1. Introduction to the Selberg Trace Formula

This is a talk about the paper H. P. McKean: Selberg’s Trace Formula as applied to a compact Riemann surface (1972). For simplicity, assume that $M$ is a compact Riemannian manifold. Consider classical mechanics on $M$, where a free particle on $M$ moves along geodesics. If $M$ has infinite fundamental group, then in each free homotopy class of a curve on $M$, there is a unique closed geodesic. If $M$ is a Riemann surface of genus $\geq 1$, we can look at the length of the (unique) closed shortest geodesic in each equivalence class from $\pi_1(M)$.

Next, consider quantum mechanics. Eigenvalues of the Laplacian on the manifold

$$0 = \gamma_1 < \gamma_2 \leq \gamma_3 \leq \ldots \uparrow +\infty$$

$\gamma_j$ is the energy of the $j^{th}$ “pure” state. We expect that there is a relation between the classical data (lengths of closed geodesics) and quantum data (eigenvalues). The Selberg trace formula provides this link. So there should be a “Selberg trace formula” on any manifold. There are many examples of this. When the manifold has a lot of symmetry (e.g., hyperbolic space mod a subgroup of $PSL(2, \mathbb{R})$), there is an example.

Starting with the 19th century: the Poisson summation formula. Let $M = \mathbb{C}/L$ be the 2-torus, where $L$ is an integral lattice. We let $L$ be the integral span of 1 and $a+ib$. Consider the Laplacian on this surface. Then the Laplacian will be

$$\Delta = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)$$

acting on $L$-periodic functions. The eigenfunctions are

$$f(x) = \exp(2\pi i \omega' \cdot x),$$

where $\omega'$ is an element of the dual lattice: that is $\omega' \cdot \omega \in \mathbb{Z}$ for all $\omega \in L$. The corresponding eigenvalues are $4\pi^2 |\omega'|^2$. The Poisson summation
formula relates the theta functions of $L$ and its dual lattice $L'$, as follows:

$$\sum_{\omega' \in L'} \exp \left( -4\pi^2 |\omega'|^2 t \right) = \frac{\text{area} ( M )}{4\pi t} \sum_{\omega \in L} \exp \left( -\frac{|\omega|^2}{4t} \right),$$
or

$$\sum_{\lambda_j} \exp (-\lambda_j t) = \frac{\text{area} ( M )}{4\pi t} \sum_{|\omega| \text{ length of closed geodesics}} \exp \left( -\frac{|\omega|^2}{4t} \right)$$

The theta function in physics language is the **partition function**. Also, this is the trace of the heat kernel:

$$\text{tr} \left( \exp (-t\Delta) \right) = \sum_{\lambda_j} \exp (-\lambda_j t),$$

where $\exp (-t\Delta)$ is the heat operator. The heat equation is

$$\frac{\partial}{\partial t} K = -\Delta_x K; \ K (0, x, y) = \delta (x - y),$$

$$\text{tr} \left( \exp (-t\Delta) \right) = \int_M K (t, x, x) \ dx$$

In the case of the torus,

$$K^T (t, x, x) = \sum_{\gamma \in L} K^{\mathbb{R}^2} (t, \gamma (x), x).$$

On the circle,

$$K^{S^1} (t, x, x) = \sum_{n \in \mathbb{Z}} K^{\mathbb{R}} (t, x + n, x).$$

To prove the Poisson summation formula, one expands the left side in terms of the Fourier series and uses the known heat kernel for $\mathbb{R}$:

$$K^{\mathbb{R}} (t, x, y) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{|x-y|^2}{4t} \right).$$

The knowledge of the spectrum of the Laplacian determines $|a|$ and $|b|$, as follows. For our torus with the lattice above, $\text{area} (M) = |b|$. After this, subtract terms from both sides the parts corresponding to geodesics with length 1, 2, 3, ... The next geodesic will be $\sqrt{|a|^2 + |b|^2}$, so we can find $|a|$. So we can determine the torus up to reflection.

Next, we generalize to a hyperbolic Riemann surface $M$ with genus $g \geq 2$. Then the spectrum $\sigma (M)$ of the Laplacian is

$$0 = \gamma_1 < \gamma_2 \leq \gamma_3 \leq ... \rightarrow +\infty.$$
The Selberg trace formula relates the trace
\[ \text{tr} \left( \exp \left( -t \Delta \right) \right) = \sum_{j=1}^{\infty} \exp \left( -t \lambda_j \right) \]
of the heat kernel to the kind of dual theta function. The role of the
lattice \( L \) is taken over by the conjugacy classes \( Q \) of \( G = \pi_1(M) \),
identified with a subgroup of \( SL(2, \mathbb{R}) \). Then
\[ M = SL(2, \mathbb{R}) \backslash G. \]
The numbers \(|w|\) are replaced by
\[ l(Q) = 2 \cosh^{-1} \left( \frac{1}{2} \text{tr}(Q) \right). \]
Here, \( Q \) is a free deformation class of closed paths on \( M \), and \( l(Q) \)
is the length of the shortest path in this class. There is a famous
(noncompact) cases that we will not cover
\[ M = SL(2, \mathbb{R}) \backslash SL(2, \mathbb{Z}) \]
or
\[ M = SL(2, \mathbb{R}) \backslash \Gamma \]
where \( \Gamma \) is an algebraic subgroup of \( SL(2, \mathbb{Z}) \). Audrey Terras and Serge
Lang have good books on the subject. Also there is a survey paper by
Werner Müller.

