s) V$. Note $Div(M)$ is a group under addition, and $Pic(M)$ is a group under tensor product, and these maps are group homomorphisms.

**Definition:** Two divisors $D$ and $D'$ are called *linearly equivalent* if $\mathcal{L}(D) \cong \mathcal{L}(D')$, or equivalently if there exists a global meromorphic function $f$ such that $D = D' + (f)$.

**Definition:** For $D \in Div(M)$, the complete linear system $|D|$ of the divisor $D$ is $|D| = \{D + (f) : D + (f) \text{ is effective for } f \text{ global meromorphic}\}$. Note that this is the same as the set of holomorphic sections of $\mathcal{L}(D)$ mod $\mathbb{C}^*$. This has the structure of a projective space, which we call $\mathbb{P}H^0(\mathcal{L}(D)) = H^0(\mathcal{L}(D)) / \mathbb{C}^*$. More generally, a family of effective divisors on $M$ corresponding to a linear subspace $W$ of $H^0(\mathcal{L}(D))$ is called a linear system.

**Motivating Classical Example:** Suppose that $j : M \hookrightarrow \mathbb{P}^N$ is a smooth analytic projective embedding. For example, with $N = 2$, consider a cubic curve is given by a cubic homogeneous polynomial $f$, and $\{f = 0\}$ is a smooth curve. This a genus 1 curve (elliptic curve), since $g = \frac{1}{2}(d-2)(d-1)$. Note that $f$ is not a global meromorphic function, but it is a global holomorphic section of a line bundle $\mathcal{L} = (\text{tautological line bundle})^3$. Imagine a line $L \subset \mathbb{P}^2$ that varies and intersects $j(M)$ in various points — in fact sets of three points (counting multiplicities). This actually defines a linear system on $M$. That is, each set of three points gives an effective divisor on $M$, and each set is linearly equivalent to each other set. If $L$ and $L'$ are intersect at $\{p, q, r\}$, $\{p', q', r'\}$, respectively. Then the equations of $L$ and $L'$ are respectively $\sum a_i x_i = 0$, $\sum a_i' x_i = 0$. So $f = \frac{\sum a_i x_i}{\sum a_i' x_i}$ is a global meromorphic function on $\mathbb{P}^2$. Note that $f$ has zeros (poles) along $L$ ($L'$). Restricted to $M$, $f$ yields a global meromorphic function on $M$, and $(f|_M) = p + q + r - p' - q' - r'$. Thus,

$$D' + (f|_M) = D.$$

Thus $\delta(L \cap M) = \delta(L' \cap M)$ is the same line bundle on $M$. What line bundle is it? It is the pullback of the tautological line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ on $\mathbb{P}^2$: $\delta(L \cap M) = j^*(\mathcal{O}_{\mathbb{P}^2}(1))$. All line bundles over $\mathbb{P}^2$ are generated by $\mathcal{O}_{\mathbb{P}^2}(1)$=tautological line bundle=hyperplane bundle. For $M \subset \mathbb{P}^n$ with $[x_0, ..., x_n]$. These are all holomorphic sections of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Restricting to $M$, $s_i = j^*(x_i)$ are holomorphic sections of the line bundle $j^*\mathcal{O}_{\mathbb{P}^n}(1)$. So

$$\langle s_0, ..., s_n \rangle \subseteq H^0(j^* \mathcal{O}_{\mathbb{P}^n}(1))$$

is a linear system.

## 2 Positivity and Chern Classes

Let $M$ be an $n$-dimensional $\mathbb{C}$-manifold. Let $D \in Div(M)$, $V \subset H^0(\mathcal{L}(D))$ subspace of holomorphic sections. Then $V/\mathbb{C}^* \cong \mathbb{P}V$ is the corresponding linear system of effective divisors linearly equivalent to $D$.

On $M$ we have the exact exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

Locally,

$$\mathbb{Z} \to \mathbb{C} \xrightarrow{e^{2\pi i x}} \mathbb{C}^*$$

We get the long exact cohomology sequence with coefficients in the sheaves above:

$$0 \to \mathbb{Z} \to \mathbb{C} \twoheadrightarrow \mathbb{C}^* \xrightarrow{0} H^1(M,\mathbb{Z}) \hookrightarrow H^1(M,\mathcal{O}) \xrightarrow{\psi} H^1(M,\mathcal{O}^*) = Pic(M) \to H^2(M,\mathbb{Z}) \to$$

Note that $H^1(M,\mathcal{O})$ is a complex vector space (no torsion), so $H^1(M,\mathbb{Z})$ is also free abelian. Thus the image of $\psi$ is $\mathbb{C}^n/$free discrete abelian group, so this is an abelian Lie group. This image is called the Jacobian of $M$ (true at least for curves). The image

$$Pic(M) \to H^2(M,\mathbb{Z}), \mathcal{L} \mapsto c_1(\mathcal{L}),$$

is called the first Chern class. So, up to a continuous holomorphic deformation, the first Chern class determines the line bundle over $M$. (In fact, the first Chern class determines the line bundle up to smooth isomorphism).

**Example:** $M = \mathbb{CP}^n$. Then $H^1(\mathbb{CP}^n,\mathcal{O}) = H^2(\mathbb{CP}^n,\mathcal{O}) = 0$. Then the map $Pic(\mathbb{CP}^n) \to H^2(\mathbb{CP}^n,\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism, so the Chern class is the degree. The generator is the hyperplane bundle (or universal line bundle) $\mathcal{O}(1)$. For example, let $D$ on $\mathbb{CP}^n$ be given by $\{f = 0\}$, $f$ homogeneous of degree $d$. Then $D$ is efective, $c_1(\mathcal{L}(D)) = d$.

**Remarks:**

1. If $n = 1$, $Pic(M)$ is quite large (eg any point). In the case of curves, the Jacobian $\mathrm{Im}\psi$ of the curve is an abelian variety of dimension $g$, where $g$ is the topological genus of $M$ as a surface.

