INTRO TO INDEX THEORY

1. Introduction to Introduction to Index Theory

Let X be a compact Riemannian manifold of dimension n. Let $V \overset{p_V}{\to} X$ and $W \overset{p_W}{\to} X$ be smooth complex vector bundles, both with rank m. Let $\mathrm{Sect}^{\infty}(V)$ denote the set of smooth sections of V. Let

$$D : \mathrm{Sect}^{\infty}(V) \to \mathrm{Sect}^{\infty}(W)$$

be called a first order differential operator if for all $s \in \mathrm{Sect}^{\infty}(V)$, we may write locally

$$Ds = \sum_{k=1}^{n} A_i(x) \frac{\partial s}{\partial x_i} + B(x)s,$$

where A_i, B are $m \times m$ matrices that varies smoothly in x. Note that we have the ball bundle

$$B(T^*X) \overset{\pi}{\to} X,$$

and so we have the vector bundle

$$\pi^*V \to B(T^*X).$$

(Note that $\pi^*V = \{(\xi, v) \in B(T^*X) \oplus V : \pi(\xi) = p(v)\}$.) We define the principal symbol $\sigma(D)$ of D as the bundle homomorphism

$$\sigma(D) : \pi^*V \to \pi^*W$$

such that

$$i\sigma(D)(\xi, v)_x = D(fs)(x) - f(x)(Ds)(x),$$

where $f \in C^{\infty}(X)$ with $df_x = \xi$. We say that D is elliptic if $\sigma(D)(\xi, v)_x$ is invertible for all $x \in X$, $v \in V$, $\xi \in T_x^*M \diagdown \{0\}$.

If D is elliptic, then $\ker D$ and $\ker D^*$ are finite dimensional and consist of smooth sections. Amazingly, though the dimensions of $\ker D$ and $\ker D^*$ may change as you deform D, the index

$$\mathrm{index}(D) := \dim\ker(D) - \dim\ker(D^*)$$

only depends on the homotopy class of $\sigma(D)$. An index theorem is a formula for the index of D in terms of $\sigma(D)$.

2. K-theory

Let X be a compact Hausdorff space, and let V a complex vector bundle over X. Let $[V]$ denote the isomorphism class of V. Let $K^0(X)$ be the Grothendieck (abelian) group of isomorphism classes (formal differences) of vector bundles over X. That is,

$$K^0(X) = \{[V_1] - [V_2] : V_1, V_2 \text{ vector bundles over } X\} \diagup \tilde{},$$

where $[V_1] - [V_2] \,\tilde{}\, [W_1] - [W_2]$ if

$$V_1 \oplus W_2 \oplus \widetilde{V} \cong W_1 \oplus V_2 \oplus \widetilde{V}$$

for some vector bundle $\widetilde{V}$. The addition operation is the direct sum. For another example, the Grothendieck completion of $\mathbb{N}$ is $\mathbb{Z}$. Note that K^0 is a contravariant functor from compact Hausdorff spaces to abelian groups. That is, a continous map $\phi : X \to Y$ induces a map $\phi^* : K^0(Y) \to K^0(X)$.

Noncompact version of K-theory: Suppose that X is locally compact and Hausdorff. Let X^+ be the one-point compactification of X, with $+$ denoting the extra point. We define

$$K^0(X) := \ker\left(K^0\left(X^+\right) \overset{\mathrm{inc}^*}{\to} K^0(+)\right),$$

so that

$$\begin{aligned} K^0\left(X^+\right) &\cong K^0(X) \oplus K^0(+) \\ &\cong K^0(X) \oplus \mathbb{Z}. \end{aligned}$$

This is the set of formal differences of vector bundles that are isomorphic outside a compact set, mod equivalence. Note that the morphisms in the locally compact category are those continous functions that are extendable to $+$. We have the following exact sequences

$$0 \to K^0(X) \to K^0\left(X^+\right) \to K^0(+) \to 0$$
$$K^0(X \setminus A) \to K^0(X) \to K^0(A)$$

Extra functorality: Let X be a locally compact Hausdorff space, and let $U \subset X$ be an open subset. Then we have an induced homomorphism

$$K^0(U) \to K^0(X) \to K^0(X \setminus U).$$

(The first map: take the difference of bundles to be $[A] - [B]$, with B trivial. Clearly B extends trivially, and then one glues [trivial] $-$ [trivial] on $X \setminus U$ to $[A] - [B]$ over U via (isomorphism) $-$ (identity), where "isomorphism" is the isomorphism between A and B outside a compact subset of U.)

