INTRO TO SUBRIEMANNIAN GEOMETRY

1. Introduction to subriemannian geometry

A lot of this talk is inspired by the paper by Ines Kath and Oliver Ungermann on the arXiv, see [3] as well as [1].

Let M be a smooth manifold and let $H \subset TM$ be a smooth distribution, where $\dim H_x = d$ for all $x \in M$. Vectors in H are called *horizontal*. Let $\Gamma(H)$ denote the space of smooth sections of H. We assume that H is **bracket-generating**. Another word for this in the literature is **nonholonomic**. That means, that for each $x \in M$ there is a $J \in \mathbb{N}$ such that the sequence

$$\Gamma_0 := \Gamma(H),\ \Gamma_{j+1} := \Gamma_j + [\Gamma_0, \Gamma_j]$$

satisfies $\{X_x : X \in \Gamma_J\} = T_x M$. Note that this is the complete opposite of a foliation, where in that case $[\Gamma(H), \Gamma(H)] \subseteq \Gamma(H)$. An example of the above is the Heisenberg distribution $H = \operatorname{span}\{\partial_y, \partial_x + y\partial_z\} \subset \mathbb{R}^3$. Note that

$$\begin{aligned}
[\partial_y, \partial_x + y\partial_z] f &= \partial_y(\partial_x + y\partial_z) f - (\partial_x + y\partial_z)\partial_y f \\
&= \partial_z f.
\end{aligned}$$

For such distributions, it is possible to get from one point on the manifold to any other point on the manifold by a tangent (horizontal) curve (Chow-Rashevskii Theorem). If g is a riemannian metric of H, then (g, H) is called a subriemannian structure on M and (M, H, g) is called a subriemannian manifold. Note that for any distribution, there exists a riemannian metric, constructed by patching together a local metric using a partition of unity. Or, using a big hammer, you could embed the manifold smoothly into $\mathbb{R}^N$, and then pull back the Euclidean metric. One can also show that given any orthonormal basis of vectors in H_x, there exists a local orthonormal horizontal frame that restricts to the given frame at x.

An example of a kind of manifold with a subriemannian structure is a contact manifold. By definition, a contact structure on a $(2m+1)$-dimensional manifold M is a one-form α such that $\alpha \wedge d\alpha \wedge ... \wedge d\alpha = \alpha \wedge (d\alpha)^m$ is a nonvanishing $(2m+1)$-form. Let $H_x = \ker \alpha_x = \{X \in T_x M : \alpha(X) = 0\}$. Let Z be a locally defined vector field such that $\alpha(Z)$ is locally nonzero, so that Z is locally transverse to H. Then for any X, Y are any local sections of H, then

$$\begin{aligned}
d\alpha(X, Y) &= X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) \\
&= -\alpha([X, Y]).
\end{aligned}$$

Suppose it were the case that there is some point x in the local neighborhood such that $[X, Y]_x$ is contained in H_x for all local sections X, Y of H. Let $X_1, ..., X_{2m}$ be a local frame of H, so that $Z, X_1, ..., X_{2m}$ is a local frame of TM. Then

$$\begin{aligned}
\alpha \wedge (d\alpha)^m (Z, X_1, ..., X_{2m}) &= \alpha(Z) \prod \pm d\alpha(X_i, X_j) \\
&= \alpha(Z) \prod \pm \alpha([X_i, X_j]) \\
&= 0,
\end{aligned}$$

which is a contradiction. Thus, $H=\ker\alpha$ is bracket-generating. The previous example of $\mathbb{R}^3$ with $\alpha = dz - ydx$ works as a contact form, because

$$\begin{aligned}\alpha\wedge d\alpha &= (dz-ydx)\wedge d(dz-ydx)\\ &= (dz-ydx)\wedge(-dy\wedge dx)\\ &= -dz\wedge dy\wedge dx\neq 0.\end{aligned}$$

Another general example of a kind of manifold with subriemannian structure is a nilmanifold. Recall the that a nilmanifold is a quotient of a nilpotent Lie group by a discrete subgroup. A nilpotent Lie group is one whose Lie algebra $\mathfrak{g}$ satisfies for some minimal integer $k>1$:

$$\mathfrak{g} > \underbrace{[\mathfrak{g},\mathfrak{g}]}_{\mathfrak{g}^{[1]}} > \underbrace{[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]}_{\mathfrak{g}^{[2]}} > \dots > \mathfrak{g}^{[k]} = 0$$

(We say that the Lie group is k-step nilpotent.) In this case, $\mathfrak{g}^{[k-1]}$ is a nontrivial subalgebra of the center $\mathfrak{z}$, so that we may find a basis of left-invariant vector fields that span a space complementary to $\mathfrak{g}^{[k-1]}$ in $\mathfrak{g}$, and by construction, this subspace is bracket-generating.