2. For $n > 1$, $Pic(M)$ can be trivial. Then, it can't be imbedded into projective space. (May need condition on $M$)

3. In general, if $M \hookrightarrow \mathbb{CP}^n$, then the map $Pic(\mathbb{CP}^n) = \mathbb{Z} \to Pic(M)$ may be onto. Lefschetz: this map is usually an isomorphism in higher dimensions (eg $n > 3$, $M$ algebraic).

4. In the case $n = 3$: $M \subset \mathbb{CP}^3$. Eg if $M$ is algebraic,

(a) degree of $M = 1:$ $M = \mathbb{CP}^2$, so $Pic\left(\mathbb{CP}^3\right) \to Pic\,(M)$ is isomorphism.

(b) degree of $M = 2:$ $M$ is a Segre imbedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$.

(c) degree of $M = 3:$ $M =$blowup of $\mathbb{CP}^2$ at 6 points, so $Pic\,(M) = \mathbb{Z}^7$.

(d) degree of $M \geq 4$: $Pic\,(\mathbb{P}^3)$ is iso to $Pic\,(M)$, for $M$ sufficiently general quartic, degree $d$ surface. (Noether-Lefschetz Theorem)

**Theorem (2008, Nollet-Brevik):** If $M$ is a sufficiently general degree $d$ surface containing $C_1, ..., C_n$ in $\mathbb{P}^3$, then

1. $M$ is smooth.

2. $Pic\,(M)$ is freely generated by $\mathcal{O}\,(1)$ and $C_1, ..., C_n$.

**Positivity:**

Back to protypical example: Let $M \overset{j}{\hookrightarrow} \mathbb{CP}^n$, with $j$ an embedding. In projective space, we have coordinates $[x_0, x_1, ..., x_n]$. This data gives a linear system on $M$ via $j^*\mathcal{O}_{\mathbb{P}^n}\,(1)$, and any linear function in $x_j$ gives a holomorphic section $\sum a_i x_i = 0$, an effective divisor. You get a line bundle by pullback: if $s_i = j^* x_i$, then $\sum a_i s_i$ is holomorphic section of $j^*\mathcal{O}_{\mathbb{P}^n}\,(1)$; thus, $\sum a_i s_i$ is a linear system(bunch of sections of line bundle linearly equivalent to each other).

**Definitions:** A divisor $D$ on $M$ is *very ample* if the associated line bundle $\mathcal{L}\,(D) = j^*\mathcal{O}_{\mathbb{P}^n}\,(1)$ for some embedding $M \overset{j}{\hookrightarrow} \mathbb{CP}^n$. We say $D$ is *ample* if $mD$ is very ample for some $m > 0$. These are different notions of positivity.

**Example:** If $D$ is a point $p$ on a Riemann surface $M$, then $D$ is ample but not very ample (unless $M = \mathbb{P}^1$; then a single point is very ample).

If $\{D_t : t \in V\}$ is a linear system ( $V$ is a subset of holomorphic sections of $\mathcal{L}\,(D)$ ). The *base locus* of a linear system is the intersection of all the $D_t$. Here is an example: if all the lines through a certain point are the linear system, the base locus is that point.

**Observe** that if this linear system comes from an embedding, then the base locus is empty.

**Bertini's Theorem:** The generic element of a linear system is smooth away from the base locus. "Generic" means off a codimension 1 set.

**Cor:** If $D$ is very ample, then the general element of the linear system $|D|$ is a smooth submanifold of $M$.

**Definition:** $K_M = \wedge^n T_M$ is called the canonical bundle on $M$. It is holomorphic.

**Kodaira Vanishing Theorem:** If $A$ is an ample divisor on a manifold $M$, then

$$H^i\,(M, \mathcal{L}\,(-A)) = 0$$

for $i < \dim_{\mathbb{C}} M$. This is equivalent to

$$H^i\,(M, \mathcal{L}\,(K_M + A)) = 0$$

for $i > 0$, by Serre duality. Note $\mathcal{L}\,(K_M + A) = \mathcal{L}\,(K_M) \otimes \mathcal{L}\,(A)$.

**Serre duality:** $H^i\,(M, \mathcal{L})$ is dual to $H^{n-i}\,(M, \mathcal{L}^* \otimes K_M)$. (nondegenerate pairing)

# 3 Igor begins

**Positive line bundles:**

Let $M$ be a complex manifold. Then $TM_{\mathbb{C}} = T^{0,1} \oplus T^{1,0}$.

Complex coordinates are $z_1, ..., z_n$; $z_1 = x_1 + iy_1$. $T^{0,1}$ =span$\left\{\frac{\partial}{\partial z_i}\right\}$.

A complex manifold is $(M, J)$, where $J$ is an almost complex structure. $J \in \text{End}\,(TM)$, $J^2 = -1$. $J$ is multiplication by $i$.

The spaces $T^{0,1}$ and $T^{1,0}$ are the $\pm i$ eigenspaces of $J$, both of the same dimension. We are given a Hermitian metric $g$ on $M$ such that $g\,(JX, JY) = g\,(X, Y)$. We can choose a basis of $T_xM$ by first picking one vector $e_1$, then $Je_1$, then *span* $\{e_1, Je_1\}$ is an invariant subspace for $J$, and then we go to the orthogonal complement, etc. etc.

**Definition:** A real $(1,1)$-form $\varphi$ on a complex manifold $(M, g, J)$ is a two form in $\Gamma\left((T^{0,1})^* \otimes (T^{1,0})^*\right)$. A positive $(1,1)$-form is called **positive** if the symmetric tensor $\varphi\,(\cdot\ , J\ \cdot)$ is a positive definite symmetric bilinear form.