Let (X, A) be a compact pair, that is, A is a nonempty closed subset of a compact Hausdorff space X. Consider triples (V, W, σ) where V and W are vector bundles over X, and $\sigma : V|_A \to W|_A$ is a bundle isomorphism (restricted to A, but it could be extended to a bundle homomorphism to X via Tietz extension theorem). We say that $(V_1, W_1, \sigma_1) \cong (V_2, W_2, \sigma_2)$ if there exist bundle isomorphisms $\alpha : V_1 \to V_2$ and $\beta : W_1 \to W_2$ such that the diagram

$$\begin{array}{ccc} V_1|_A & \overset{\sigma_1}{\to} & W_1|_A \\ \downarrow \alpha|_A & & \downarrow \beta|_A \\ V_2|_A & \overset{\sigma_2}{\to} & W_2|_A \end{array}$$

commutes. Let

$$\mathcal{V}(X, A) = \text{isomorphism classes } [V, W, \sigma],$$

with the addition

$$[V_1, W_1, \sigma_1] + [V_2, W_2, \sigma_2] := [V_1 \oplus V_2, W_1 \oplus W_2, \sigma_1 \oplus \sigma_2].$$

Let

$$\mathrm{Triv}(X, A) = \{[V, V, \mathrm{id}] : V \text{ a v.b. over } X\}.$$

Then

Theorem 2.1. *The monoid quotient*

$$K^0(X,A) := \mathcal{V}(X,A) \diagup \mathrm{Triv}(X,A)$$

is an abelian group.

As a consequence,

$$-[V,W,\alpha] = \left[W,V,\alpha^{-1}\right].$$

Theorem 2.2. *The following sequence is exact:*

$$K^0(X,A) \to K^0(X) \to K^0(A),$$

where the first map is $[V,W,\alpha] \mapsto [V]-[W]$.

Theorem 2.3. $K^0(X,A) \cong K^0(X \setminus A)$.

Theorem 2.4. *(Thom isomorphism) Let* $V \to X$ *is a complex vector bundle over a compact Hausdorff space* X. *Then*

$$K^0(V) \cong K^0(X).$$

3. The Atiyah-Singer Index Theorem

3.1. **Pre-construction of the topological index.** Let X be a (not necessarily compact manifold). Let $Y \subset X$ be a submanifold of lower dimension. Let N be the normal bundle of $Y \subset X$, thought of as an open tubular neighborhood of Y in X. The normal bundle to TY in TX looks like $N \oplus N \cong N \otimes \mathbb{C}$. By the Thom isomorphism theorem,

$$K^0(TY) \cong K^0(N \otimes \mathbb{C}) \hookrightarrow K^0(TX).$$

This composition $i_! : K^0(TY) \to K^0(TX)$. Next, embed M into $\mathbb{R}^N$. Then

$$i_! : K^0(TM) \to K^0\left(T\mathbb{R}^N\right) \cong K^0\left(\mathbb{R}^{2N}\right) \cong K^0\left(\mathbb{C}^N\right) \cong K^0(\{0\}) \cong \mathbb{Z}$$

