GROUPS OF HOMOTOPY SPHERES, SURGERY, AND THE KERVAIRE INVARIANT

Milnor was one of the pioneers of the classification of manifolds field. It turns out that higher dimensional manifolds can't be classified. (For example: a 4-manifold can be constructed that has any finitely generated group as the fundamental group. Since those groups can't be classified, 4-manifolds can't be classified.) So the next attempt is to classify all manifolds of a given homotopy type (eg Poincare conjecture).

Theorem 1. *(Poincaré Conjecture) If M^n is a closed manifold homotopy equivalent to S^n, then M is homeomorphic to S^n.*

This is known for all n. ($n = 0, 1$ trivial). For $n = 2$, it is the classification of surfaces. For $n = 3$, due to Perelman, $n = 4$ Mike Freedman, $n \geq 5$ Stephen Smale (smooth case), Newman, Connell (topological case).

Conjecture 2. *(Smooth Poincaré Conjecture) If M is a closed smooth manifold h.e. to S^n, is M diffeomorphic to S^n?*

Cases: $n = 0, n = 1$ trivial. For $n \leq 3$, an n-manifold has a unique diff. structure.
$n = 4$ – noone has a clue. (eg $\mathbf{R}^4$ has an uncountable number of diff'ble structures, only one in every other $\mathbf{R}^n$)
$n = 5$ s.p.c. true.
$n = 6$ s.p.c. true.
$n = 7$ false - in fact, there exist M^7 homeo but not diffeo to S^7 (28 different classes)
$n \geq 8$ known to some extent through surgery and stable homotopy groups of sphere.
Double-suspension Theorem: Given a (PL) homology sphere Ω, $\Sigma(\Sigma\Omega)$ is homeomorphic to a sphere.
Question: How many smooth homotopy spheres are there in each n?

Theorem 3. *(The h-cobordism Theorem) $n \geq 5$. Suppose that Y^{n+1} is a smooth, simply-connected compact manifold and $\partial Y = Y_1 \sqcup (-Y_2)$ and the inclusions $Y_1 \hookrightarrow Y$ and $Y_2 \hookrightarrow Y$ are homotopy equivalences. Then Y is diffeomorphic to $Y_1 \times I$ and $Y_2 \times I$. In particular, $Y_1 \cong Y_2$.*

A consequence: simply connected smooth manifolds are diffeomorphic if and only if they are h-cobordant.
Let Θ^n be the set of smooth n-dimensional homotopy spheres up to h-cobordism.

Theorem 4. *The set Θ^n is a group (under connected sum).*

Note that the identity is the standard S^n. The manifold with reversed orientation is the inverse (?).
Why is this the right thing to do? boundary of $E \times I$ is $(-E) \sqcup E$ then cut out a ball, and you have an h-cobordism to the sphere.
We have a map from $\Theta^n \to \Pi_n$. This is the stable n-stem $= \lim\limits_{k\to\infty} \pi_{n+k}\left(S^k\right) = \lim\limits_{k\to\infty}\left[S^{n+k}, S^k\right]$. (use suspensions).

To see where this goes,

$$0 \to bP_{n+1} \to \Theta^n \xrightarrow{p} \frac{\Pi_n}{im\,(J)}$$

bP_{n+1} is the subgroup of homotopy spheres that bound parallelizable manifolds. A manifold is called parallelizable if TM is trivial bundle. For example, D^{n+1}, $\partial D^{n+1} = S^n \subset bP_{n+1}$. An n-manifold M is stably parallelizable if $TM \oplus \mathbf{R}^k$ is trivial for sufficiently large k.

Theorem 5. *Homotopy spheres are stably parallelizable.*

It is all about looking at $\pi_{n-1}(SO_n)$. Proved by complicated homotopy theory.

Take a homotopy sphere E, and embed it in S^{n+k}. So the normal bundle and tangent bundle are stably trivial. Up to homeomorphism, $NE \cong S^n \times \mathbf{R}^k$. Let ϕ be the trivialization of the bundle (lots of choices). The map p above is $p(E,\phi)$. Collapse NE to S^k, then get an embedding.

Question: Compute $|\Theta^n|$; Θ^n is the group of smooth homotopy n-spheres, $n \geq 5$, with operation connected sum.