# 4 Hodge Theory

Consider the following complex $(V, D)$ of vector spaces and linear operators.

$$0 \to V_0 \overset{D_0}{\to} V_1 \overset{D_1}{\to} V_2 \overset{D_2}{\to} ... \overset{D_{n-1}}{\to} V_n \to 0$$

We assume

$$D_p \circ D_{p-1} = 0.$$

Let

$$V = \bigoplus V_i,\ \ D = \bigoplus D_i$$

We can define cohomology

$$H^p\,(V, D) = \ker D_p \big/ \text{Im} D_{p-1}$$

Our goal is to identify $H^p\,(V, D)$ with a specific subspace of $\ker D_p$ by choosing a specific representative inside $\ker D_p$, which is the **harmonic** element in each cohomology class.

In order to do this, each $V_p$ must be equipped with an inner product $\langle\ \bullet\ ,\ \bullet\ \rangle_p$.

**Remark:** In applications, $V_p$'s are infinite dimensional, and $D_p$'s are unbounded operators, so there are some analytic difficulties. Thus we need an additional condition.

We say that the complex $(V, D)$ is **elliptic**: this condition is algebraic. This condition will take care of analytic difficulties.

**Example 1:** Let $M$ be a compact manifold with empty boundary. Let $V_p = \Omega^p\,(M) = \Gamma\,(\wedge^p T^* M)$ be the vector space of smooth real-valued $p$-forms on $M$. Then $D_p = d$ is the standard exterior derivative. Then

$$H^p\,(\Omega, d) = H_{de\ Rham}\,(M)\,.$$

If $M$ is Riemannian, then $\wedge^p\,(M)$ has an inner product $(\ \bullet\ ,\ \bullet\ )_p$, and the inner product

$$\langle \omega_1, \omega_2 \rangle_p = \int_M (\omega_1, \omega_2)_p\ \ dV.$$

**Example 2:** Let $M$ be a Kähler manifold,

$$\begin{aligned} V_p &= \Gamma\left(\wedge^{q,p} M\right), \\ D_p &= \overline{\partial}_p, \end{aligned}$$

where $\overline{\partial}_p$ is the Dolbeault differential. The corresponding cohomology is Dolbeault cohomology:

$$H^{q,p}(M) = \ker \overline{\partial}_p \diagup \mathrm{Im}\overline{\partial}_{p-1}.$$

**The Laplacian (Laplace Operator):** The inner products $\langle \bullet , \bullet \rangle_p$ allow us to define adjoint operators

$$D_p^* : V_{p+1} \to V_p,$$

defined by

$$\left\langle D_p^* u, v \right\rangle_p = \langle u, D_p v \rangle_{p+1}$$

for all $u \in V_{p+1}$ and $v \in V_p$. The Laplacian $\Delta_p$ is defined as

$$\Delta_p = D_p^* D_p + D_p D_{p-1}^*.$$

In Example 1, $\Delta = dd^* + d^* d$. In Example 2, $\Delta = \overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}$. Note the geometry comes in by defining the adjoints.

On the circle, $\Delta_0 = -\frac{d^2}{d\theta^2}$, eigenvalues are $n^2$ corresponding to eigenfunctions $e^{in\theta}$.

**Properties of the Laplace operator:** Spectrum "=" eigenvalues for this discussion.

1. $\Delta_p$ is positive semi-definite, meaning that $\langle \Delta_p u, u \rangle \geq 0$. Proof:

$$\langle (D^* D + DD^*) u, u \rangle = \langle Du, Du \rangle + \langle D^* u, D^* u \rangle .$$

   This implies its spectrum is nonnegative.

2. $\Delta_p$ is self-adjoint ($\Delta_p^* = \Delta_p$).

3. The spectrum of $\Delta_p$ consists of nonnegative eigenvalues

$$0 \leq \lambda_1 < \lambda_2 < ... \to \infty$$

   with multiplicities. Each eigenspace is finite dimensional. Eigenspaces corresponding to distinct eigenvalues are orthogonal.

4. There is an orthonormal basis of $V_p$ consisting of eigenvectors of $\Delta_p$.

In this basis, $\Delta_p = diag\left(\lambda_1, \lambda_2, ...\right)$. From this discussion, it is clear that

$$\Delta_p : \left(\ker \Delta_p\right)^{\perp} \to \mathrm{Im}\left(\Delta_p\right)$$

is an isomorphism.

This implies that

$$V_p = \ker \Delta_p \oplus \mathrm{Im}\Delta_p$$

as an orthogonal direct sum. But what is $\mathrm{Im}\Delta_p$? We have

$$\begin{aligned}\Delta_p u &= D_p^*(D_p u) + D_{p-1}\left(D_{p-1}^* u\right) \\ &= v + w\end{aligned}$$

is the orthogonal direct sum of $\mathrm{Im}\left(D_p^*\right)$ and $\mathrm{Im}\left(D_{p-1}\right)$.

Thus, we have

**Theorem. (Hodge decomposition)** We have the following orthogonal decomposition:

$$V_p = \ker \Delta_p \oplus \mathrm{Im} D_p^* \oplus \mathrm{Im} D_{p-1}$$

Recall that

$$H^p(V, D) = \ker D_p \big/ \mathrm{Im} D_{p-1}$$

We claim that

$$\left(\mathrm{Im} D_p^*\right)^{\perp} = \ker D_p.$$

The conclusion is:

**Theorem (Hodge Theorem):**

$$\ker \Delta_p \cong H^p(V, D).$$

There is a recipe: Given any $[u] \in H^p(V, D)$. Pick any representative $u$, and map it to its orthogonal projection to $\ker \Delta_p$.

**Proof of claim:**

$$\begin{aligned} u &\in \ker D_p \Longleftrightarrow D_p u = 0 \\ &\Longleftrightarrow \langle D_p u, v\rangle = 0 \text{ for all } v \\ &\Longleftrightarrow \left\langle u, D_p^* v\right\rangle = 0 \text{ for all } v \\ &\Longleftrightarrow u \in \left(\mathrm{Im} D_p^*\right)^{\perp}. \end{aligned}$$

# 5 Shocking revelations about Kähler manifolds and you

A **Kähler manifold** is a complex Hermitian manifold which is also endowed with a compatible symplectic structure. These manifolds generalize nonsingular complex algebraic varieties. We will see that any complex submanifold of $\mathbb{C}P^n$ is Kähler. By Chow's Theorem, it is also algebraic. In differential geometry, there are two approaches: one without coordinates, one with coordinates. We will use both.