(last one by Thom isomorphism). We call the composition of isomorphisms $(q_!)^{-1}$. The topological index

$$\mathrm{t-ind} : K^0(TM) \to \mathbb{Z}$$

is defined to be

$$\mathrm{t-ind} = (q_!)^{-1} \circ i_!.$$

Note

$$K^0(TX) \cong K^0(T^*X) \cong K^0(B^*X, S^*X).$$

4. The analytic index

Given an elliptic differential operator

$$D : C^\infty(V) \to C^\infty(W)$$

over a compact manifold X with symbol σ, we have $[\pi^*V, \pi^*W, \sigma] \in K^0(B^*X, S^*X)$, where $\pi : T^*X \to X$ is the vector bundle projection. So we define

$$\mathrm{t-ind}(D) = \mathrm{t-ind}([\pi^*V, \pi^*W, \sigma]).$$

Given any element $\left[\widetilde{V}, \widetilde{W}, \alpha\right] \in K^0(B^*X, S^*X)$, choose any elliptic pseudodifferential operator $P : C^\infty\left(\widehat{V}\right) \to C^\infty\left(\widehat{W}\right)$ such that $\left[\widetilde{V}, \widetilde{W}, \alpha\right] = [\pi^*V, \pi^*W, \sigma(P)] \in K^0(B^*X, S^*X)$.

One can always choose such a P (see original Atiyah-Singer papers in the Annals). The analytic index

$$\mathrm{a-ind} : K^0(B^*X, S^*X) \to \mathbb{Z}$$

is defined to be

$$\left[\widetilde{V}, \widetilde{W}, \alpha\right] \mapsto \dim\ker P - \dim\ker P^*.$$

Theorem 4.1. *(Atiyah-Singer Index Theorem) We have* $\mathrm{a-ind} = \mathrm{t-ind}$.

This may be translated as an integral of characteristic forms by using the Chern character from K-theory to cohomology. Actually, the Chern character is a natural transformation from the K-theory functor to the cohomology functor.

5. K-homology

Let $D : C^\infty(V) \to C^\infty(W)$ be an elliptic operator over X. Atiyah realized that this should give some kind of K-homology theory. Let $\Theta^n(X)$ be the trivial bundle of rank n over X. Then you can extend it by

$$D_{\Theta^n(X)} := D \otimes I_n : C^\infty(V \otimes \Theta^n(X)) \to C^\infty(W \otimes \Theta^n(X)).$$

This is called by "twisting" by a trivial bundle. If $\widetilde{V}$ is an arbitrary vector bundle over X, embed $\widetilde{V}$ into $\Theta^N(X)$ for N sufficiently large. Let

$$P : \Theta^N(X) \to \widetilde{V}$$

be the orthogonal projection (equivalent to choosing a connection). Then you can define

$$D_V = P\left(D_{\Theta^N(X)}\right)P : C^\infty\left(V \otimes \widetilde{V}\right) \to C^\infty\left(W \otimes \widetilde{V}\right),$$

which is D twisted by V.

Let the K-homology $K_0(X)$ of X is the dual theory to $K^0(X)$. It should be true that $K_0(X)$ is generated by elliptic operators. In fact, there is a pairing

$$K_0(X) \otimes K^0(X) \to \mathbb{Z}$$

given by

$$[D] \otimes [V] \mapsto \mathrm{index}(D_V),$$

which is nondegenerate. But there needs to be some K-theory orientation, which amounts to X being spin^c. In that case, the associated Dirac operator ∂ gives a map

$$[\partial] \otimes \cdot : K^0(X) \to K_0(X),$$

which is an isomorphism.

6. The cohomological form of the index theorem

6.1. **Characteristic classes of vector bundles.** Let $E \to M$ be a complex vector bundle of rank n, and let $P \to M$ be the corresponding bundle of unitary frames of E. Let ω be the $\mathfrak{u}(n)$-valued connection one-form on M, thus a skew-Hermitian matrix-valued one-form on M. This connection is related to the other notions of connection, such as

$$\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$$

via the following formula. If $(e_\beta) = (e_1, ..., e_m)$ is a local unitary frame field for E, then

$$\nabla e_\beta = \omega^\alpha{}_\beta \otimes e_\alpha$$

The curvature Ω of the connection ω is a $\mathfrak{u}(n)$-valued 2-form on M defined by the formula

$$\Omega = d\omega + \omega \wedge \omega,$$

or more precisely

$$\Omega^{\alpha}{}_{\beta} = d\omega^{\alpha}{}_{\beta} + \sum_{k} \omega^{\alpha}{}_{\gamma} \wedge \omega^{\gamma}{}_{\beta}.$$