2. Metrics and Cometrics

2.1. **The matrix for the musical isomorphisms.** Recall the following standard construction in riemannian geometry. Let $\langle\bullet,\bullet\rangle$ be a riemannian metric on the tangent bundle to a manifold. This induces a metric on cotangent vectors in each T_x^*M via the formula $\langle v^*,w^*\rangle=\langle v,w\rangle$, where v^* is the dual vector defined by $v^*(X)=\langle v,X\rangle$ (and similarly for w^*). We let $\#: T^*M\to TM$ be the bundle map defined by $(v^*)^{\#}=\#(v^*)=v$. If $G=(g_{ij})=(\langle\partial_i,\partial_j\rangle)$ is the matrix for a metric on TM with respect to the coordinate vectors $\partial_1,\dots,\partial_n$, then $G^{-1}=(g^{ij})$ is the matrix for the induced metric on the cotangent bundle. To see this, observe that if $(\partial_i)^*=\sum_k B_{ik}dx^k$ for some smooth functions B_{ik}, then

$$\begin{aligned}(\partial_i)^*(\partial_j) &= \langle\partial_i,\partial_j\rangle = g_{ij}\\ &= \sum_k B_{ik}dx^k(\partial_j)=\sum_k B_{ik}\delta_j^k = B_{ij}.\end{aligned}$$

Thus, $(\partial_i)^*=\sum_k g_{ik}dx^k$. Then

$$\begin{aligned}g_{ij} &= \langle\partial_i,\partial_j\rangle = \langle(\partial_i)^*,(\partial_j)^*\rangle\\ &= \left\langle\sum_k g_{jk}dx^k,\sum_\ell g_{j\ell}dx^\ell\right\rangle\\ &= \sum_{k,\ell} g_{jk}g_{j\ell}\left\langle dx^k,dx^\ell\right\rangle.\end{aligned}$$

As a matrix equation, this is

$$G=G^2H,$$

where H is the matrix $(\langle dx^i,dx^j\rangle)$. We see that $H=G^{-1}=(g^{ij})$.

I claim the matrix for $\#$ in terms of the bases $dx^1,\dots,dx^n$ of T^*M and $\partial_1,\dots,\partial_n$ of TM is $G^{-1}=(g^{ij})$. To see this, suppose that

$$\#\left(dx^j\right)=\sum_k S^{jk}\partial_k$$

for some smooth functions S^{jk}, associated to the matrix $S = \left(S^{jk}\right)$. Observe that

$$\begin{aligned} \left\langle \# \left(dx^j\right), \partial_\ell \right\rangle &= \sum_k S^{jk} \left\langle \partial_k, \partial_\ell \right\rangle = \sum_k S^{jk} g_{k\ell} = (SG)^j_\ell \\ &= \left\langle dx^j, (\partial_\ell)^* \right\rangle = \sum_m \left\langle dx^j, g_{\ell m} dx^m \right\rangle \\ &= \sum_m g_{\ell m} \left\langle dx^j, dx^m \right\rangle = \sum_m g_{\ell m} g^{jm} = \delta^j_\ell \end{aligned}$$

Thus, as matrices, $SG = I$, and so $S = G^{-1}$. Therefore,

$$\# \left(dx^j\right) = \sum_k g^{jk} \partial_k.$$

Similarly, the flat musical isomorphism has matrix (g_{ij}):

$$\flat \left(\partial_i\right) = \left(\partial_i\right)^* = g_{ij} dx^j.$$

2.2. **Taming metrics.** Note that given any subriemannian manifold (M, H, g), it is always possible to find a riemannian metric h on TM that restricts to g on H. In this case, we say that h **tames** g. But there is no canonical way of choosing h. However, there is a way to canonically choose a degenerate metric on the cotangent bundle.

2.3. **Defining the cometric from the subriemannian metric.** Given a subriemannian manifold (M, H, g), recall that g is a positive definite inner product on the subbundle H and is not defined on nonhorizontal tangent vectors. Without further information such as a metric on all of TM, we are not able to extend g uniquely to be defined on TM, since it is not clear how to designate the complimentary subspace to H on which g vanishes. We define the bundle map $r : T^*M \to TM$ as follows: for any $\alpha \in T^*_xM$, the linear map $v \mapsto \alpha(v)$ for $v \in H_x$ can be represented uniquely by $\alpha(v) = g(v.Y_\alpha)$ for some $Y_\alpha \in H_x$. We define the map r by

$$r(\alpha) = Y_\alpha,$$

which can be seen to be linear in α. The image of $r|_{T^*_xM}$ is all of H_x, and r coincides with the map $\#$ in the case where H_x is replaced by T_xM. The transformation r varies smoothly with $x \in M$ and is symmetric and nonnegative; let (g^{ij}) be the corresponding matrix, in analogy with the riemannian case. Let $N^*_x \subset T^*_xM$ denote the kernel of r; this is the annihilator of H_x in T^*_xM.