## 5.1 Complex manifolds

A complex manifold $M$ of complex dimension $n$ is a manifold where every point $p \in M$ has a neighborhood with complex coordinates $z^1, ..., z^n$, $z^j = x^j + iy^j$. All the transition functions between the two coordinate systems must be holomorphic. That is, a function $f$ is holomorphic if $\overline{\partial_j} f = 0$ for all $j$, where

$$\begin{aligned} \overline{\partial_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right), \\ \partial_j &= \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right). \end{aligned}$$

Let the complex manifold $M$ be considered as a real $2n$-dimensional manifold. Then $TM \otimes \mathbb{C}$ is a rank $2n$ complex vector bundle over $M$. Then a local basis of $TM \otimes \mathbb{C}$ :

$$\partial_1, ..., \partial_n, \overline{\partial_1}, ..., \overline{\partial_n}$$

Thus, we have a decomposition

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$

Then $T^{1,0}M$ is the holomorphic tangent bundle (isomorphic as a complex bundle to $TM$), and $T^{0,1}M$ is the anti-holomorphic tangent bundle. Similarly,

$$\begin{aligned} T^*M \otimes \mathbb{C} &= \wedge^{1,0}M \oplus \wedge^{0,1}M \\ &= span\left\{dz^1, ..., dz^n\right\} \oplus span\left\{d\overline{z^1}, ..., d\overline{z^n}\right\}. \end{aligned}$$

Forms:

$$\wedge_{\mathbb{C}}^r M = \bigoplus_{p+q=r} \wedge^{p,q} M,$$

with $\wedge^{p,q}M$ spanned locally by

$$dz^{i_1} \wedge ... \wedge dz^{i_p} \wedge d\overline{z^{j_1}} \wedge ... \wedge d\overline{z^{j_q}}.$$

A $(p,q)$ form $\omega \in \Omega^{p,q}(M)$ is a smooth section of $\wedge^{p,q}M$. Similarly,

$$\Omega^r(M, \mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}(M).$$

Let $d : \Omega^r(M) \to \Omega^{r+1}(M)$ be the exterior derivative that maps

$$d : \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1}.$$

we decompose

$$d = \partial + \overline{\partial}$$

according to the appropriate images above. Any $\omega \in \Omega^{p,q}(M)$ can be written

$$\omega = \sum_{|I|=p, |J|=q} \omega_{I,J} dz^I \wedge d\overline{z}^J,$$

and

$$\overline{\partial}\omega = \sum_{k=1}^{p} \left(\overline{\partial}_k \omega_{I,J}\right) d\overline{z}_k \wedge dz^I \wedge d\overline{z}^J.$$

On a complex manifold, we have

$$d^2 = \overline{\partial}^2 = \partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

The Dolbeault complex is (for all $p$)

$$0 \to \Omega^{p,0} \overset{\overline{\partial_0}}{\to} \Omega^{p,1} \overset{\overline{\partial_1}}{\to} \ldots \overset{\overline{\partial_{2n-p-1}}}{\to} \Omega^{p,2n-p} \overset{\overline{\partial_{2n-p}}}{\to} 0.$$

The corresponding cohomology is Dolbeault cohomology:

$$H^{p,q}(M) = \ker \overline{\partial_q} \diagup \mathrm{Im}\overline{\partial_{q-1}}.$$

If $E \to M$ is a holomorphic rank $k$ complex vector bundle, we can consider $E$-valued $(p,q)$ forms

$$\Omega^{p,q}(M,E) = \Omega^{p,q}(M) \otimes \Gamma(E).$$

Then $\overline{\partial}$ can be extended to $\Omega^{p,q}(M,E)$. In local coordinates: let $s_1, \ldots, s_k$ be a basis of local holomorphic sections. Then the Dolbeault differential is

$$\overline{\partial}\left(\sum_{r=1}^{k} \omega_r \otimes s_r\right) = \sum_{r=1}^{k} \overline{\partial}(\omega_r) \otimes s_r.$$

This is well-defined. Again, the extended $\overline{\partial}^2 = 0$, so we can define

$$H^{p,q}(M,E).$$

We want to do Hodge theory with these cohomology groups. The appropriate metric is a Hermitian metric. A **Hermitian metric** over a complex vector bundle $E \to M$ is a smoothly varying positive definite Hermitian form $H(\bullet,\bullet)$ on each fiber. By definition, this means

$$H \in \Gamma(\mathrm{End}(E,E)) \cong \Gamma\left(E \otimes \overline{E}\right)$$

such that if $Z$ and $W$ are holomorphic sections of $E$ and $\lambda \in \mathbb{C}$, then on each fiber

$$\begin{aligned} H\left(Z,\overline{W}\right) &= \overline{H\left(W,\overline{Z}\right)} \\ H\left(\lambda Z,\overline{W}\right) &= \lambda H\left(Z,\overline{W}\right) \\ H\left(Z,\overline{Z}\right) &> 0 \text{ for all } Z \neq 0 \end{aligned}$$

at the point in question. If $E = TM$, then in local holomorphic coordinates,

$$H = \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}} dz^\alpha \otimes d\overline{z}^\beta,$$

so $H$ extends to $TM\otimes\mathbb{C}$. In particular,

$$\begin{aligned} H\left(\partial_\alpha,\partial_\beta\right) &= 0=H\left(\overline{\partial}_\alpha,\overline{\partial}_\beta\right), \\ h_{\alpha\overline{\beta}} &= H\left(\partial_\alpha,\overline{\partial}_\beta\right), \end{aligned}$$

with the matrix $\left(h_{\alpha\overline{\beta}}\right)$ Hermitian and positive definite. Note that we can write as real and imaginary parts

$$H=g-i\omega.$$

Then $g$ is a Riemannian metric on $TM$, and $\omega$ is called the **Kähler form**. Note that

$$\begin{aligned} g &= \frac{1}{2}\left(H+\overline{H}\right)=\sum\frac{1}{2}\left(h_{\alpha\overline{\beta}}dz^\alpha\otimes d\overline{z}^\beta+\overline{h_{\alpha\overline{\beta}}}\overline{dz^\alpha}\otimes dz^\beta\right) \\ &= \sum\frac{1}{2}h_{\alpha\overline{\beta}}\left(dz^\alpha\otimes d\overline{z}^\beta+\overline{dz^\beta}\otimes dz^\alpha\right) \end{aligned}$$