Thom-Pontryagin Construction:

Suppose we have a smooth map $f : S^{n+k} \to S^k$. If p is the south pole and is a regular point, $f^{-1}(p)$ is a manifold. But we can actually do better. Because the point has a nice neighborhood, the neighborhood pulls back to a trivial normal bundle. One can suspend $f \mapsto \Sigma f \mapsto \Sigma^2 f$ ... Given f, we have a manifold with a trivial normal bundle inside a sphere. If we change f via smooth homotopy, then you can make the homotopy regular at the south pole, then we have a framed bordism (trivialization of normal bundle) between $f^{-1}(p)$ to $g^{-1}(p)$. If we suspend the whole picture, we get an assignment from {stable homotopy classes of $f : S^{n+k} \to S^k$} $\to$ {framed bordism classes of framed manifolds}. A manifold is frameable if it has a trivial normal bundle for some embedding into some $\mathbf{R}^m$. (Note stably parallelizable implies frameable.). If M is a manifold with nonempty boundary, then frameable implies parallelizable. This assignment mentioned above is actually an isomorphism. (bordism is a group under disjoint sum). Thus there is an isomorphism

$$\Pi_n \cong \Omega_n^{fr}.$$

Given an element $E \in \Theta^n$, let ϕ be a framing of its normal bundle. Then if the fiber dimension is k, then we can map into S^k, with E going to south pole, and the framing gets wrapped around the sphere. This is the map from $\Theta^n \xrightarrow{p} \Pi_n$. The kernel of this and image of this are important. Given E and $\phi : NE \to E \times \mathbf{R}^k$ the framing, let $p(E,\phi)$ be the stable homotopy class of $S^{n+k} \to S^k$. Let $p(E) = \{p(E,\phi)\}$ be the set of all possible framings. It turns out that this set map preserves connected sums, in that $p(E_1) + p(E_2) \subset p(E_1 \# E_2)$. Since $p(S^n) + p(S^n) \subset p(S^n)$, we have that $p(S^n)$ is a subgroup of Π_n. Also, $p(S^n) + p(E) \subset p(S^n \# E) = p(E)$ implies $p(E)$ is a union of cosets. But $p(E) + p(-E) \subset p(S^n)$ implies $p(E)$ is a coset of $p(S^n) \subset \Pi_n$. So $\overline{p} : \Theta^n \to \Pi_n / p(S^n)$ is well-defined. Note that $\Pi_n / p(S^n) \cong \Pi_n / \mathrm{Im}(J)$. Note that $J : \pi_n(SO(r)) \to [S^{n+r}, S^r]$ acts by rotations. For $\alpha \in \pi_n(SO(r))$, write $S^{n+r} = (S^n \times D^r) \cup (D^{n+1} \times S^{r-1})$, and define

$$J(\alpha)(x,y) = \begin{cases} \alpha(x)\,y & x,y \in S^n \times D^r \\ \text{base pt} & (x,y) \in D^{n+1} \times S^{r-1} \end{cases}$$

For r sufficiently large, you get the isomorphism above. Note that $\overline{p}$ is not always onto.

The kernel of $\overline{p}$: Note that $p(E)$ contains the zero element of $\Pi_n \cong \Omega_n^{fr}$. Given E and framing of E that represents zero, ie the empty manifold, so that means that there is a framed manifold that has E as its boundary, iff E is the boundary of a parallelizable manifold (ie an element of bP_{n+1}). Then you get the exact sequence

$$0 \to bP_{n+1} \to \Theta^n \xrightarrow{p} \frac{\Pi_n}{im\,(J)}$$

Note that bP_{n+1} can be computed using surgery theory, and its dimension mod 4 is important. We will have an exact formula for bP_{n+1}, and also the right side is computable via surgery theory.

Consider

$$\begin{array}{rcl} 0 & \to & bP_{n+1} \to \Theta^n \xrightarrow{p} \dfrac{\Pi_n}{p\,(S^n)} \\ 0 & \to & bP_{4m} \to \Theta^{4m-1} \xrightarrow{p} \dfrac{\Pi_{4m-1}}{p\,(S^{4m})} \end{array}$$

There are several cases depending on what n is mod4. Surgery theory was invented for this purpose (they called it *spherical modification*). To try to show that $\overline{p}$ is onto, we want a framed manifold to be framed-bordant to a homotopy sphere. How can we recognize something to be a homotopy sphere? What is enough: $\pi_1 = 0$, reduced homology=0. Notice we get a map $M^n \to S^n$ by collapsing the outside. If M is s.c. and $H_*(M) = H_*(S)$ then the Whitehead theorem implies that this is a homotopy equivalence.