Note that (g^{ij}) is nonnegative definite but is not positive definite since it is not onto. Since (g^{ij}) is not invertible, there is no analogue of the metric g_{ij} for subriemannian geometry. As a general rule of thumb, any formula of riemannian geometry that can be expressed in terms of raised indices alone will remain valid in subriemannian geometry. For example, the *raised Christoffel symbols* are (Einstein summation used)

$$\Gamma^{kpq} = \frac{1}{2}\left(g^{jp}\partial_j g^{kq} + g^{jq}\partial_j g^{kp} - g^{jk}\partial_j g^{pq}\right),$$

and the function

$$\begin{aligned} \Gamma^k(\xi, \beta) &= \Gamma^{kpq}\xi_p\beta_q, \\ \Gamma(\xi, \beta) &= \Gamma^{kpq}\xi_p\beta_q\partial_k \in T_xM \end{aligned}$$

for $\xi \in T^*_xM$, $\beta \in N^*_x$, is actually tensorial.

3. The horizontal gradient and geodesics in subriemannian geometry

In subriemannian geometry, horizontal curves are those whose tangent vectors lie in the distribution H. We define the length of a horizontal curve $\gamma : [a,b] \to M$ by

$$\ell(\gamma) = \int_a^b g(\gamma', \gamma')^{1/2},$$

and the energy of the curve is

$$E(\gamma) = \int_a^b g(\gamma', \gamma').$$

As in the riemannian case, the energy is minimized exactly when it minimizes length and has constant speed. The distance between points of M (called the *Carnot-Carathéodory distance*) is

$$d_C(p,q) = \inf\{\ell(\gamma) : \gamma \text{ is horizontal and connects } p \text{ with } q\}.$$

The **horizontal gradient** $\nabla^H f$ of a differentiable function $f : M \to \mathbb{R}$ is the horizontal vector field defined by

$$g\left(\nabla^H f, X\right) = X(f)$$

for all $X \in \Gamma(H)$. Then

$$\nabla^H f = \sum_{i=1}^{k} e_i(f)\, e_i \;,$$

if $e_1, ..., e_k$ is an orthonormal frame for H at the point in question.

Lemma 3.1. *If f is a smooth function and $g\left(\nabla^H f, \nabla^H f\right) = 0$, then f is constant.*

Let (M, H, g) be a subriemannian manifold. For any $p_q \in T_q^* M$, let

$$H(q,p) := \frac{1}{2}(p,p)_q$$

where $(\bullet, \bullet)_q$ is the cometric on $T_q^* M$. This is called the subriemannian Hamiltonian (kinetic energy).

For any horizontal curve $\gamma(t)$, then $\gamma'(t) = r_{\gamma(t)}(p)$ for some $p \in T^*_{\gamma(t)}M$. We define the Hamiltonian function

$$H(q,p) = \frac{1}{2}\left\|\gamma'(t)\right\|^2.$$

The function H uniquely determines the sharp map r by polarization.

Lemma 3.2. *The subriemannian structure is uniquely determined by its Hamiltonian. Conversely, any nonnegative fiber-quadratic Hamiltonian of constant fiber rank $k < n$ gives rise to a subriemannian structure whose distribution has rank k.*

Normal geodesic equations:

$$\left(q^i\right)' = \frac{\partial H}{\partial p_i},\ p_i' = -\frac{\partial H}{\partial q^i}.$$

4. The subDirac operator

Let $\nabla : \Gamma(H) \times \Gamma(H) \to \Gamma(H)$ be a metric connection on H. Note that ∇ is not an ordinary derivation, because it only accepts vectors from H. Suppose that H is oriented and admits a spin structure with associated spinor bundle S (bundle of irreducible $\mathbb{C}\mathrm{l}(H)$-modules), with spin connection ∇^S. We will think more generally of Clifford modules that are not necessarily irreducible at each point. Given a local orthonormal frame $(e_1, ..., e_d)$ of H, let

$$\omega_{ij}(\bullet) = g(\nabla_\bullet e_i, e_j).$$

We define

$$\nabla^S_X \varphi := X(\varphi) + \frac{1}{2}\sum_{i<j} \omega_{ji}(X)\, e_i \cdot e_j \cdot \varphi,$$

where $\cdot$ denotes Clifford multiplication.

We define the sub-Dirac operator as

$$D = \sum e_j \cdot \nabla^S_{e_j} : \Gamma(S) \to \Gamma(S).$$

In particular, when D acts the bundle of differential forms, the result of D^2 is the subLaplacian.

References

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