This is a Riemannian metric on the real tangent bundle. Also,

$$\omega=\frac{i}{2}\left(H-\overline{H}\right)=\sum\frac{i}{2}h_{\alpha\overline{\beta}}\overline{dz^\alpha}\wedge dz^\beta.$$

**Remark:** (relation to an almost complex structure). Each complex manifold has a canonical complex structure $J\in\mathrm{End}\,(TM)$, such that $J^2=-\mathbf{1}$. In holomorphic coordinates $(z^\alpha=x^\alpha+iy^\alpha)$. We define

$$J\left(\frac{\partial}{\partial x^\alpha}\right)=\frac{\partial}{\partial y^\alpha},\ J\left(\frac{\partial}{\partial y^\alpha}\right)=\frac{\partial}{\partial x^\alpha},$$

then by extending to complex vectors,

$$J\left(\frac{\partial}{\partial z^\alpha}\right)=i\frac{\partial}{\partial z^\alpha},\ J\left(\frac{\partial}{\partial \overline{z}^\alpha}\right)=-i\frac{\partial}{\partial \overline{z}^\alpha}.$$

Note that

$$\begin{aligned} \omega(u,v) &= g(Ju,v) \\ g(u,v) &= \omega(u,Jv) \\ H(Ju,Jv) &= H(u,v) \\ g(Ju,Jv) &= g(u,v) \\ \omega(Ju,Jv) &= \omega(u,v). \end{aligned}$$

Note that a Hermitian manifold has a canonical volume form:

$$\begin{aligned} dV &= \frac{\omega^n}{n!} \\ &= \left(\frac{i}{2}\right)^n\det\left(h_{\alpha\overline{\beta}}\right)dz^1\wedge d\overline{z}^1\wedge\ldots\wedge dz^n\wedge d\overline{z}^n. \end{aligned}$$

The complex manifold $M$ is **Kähler** if it has a Hermitian metric such that

$$d\omega = 0.$$

Then $\omega$ is called the Kähler form, and $(M, \omega)$ is a symplectic manifold.

Given a Kähler form $\omega$ (a $(1,1)$ real form such that $d\omega = 0$), it can be written locally

$$\omega = \frac{1}{2} d\left(\phi + \overline{\phi}\right)$$

with $\phi$ a one-form, which can be chosen so that

$$\phi = \sum \phi_\alpha dz^\alpha.$$

Since $d = \partial + \overline{\partial}$, we have

$$\overline{\partial \phi} = 0, \;\; \partial \overline{\phi} = \omega.$$

Then we have a $\overline{\partial}$-Poincare lemma that says that locally

$$\overline{\phi} = c \cdot \overline{\partial} F,$$

so that

$$\omega = \frac{i}{2} \partial \overline{\partial} F = -\frac{i}{2} \overline{\partial} \partial F$$

The function $F$ is called a **Kähler potential**, and it is defined up to $\partial G + \overline{\partial} W$.

**Examples of Kähler manifolds.**

1. $M = \mathbb{C}^n$, $H = \sum_j dz_j \otimes d\overline{z}_j$, $\omega = \sum dx_j \wedge dy_j$, $g = \sum_j \left(dx_j \otimes dx_j + dy_j \otimes dy_j\right)$.

2. Let $M$ be any complex submanifold of $\mathbb{C}^n$ (or any other Kähler manifold) will pull back the Hermitian structure to produce a (canonical) Kähler structure.

3. $M = \mathbb{C}P^n$: the Kähler form is given in terms of the Kähler potential:

   $$F = \ln\left(1 + \|z\|^2\right) = \ln\left(\|w\|^2\right),$$

   where $z_j = \frac{w_j}{w_0}$, so

   $$\omega = \frac{i}{2} \partial \overline{\partial} \ln\left(\|w\|^2\right).$$

   The hard part is to show that the resulting matrix $\left[h_{\alpha\overline{\beta}}\right]$ is positive definite. For example, if $n = 1$, $S^2 = \mathbb{C}P^1$,

   $$\begin{aligned} \omega &= \frac{i}{2}\left(\frac{dz \wedge d\overline{z}}{\left(1 + z\overline{z}\right)^2}\right) = \frac{dx \wedge dy}{\left(1 + x^2 + y^2\right)^2}, \\ g &= \frac{dx \otimes dx + dy \otimes dy}{\left(1 + x^2 + y^2\right)^2} \end{aligned}$$

   This is the metric that comes from stereographic projection. This the Euclidean metric $+\mathcal{O}\left(|z|^2\right)$. These are holomorphic geodesic normal coordinates. This happens only for Kähler manifolds.

# 6 Chern class warfare (Chern-Weil-Marx-Lenin theory)

Comrade Einstein summation convention:

$$R_i^j R_j^k \text{ means } \sum_{j=1}^{n} R_i^j R_j^k$$

(sum over repeated indices if one is up and the other is down).

Let $E$ be a complex vector bundle over a manifold over $M$. Chern-Weil theory allows one to express the image in $H^\bullet(M,\mathbb{R})$ of the Chern classes of $E$ ($\mathbb{Z}$-coefficients) using curvature of an arbitrary connection $\nabla$ of $E$.