Observe that the curvature operator $R(\cdot\,,\cdot)$ is related to this form, as follows. For any X, $Y \in \Gamma(M, TM)$ and $Z \in \Gamma(M, E)$, we have

$$\Omega(X \wedge Y) Z = R(X, Y) Z.$$

The *Chern classes* $c_k(E)$ of the vector bundle E are defined by the formula

$$\det\left(I + \frac{it}{2\pi}\Omega\right) = \sum c_k(E)\, t^k.$$

The formula only defines $c_k(E)$ as a form of degree $2k$, but in fact the cohomology class of this form is independent of the choice of connection. If the complex bundle E has rank n, one obtains n different characteristic classes (unless one counts $c_0(E) = 1$).

The *total Chern class* $c(E)$ is the class in $H^*(M)$ defined as

$$c(E) = \sum c_k(E).$$

The *Chern numbers* are the numbers obtained by integrating linear combinations of products of Chern classes over M. The top Chern class $c_n(E)$ of a vector bundle is the same as the Euler class $e(E_{\mathbb{R}})$ of its realization as a real vector bundle. The top Chern number is the obstruction to finding a nonzero section of E (ie if there exists a section of E, then $c_n(E) = 0$). If $c_n(E) = 0$, then $c_{n-1}(E)$ is the obstruction to find a pair of nonzero sections of E that are linearly independent at each point. And so on ....

Example 6.1. *Consider the Riemann sphere $\mathbb{C}P^1$ with the Kähler metric*

$$h = \frac{dz d\overline{z}}{1 + |z|^2}.$$

The tangent bundle $T\mathbb{C}P^1$ is a complex line bundle, and the curvature of the Levi-Civita connection is

$$\Omega = \frac{2dz \wedge d\overline{z}}{\left(1 + |z|^2\right)^2}.$$

The resulting Chern class is

$$c_1\left(T\mathbb{C}P^1\right) = \left[\frac{i}{2\pi}\Omega\right] = e\left(\left(T\mathbb{C}P^1\right)_{\mathbb{R}}\right).$$

The Chern number corresponding to this class is the Euler characteristic.

$$\int c_1 = \frac{i}{2\pi}\int \Omega = 2.$$

Example 6.2. *The tautological line bundle T over $\mathbb{C}P^k$ has total Chern class*

$$c(T) = 1 - \widetilde{H},$$

where $\widetilde{H}$ is the form/class that is the Poincaré dual of the submanifold $\mathbb{C}P^{k-1} \subset \mathbb{C}P^k$.

Other characteristic classes are obtained from the Chern forms by taking wedge products and linear combinations of the forms $c_k(E)$. One useful way of defining other characteristic classes is to pretend that the bundle is a direct sum of line bundles. Note that the only nontrivial class of a line bundle L is

$$x = c_1(L).$$

If

$$E = L_1 \oplus L_2 \oplus ... \oplus L_n,$$

then let

$$x_j = c_1(L_j)$$

for each j, and we see that

$$\begin{aligned} c_1(E) &= \sum x_j, \\ c_2(E) &= \sum x_j x_k, \end{aligned}$$

and in general $c_j(E)$ is the j^{th} elementary symmetric polynomial in the x_k's. Because of the *splitting principle*, one may often assume that the vector bundle is a direct sum of line bundles; if one is able to prove an identity in the Chern classes by using this assumption, then the result is true in general. In any case, expressing characteristic classes in terms of the x_k's instead of the c_k's is often convenient. Here is a list of useful characteristic classes. In all cases, we expand by a Taylor series and truncate (such as when the degree exceeds the dimension of the manifold). As before n is the rank of E.