Idea : surgery theory. Motivation: CW complexes. In homotopy, we glue cells to kill things, but then we don't have a manifold anymore. Now, suppose that we have a manifold, take the trivial cylinder on it, and have a nontrivial element of a homotopy group. So we will glue on a thickened disk $D^2 \times S^{n-1}$. We need the generated to be embedded and framed in M, then we can do this, and everything stays a manifold. The new boundary piece is $M' = M - (S^1 \times D^{n-1}) \cup (D^2 \times S^{n-2})$ on top. Then this kills off π_1. Next step: to kill $H_2 \cong \pi_2$ and so is represented by a sphere — keep going. This works well until you hit trouble in the middle dimension. Might have trouble embedding S^1 in a 2-manifold, for instance. If n is odd, there is no middle dimension. Poincaré duality tells us that we kill everything through $\frac{n-1}{2}$, and the rest is history. We get a sequence of framed cobordisms to a homology sphere. This tells us that we are onto (in some cases).

If $n = 4m$, the obstruction is the signature of M (which is a bordism invariant). For our case, M is framed, which implies that all the characteristic classes are zero, so the signature is zero. So we can still prove that the map is onto. The last case is the $n = 4m + 2$ — this part is harder. (more later)

Can we say something about bP? Suppose that E is a homotopy sphere bounding a parallelizable manifold. If we can kill $\pi_1(M)$ and homology of M, then $M - S^n$ is a cylinder, so that E is diffeo to S^n. but then the same arguments say that if $n+1$ is odd, we can do this. Then $bP_{odd} = 0$. Thus, $\Theta_{4m} \cong \frac{\Pi_{4m}}{p(S^{4m})}$. Next, bP_{4m} — obstruction is the signature. It turns out that for $m > 1$, bP_{4m} is cyclic of order

$$\begin{aligned}
|bP_{4m}| &= a_m 2^{2m-2}\left(2^{2m-1}-1\right)\operatorname{num}\left(\frac{\beta_m}{4m}\right); \\
a_m &= \begin{cases} 1 & m \text{ even} \\ 2 & m \text{ odd} \end{cases} \\
\beta_m &= \text{bernoulli number, determined by} \\
\frac{z}{e^z-1} &= 1-\frac{z}{2}+\frac{\beta_1}{2!}z^2-\frac{\beta_2}{4!}z^4+\ldots
\end{aligned}$$

Now we need to look at

$$\frac{\Pi_{4m+2}}{p\left(S\right)} \overset{\Phi_{m+1}}{\to} \mathbb{Z}_2 \overset{b}{\to} bP_{4m+2} \to 0.$$

Now we need the Kervaire invariant. The obstruction to $4m-2$ dimensional surgery is the Arf invariant. Suppose that V is $\mathbb{Z}_2$ vector space, and we have a nondegenerate anti-symmetric inner product. A quadratic refinement of the inner product is a function $V \overset{\xi}{\to} \mathbb{Z}_2$ such that $\xi\left(x+y\right)-\xi\left(x\right)-\xi\left(y\right)=\left(x,y\right)$. (not uniquely determined!) If α_i, β_j is a symplectic basis for V, the Arf invariant is

$$Arf\left(\xi\right)=\sum^{(\dim V)/2}\xi\left(\alpha_j\right)\xi\left(\beta_j\right)\in\mathbb{Z}_2$$

Also called the democratic invariant: the elements of V vote via ξ.

Given M^{4m+2}, $H_{2n+1}\left(M\right)$ has an antisymmetric cup product. Given, $w \in H_{2n+1}\left(M\right)$, let $\xi\left(w\right)$ be the self-intersection number of w (assuming M is $2n$-connected). Let $c\left(M\right)$ be Arf(ξ). It turns out that $c\left(M\right)$ is the only obstruction to surgery on M. Then we just need to show that $c\left(M\right)$ is this map Φ above. There are two related arf-invariant problems going on. This is enough to show that $bP_{4m+2}=0$ or $\mathbb{Z}_2$. Why? Suppose E is not S^n and E bounds a parallelizable manifold. Then E bounds some M with $c\left(M\right)=1$. Let E' be any other $E'=\partial M'$ with $c\left(M'\right)=1$. Then the boundary connect sum $E\#E'$ yields $c\left(E\#E'\right)=0$. So $E\#E'$ is S^n. So there is at most one nonzero element of bP_{4m+2} .

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129, USA