A connection on $E \to M$ is a device for computing directional derivatives of sections of $E$. Another interpretation: a connection for identifying fibers of $E$ over different points, ie a device for lifting paths from $M$ to $E$. This directional derivative is called a covariant derivative and is defined as follows:

**Definition:** A covariant derivative on $E$ is an $\mathbb{R}$-linear operator $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$, say $(X,\sigma) \mapsto \nabla_X \sigma$ such that for all smooth $f : M \to \mathbb{R}$

1. $\nabla_{fX}\sigma = f\nabla_X\sigma$ ($\nabla_X$ is tensorial)
2. $\nabla_X(f\sigma) = f\nabla_X\sigma + (Xf)\sigma$ (Leibniz rule)

**Example:** Connections on $TM$ (local coordinates). Let $x^1, ..., x^n$ be local coordinates on $U \subset M$, so that $\left\{\frac{\partial}{\partial x^j}\right\}$ form a basis of $T_UM$. We have

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^i\partial_i} Y^j \partial_j \\
&= X^i \nabla_{\partial_i} Y^j \partial_j \\
&= X^i \frac{\partial Y^j}{\partial x_i}\partial_j + X^i Y^j \nabla_{\partial_i}\partial_j \\
&= X^i \frac{\partial Y^j}{\partial x_i}\partial_j + X^i Y^j {\Gamma_{ij}}^k \partial_k
\end{aligned}$$

where the Christoffel symbols are defined by

$$\nabla_{\partial_i}\partial_j = {\Gamma_{ij}}^k \partial_k$$

In local coordinates, the connection is defined by the set of Christoffel symbols.

Sometimes it is helpful to study covariant derivatives in the direction of all vectors at once. We think of this as

$$\nabla : \Gamma(TM) \to \Omega^1(M) \otimes \Gamma(TM) = \Omega^1(TM),$$

where

$$\nabla(fY) = df \otimes Y + f\nabla Y.$$

Then

$$\begin{aligned}\nabla\frac{\partial}{\partial x^k} &= \omega_k{}^l\otimes\frac{\partial}{\partial x^l},\ \text{and}\\ \omega_k{}^l\left(\frac{\partial}{\partial x^j}\right) &= \Gamma_{jk}{}^l.\end{aligned}$$

Then

$$\Omega=\left(\omega_k^l\right)\in\Omega^1(M)\otimes\mathrm{End}(TM).$$

So one may also think of the connection as determined by $\Omega$.

**Levi-Civita connection:** If $M$ is a Riemannian manifold, then $TM$ has a unique torsion-free connection compatible with the Riemannian metric.

1. (Torsion free) $\nabla_X Y-\nabla_Y X=[X,Y]$. (Which implies $\Gamma_{ij}^k=\Gamma_{ji}^k$)

2. (metric) $\frac{\partial}{\partial x^j}\langle Y,Z\rangle=\left\langle\nabla_{\frac{\partial}{\partial x^j}}Y,Z\right\rangle+\left\langle Y,\nabla_{\frac{\partial}{\partial x^j}}Z\right\rangle$ for any $j$. (In general $X\langle Y,Z\rangle=\langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle$. (This implies that the tensor corresponding to the Riemannian metric is "constant" in our connection.)

We get

$$\Gamma_{jk}^i=\frac{1}{2}g^{im}\left(\partial_j g_{mk}+\partial_k g_{jm}-\partial_m g_{jk}\right),$$

where $(g_{ij})=(\langle\partial_i,\partial_j\rangle)$ is the matrix of the metric and $(g^{ij})$ is the matrix for the inverse of $(g_{ij})$.

Let $E\to M$ be an arbitrary bundle on $E$. A connection on $E$ is an $\mathbb{R}$ or $\mathbb{C}$-linear differential operator $\nabla:\Gamma(E)\to\Omega^1(E)$ satisfying the Leibnitz rule

$$\nabla(f\sigma)=df\otimes\sigma+f\nabla\sigma.$$

On $\Omega^p(E)$, the connection may be extended by

$$\nabla(\omega\otimes\sigma)=d\omega\wedge\sigma+(-1)^p\,\omega\wedge\nabla\sigma.$$

In a local basis of sections $(s_i)$, $\nabla$ is determined by

$$\nabla s_i=\omega_i^j s_j.$$

The connection matrix of locally-defined one-forms is

$$\Omega=\left(\omega_i^j\right).$$

If $E=TM$,

$$\omega_i^j\left(\frac{\partial}{\partial x_k}\right)=\Gamma_{ik}^j$$

The curvature operator of $\nabla$ is the $\mathrm{End}(E)$-valued 2-form $R^\nabla$ defined by

$$R^\nabla(\sigma):=\nabla(\nabla\sigma)$$

We need to check that the result is actually an endomorphism! We check that $R^{\nabla}(f\sigma) = fR^{\nabla}(\sigma)$. We compute

$$\begin{aligned} R^{\nabla}(f\sigma) &= \nabla(\nabla(f\sigma)) \\ &= \nabla(df \otimes \sigma + f\nabla\sigma) \\ &= -df \wedge \nabla\sigma + df \wedge \nabla\sigma + f\nabla(\nabla(\sigma)) \\ &= fR^{\nabla}(\sigma). \end{aligned}$$

Also,

$$R^{\nabla} = d\Omega + \Omega \wedge \Omega.$$

Let

$$\omega := Tr\left(R^{\nabla}\right) = \sum_i \left(d\omega_i^i + \sum_j \left(\omega_j^i \wedge \omega_i^j\right)\right).$$

This does not change when coordinates are changed. Then note that

$$\omega = d\left(\sum \omega_i^i\right),$$

so $\omega$ is closed (but not necessarily exact) 2-form. Thus, $[\omega]$ defines a class in $H^2(M, \mathbb{C})$. It turns out that the class does not depend on the connection. Let $\widetilde{\nabla}$ be another connection. Then let $A = \widetilde{\nabla} - \nabla$. Then $A(fs) = f(As)$. So $A$ is an $\operatorname{End}(E)$-valued 1-form. Then $a = tr(A)$ is a well-defined 1-form. Then

$$Tr\left(R^{\widetilde{\nabla}} - R^{\nabla}\right) = da,$$

which shows that

$$\widetilde{\omega} = \omega + da,$$

so that $[\widetilde{\omega}] = [\omega]$.