$$\begin{aligned} \text{Total Chern class } c(E) &= \prod (1+x_j) = \sum c_j(E) \\ \text{Euler class } e(E_{\mathbb{R}}) &= \prod x_j = c_n(E) \\ \text{Chern character } ch(E) &= \sum e^{x_j} = n + c_1(E) + \left(\frac{c_1(E)^2}{2} - c_2(E) \right) + ... \\ \text{A-roof class } \widehat{A}(E) &= \prod \frac{x_j/2}{\sinh(x_j/2)} \\ \text{Todd class } Td_{\mathbb{C}}(E \otimes \mathbb{C}) &= \widehat{A}(E)^2 = \prod \frac{x_j}{1-e^{-x_j}} \\ \text{L class } L(E) &= \prod \frac{x_j}{\tanh(x_j)} \end{aligned}$$

The most important one of these is the Chern character, which is a ring homomorphism

$$ch : K(X) \to H^{\text{even}}(X),$$

which is an isomorphism onto rational cohomology. This homomorphism extends to the compactly supported versions. The reason for this is that if E_1 and E_2 are two vector bundles, then

$$\begin{aligned} ch(E_1 \oplus E_2) &= ch(E_1) + ch(E_2), \\ ch(E_1 \otimes E_2) &= ch(E_1)\, ch(E_2). \end{aligned}$$

Note that the following diagram does NOT commute:

$$\begin{array}{ccc} K_{cpt}(X) & \xrightarrow{ch} & H^{\text{even}}_{cpt}(X) \\ \downarrow^{i!} & & \downarrow^{i!} \\ K_{cpt}(E) & \xrightarrow{ch} & H^{\text{even}}_{cpt}(E) \end{array}$$

where $i!$ is the Thom isomorphism. The commutativity defect is measured by the Todd class:

$$(i!)^{-1} \circ ch \circ i! = (-1)^n \, Td_{\mathbb{C}} \left(\overline{E}\right)^{-1}.$$

When the vector bundle is the tangent bundle of a manifold M, the classes are often denoted

$$\begin{aligned} c(M) &= c(TM), \\ \widehat{A}(X) &= \widehat{A}(TX), \text{ etc.} \end{aligned}$$

6.2. **The index theorem.** The topological index may be reexpressed using the Chern classes.

Theorem 6.3. *(Cohomological form of the Atiyah-Singer Index Theorem) If σ is the principal symbol of D, then*

$$\begin{aligned} \mathrm{a} - \mathrm{ind}\left(D : \Gamma(X, V) \to \Gamma(X, W)\right) &= \mathrm{t} - \mathrm{ind}\left([\pi^* V, \pi^* W, \sigma]\right) \\ &= (-1)^{n(n+1)/2} \left[\pi!\left(ch(\sigma)\right) \widehat{A}(X)^2\right][X], \end{aligned}$$

where $\pi!$ is the Thom isomorphism, with $\pi : TX \to X$.

This theorem reduces to many other known theorems when the indices of specific differential operators are computed. The results are as follows:

Corollary 6.4. *(Gauss-Bonnet)*

$$\begin{aligned} \chi(X) &= \mathrm{index}\left(d + d^*|_{\Omega^{\text{even}} \to \Omega^{\text{odd}}}\right) \\ &= \int Pf(X). \end{aligned}$$

Corollary 6.5. *(Hirzebruch-Riemann-Roch)*

$$\begin{aligned} \chi(X, V) &= \mathrm{index}\left(\overline{\partial} + \overline{\partial}^* \Big|_{\Omega^{*,0}_{\text{even}} \otimes V \to \Omega^{*,0}_{\text{odd}} \otimes V}\right) \\ &= \left(ch(V)\, Td_{\mathbb{C}}(X)\right)[X]. \end{aligned}$$

Corollary 6.6. *(Hirzebruch Signature Theorem)*

$$\begin{aligned} \mathit{signature}(X) &= \mathrm{index}\left(d + d^*|_{\Omega^+ \to \Omega^-}\right) \\ &= L(X) = \left(L(TX)\right)[X] \end{aligned}$$

Corollary 6.7. *(spin Dirac operator index)*

$$\mathrm{index}\left(D^+\Big|_{S^+ \otimes E \to S^- \otimes E}\right) = \left(ch(E)\, \widehat{A}(X)\right)[X].$$

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129, USA
E-mail address: e.park@tcu.edu