Next, we claim that $[\omega]$ is represented by a purely imaginary form. To see this, choose a Hermitian structure $\langle \bullet, \bullet \rangle$ on $E$, and choose $\nabla$ to be compatible with $E$. This is always possible, and it means

$$\partial_j \langle s_1, s_2 \rangle = \left\langle \nabla_{\partial_j} s_1, s_2 \right\rangle + \left\langle s_1, \nabla_{\partial_j} s_2 \right\rangle.$$

Let $\{s_i\}$ be an orthonormal basis. Then

$$\omega_j^j + \overline{\omega_j^j} = 0 \text{ (no sum)}$$

Thus, each $\omega_j^j$ is purely imaginary. Thus $Tr\left(\sum_j d\left(\omega_j^j\right)\right)$ is purely imaginary.

**Theorem.** $c_1(E) = \left[\frac{i}{2\pi}\omega\right] \in H^2(M, \mathbb{R})$.

**Definition.** A Chern class $c_1(E) \in H^2(M, \mathbb{Z})$ must satisfy the following axioms:

1. (Naturality) For every smooth $f : M \to N$,

$$f^*(c_1(E)) = c_1(f^*E)$$

2. (Whitney Sum)
$$c_1 (E \oplus F) = c_1 (E) + c_1 (F)$$

3. (Normalization) If $L \to \mathbb{C}P^1$ is a tautological line bundle, then $c_1 (L) = -1$. (ie $\int_{\mathbb{C}P^1} \omega = -1$.)

**Proof:** (1) and (2) are automatic. There is a special connection called the **Chern connection**. Let $E \to M$ be a complex vector bundle over a complex manifold $M$. Then $\Lambda^1 (E) = \Lambda^{1,0} (E) \oplus \Lambda^{0,1} (E)$ (first is span $dz_j$, second is span of $d\bar{z}_j$). So we may decompose

$$\begin{aligned} \nabla &= \nabla^{1,0} \oplus \nabla^{0,1}, \\ \nabla^{1,0} &= \Pi^{1,0} \nabla, \nabla^{0,1} = \Pi^{0,1} \nabla. \end{aligned}$$

**Theorem:** If $E$ is a holomorphic vector bundle with a Hermitian structure $H (s_1, \bar{s}_2) = \langle s_1, \bar{s}_2 \rangle$. Then there is a unique connection on $E$ satisfying

1. $\nabla$ is compatible with $H$.

2. $\nabla^{0,1} = \bar{\partial}$.

This connection is given by

$$\nabla = \bar{\partial} + H^{-1} \circ \bar{\partial} \circ H,$$

where $H : E \to E^*$ is the metric isomorphism. This is called the *Chern connection*.

**Theorem:** $M$ is Kähler iff the Chern connection of $TM \otimes \mathbb{C}$ is the same as the Levi-Civita connection extended by complex linearity.

**Last step in proof: The Chern class of $CP^1$:** let $L \xrightarrow{\pi} \mathbb{C}P^1$ be the tautological line bundle, whose fiber is

$$L_{[z]} = L_{[z_0, z_1]}$$

is the complex line $\langle z_0, z_1 \rangle$ in $\mathbb{C}^2$. Let $(U_0, \phi_0)$ and $(U_1, \phi_1)$ be canonical charts. The canonical trivialization

$$\psi_\alpha : \pi^{-1} (U_\alpha) \to U_\alpha \times \mathbb{C}$$

is given by

$$\psi_\alpha ([z], w) = ([z], w_\alpha).$$

The Hermitian structure $H$ on $L$ comes from $\mathbb{C}^2$. Let $\sigma$ be a local holomorphic section, and let $u = H (\sigma, \bar{\sigma}) = \|\sigma\|^2$. Then for every $X \in T\mathbb{C}P^1$. The derivative

$$\begin{aligned} du (X) &= \partial_X u = \partial_X (H (\sigma, \bar{\sigma})) \\ &= H (\nabla_X \sigma, \bar{\sigma}) + H \left(\sigma, \overline{\nabla_X \sigma}\right) \\ &= \omega (X) u + \overline{\omega (X)} u. \end{aligned}$$

Note

$$\omega + \bar{\omega} = d (\log (u)).$$

On the other hand, since $\sigma$ is holomorphic, and $\nabla^{0,1} = \overline{\partial}$, $\omega$ is a $(1,0)$-form. Then

$$d = \partial + \overline{\partial},$$

so

$$\partial\left(\log\left(u\right)\right) = \omega, \overline{\partial}\left(\log\left(u\right)\right) = \overline{\omega}$$

Then

$$\begin{aligned} R^{\nabla} &= d\omega = d\left(\partial \log u\right) = \left(\partial + \overline{\partial}\right)\partial \log u \\ &= \overline{\partial}\partial \log u. \end{aligned}$$

Then, letting $s = 1$ on $U_0$

$$\frac{i}{2\pi}\int_{\mathbb{C}P^1} \overline{\partial}\partial \log u = \frac{i}{2\pi}\int_{\mathbb{C}P^1} \frac{i}{2}\Delta \log\left(1 + x^2 + y^2\right) = -1.$$

# 7 Kodaira Vanishing Theorem

The Kodaira Vanishing Theorem has many variants. This is used in embeddings of complex manifolds into projective space, etc.

**Theorem** (Kodaira) Let $M^m$ be a complex compact manifold, and let $L$ be a holomorphic line bundle that is positive, i.e. it admits a Hermitian structure with positive curvature. Then, letting $K_M = \Lambda^m T^{(1,0)}M$ be the canonical bundle the cohomology groups, we have

$$H^q\left(M, K_M \otimes L\right) = 0, \; q > 0.$$

**Remark:** Note that $L$ positive implies $M$ is Kähler (but not conversely).

Recall that a line bundle $L$ is ample iff some tensor power of $L$ is very ample. A line bundle $L'$ is very ample iff $L'$ has enough global sections to embed $M$ into projective space.

**Corollary:** If $L$ is positive iff $L$ is ample.

Review of ingredients in the proof:

- **Dolbeault cohomology:** If $M$ is a complex manifold, let $\Omega^k\left(M\right)$ be the space(sheaf) of complex-valued smooth $p$-forms. In holomorphic coordinates, and form $\omega \in \Omega^k\left(M\right)$ can be written as

  $$\omega = \sum_{|I|+|J|=k} \omega_{I,J}\, dz_I \wedge d\overline{z}_J.$$

  If $|I| = p$, $|J| = q$, $\omega \in \Omega^{p,q}\left(M\right)$. So $\Omega^{p,q}\left(M\right)$ is a globally defined subsheaf in the sheaf $\Omega^k$. Then

  $$\begin{aligned} d &= \partial + \overline{\partial}, \partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}, \\ d^2 &= \partial^2 = \overline{\partial}^2 = 0. \end{aligned}$$

If $E \to M$ is a holomorphic vector bundle, then since $\overline{\partial}$ annihilates holomorphic functions, it extends to

$$\begin{aligned} \overline{\partial} &: \Omega^{p,q}(M) \otimes E \to \Omega^{p,q+1}(M) \otimes E, \\ \overline{\partial}^2 &= 0. \end{aligned}$$

For all $p$, one has the Dolbeault complex

$$0 \to \Omega^{p,0} \otimes E \to \Omega^{p,1} \otimes E \to \Omega^{p,2} \otimes E \to ... \to \Omega^{p,m} \otimes E \to 0$$

and then Dolbeault cohomology $H^{p,q}(M, E)$.

**Theorem** (Dolbeault isomorphism theorem)

$$H^{p,q}(M, E) = H^q(M, \Omega^p \otimes E),$$

where $\Omega^p \subset \Omega^{p,0}$ is the sheaf of holomorphic sections.

- **Curvature:** Recall a connection $\nabla$ on a vector bundle $E$ is a $\mathbb{C}$-linear map $\nabla : E \to \Omega^1(M) \otimes E$ that satisfies the Leibniz rule $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$. Recall that $\nabla$ can be extended in a unique natural way to an operator

$$\nabla : \Omega^p(M) \otimes E \to \Omega^{p+1}(M) \otimes E$$

There is a natural decomposition $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where for example

$$\nabla^{1,0} : E \to \Omega^{1,0}(M) \otimes E.$$

Note that $\overline{\partial}$ and $\nabla^{0,1}$ have similar properties - map between the same spaces, both $\mathbb{C}$-linear, both obey Leibniz rule.

**Proposition:** Choose a Hermitian metric $h$, where $h(\sigma, \mu) = \langle \sigma, \mu \rangle$ on $E$. Then there exists a unique connection $\nabla$ called a *Chern connection* with the following properties:

1. $X\langle \sigma, \mu \rangle = \langle \nabla_X \sigma, \mu \rangle + \langle \sigma, \nabla_X \mu \rangle$ (metric compatibility)
2. $\nabla^{0,1} = \overline{\partial}$.

Any connection has a two-form associated to it called curvature. The curvature $R^\nabla$ of $\nabla$ is an $\mathrm{End}(E)$-valued two-form defined as $R^\nabla = \nabla \circ \nabla$.

If $L$ is a holomorphic line bundle. Its **first Chern class** is

$$c_1(L) = \left[\frac{i}{2\pi} R^\nabla\right] \in H^2(M, \mathbb{R}),$$

where $\frac{i}{2\pi} R^\nabla$ is a well-defined real $(1,1)$-form whose class is independent of the metric. We say $L$ is **positive** if $\frac{i}{2\pi} R^\nabla$ is positive, i.e.

$$\frac{i}{2\pi} R^\nabla \left(Z, \overline{Z}\right) > 0$$

for any nonzero section $Z$ of $T^{1,0}M$.

- **Hodge theory:** Let $M$ be a compact complex Hermitian manifold. Then one has an $L^2$-inner product on $\Omega^p \otimes E$. That is,

$$(\sigma_1, \sigma_2) = \int_M \langle \sigma_1, \sigma_2 \rangle \; dV$$

Since $\nabla = \nabla^{1,0} + \nabla^{0,1}$, we abuse notation by saying $\partial = \nabla^{1,0}$, $\overline{\partial} = \nabla^{0,1}$. We can form two Laplacians.

$$\Delta_{\overline{\partial}} = \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^* : \Omega^{p,q} \otimes E \to \Omega^{p,q} \otimes E$$

and $\Delta_\partial$ similarly. Let

$$\mathcal{H}^{p,q}(M, E) = \ker \Delta_{\overline{\partial}}.$$

**Theorem:** $\mathcal{H}^{p,q}(M, E) \cong H^{p,q}(M, E)$.

**Remark:** If $M$ is an arbitrary complex manifold and $E$ is trivial, we have $d = \partial + \overline{\partial}$, but there is no relation between the standard $\Delta$ and $\Delta_{\overline{\partial}}$. In particular, $H^k(M) \neq \bigoplus_{p+q=k} H^{p,q}(M)$. However, if $M$ is Kähler, then $\Delta = 2\Delta_{\overline{\partial}} = 2\Delta_\partial$. And then the previous relation is true. Recall that a closed, real-valued $(1,1)$-form $\omega$ is called a Kähler form. A Hermitian metric $h$ is called Kähler if the two-form $\omega = \frac{i}{2}\left(h - \overline{h}\right)$ is a Kähler form. A complex manifold is called Kähler if it admits a Kähler metric.