2. Riemann surface formula

Let \( M \) be a compact Riemann surface of genus \( g \geq 2 \). By the
Riemann uniformization theorem, the universal cover is the upper half
plane \( \mathbb{H} \).
\[ \mathbb{H} = \{(x_1 + ix_2) : x_2 > 0\} \]
This is also called the Poincaré hyperbolic plane, with metric
\[ ds^2 = \frac{dx_1^2 + dx_2^2}{x_2^2} \]
(You can also realize this as the Poincare disk.) The fundamental
group \( \pi_1(M) \) acts by deck transformations on \( \mathbb{H} \) that are isometries
\[ z \mapsto \frac{az+b}{cz+d}, \]
such that \( \det \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = 1 \) with \( a, b, c, d \in \mathbb{R} \). So \( SL(2, \mathbb{R}) \)
is the group of isometries of \( \mathbb{H} \). Thus, we can identify \( \pi_1(M) \) with a
subgroup \( G \) of \( SL(2, \mathbb{R}) \). The \( G \) has a fundamental region in \( \mathbb{H} \) that is
a hyperbolic polygon with \( 4g \) sides.
What must be true about $G$ in order that $\mathbb{H}/G$ is a compact Riemann surface? Note that $SL(2,\mathbb{R}) = KAN$, where $K = SO(2) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \right\}$ is the stabilizer of $i$. The group $A$ is the group of magnifications $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\}$. The group $N$ is the group of horizontal translations, $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$.

**Proposition 1.** For any $g \in SL(2,\mathbb{R})$, $g$ is conjugate to

- rotation iff $\text{tr} \ (g) < 2$ (elliptic)
- magnification iff $\text{tr} \ (g) > 2$ (hyperbolic)
- translation iff $\text{tr} \ (g) = 2$ (parabolic)

The hyperbolic distance $d(x,y)$ satisfies

$$d(x,gx) = d(kx,kgx) = d(kx,(kgk^{-1})kx),$$

for all $k,g$ in $SL(2,\mathbb{R})$. So

$$\inf_x d(x,gx) = \inf_x d(kx,(kgk^{-1})kx) = \inf_x d(x,(kgk^{-1})x)$$

Think of elements of $G$ as homotopy classes of closed paths on $M$ with fixed base point.

Free homotopy classes of $M$ are identified with the conjugacy class $Q = \{kgk^{-1} : k \in G\}$.

Every nontrivial element of $G$ is conjugate to a hyperbolic element. (Proof: if an element $g$ of $G$ is not the identity, then there is a geodesic of minimum length connecting some $x$ to $gx$, but if $g$ is parabolic or elliptic, this length can go to zero.) Thus it is conjugate to a magnification, and thus $\ell(g^n) = |n|\ell(g)$ is true, where $\ell(g) = \inf_x d(x,gx)$ is the length of the shortest path.

For every $g \in G$ that is nontrivial, it can be expressed in a unique way as the positive power of a primitive element $p \in G$ (primitive: it is not the power of any other element of $G$).

**Proposition 2.** As $p$ runs through the inconjugate primitive elements in $G$ and $n$ through the positive integers, the conjugacy class $Q = \{kp^nk^{-1} : k \in G/G_p\}$ runs through the conjugacy classes of $G$. Here, $G_p$ is the centralizer of $p$. Moreover, for fixed $p$, $n$, elements $kp^nk^{-1}$ run once through $Q$ as $k$ runs through $G/G_p$. 
Note that \( d(x,y) = \cosh^{-1}\left(1 + \frac{\|x-y\|}{2x_2y_2}\right) \), \( \ell(p^n) = n|\log m|^2 \) where \( p \sim \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \).

**Theorem 3.** (Selberg trace formula) Start with a function \( K : \mathbb{R} \to \mathbb{R} \) that decays sufficiently rapidly as \( x \to \infty \). Then \( K_H(x,y) = K(\cosh d(x,y)) \) is a function on \( \mathbb{H} \times \mathbb{H} \). It induces a symmetric kernel on \( M \times M \) via

\[
K_M(x,y) = \sum_{g \in G} K_H(x,gy).
\]

Then

\[
K_M(x,hy) = K_M(x,y)
\]

for all \( h \in G \). Then, with \( dx = \frac{dx_1dx_2}{x_2^2} \) = hyperbolic volume element

\[
\text{tr}K_M := \int_M K_M(x,x) \, dx
\]

\[
= \text{area}(M) K(1) + \sum_{n=1}^{\infty} \sum_{\text{inconjugate } p \text{ primitive } k \in G/G_p} \frac{\ell(p)}{\sqrt{\cosh \ell(p^n) - 1}} \int_0^\infty \frac{K(b) \, db}{\sqrt{b - \cosh \ell(p^n)}}.
\]

**Proof.** Let \( F \) be a fundamental domain of \( M \). We have

\[
\text{tr}K_M = \int_F K_M(x,x) \, dx
\]

\[
= \sum_{g \in G} \int_M K(\cosh d(x,gx)) \, dx
\]

\[
= \text{area}(M) K(1) + \sum_{n=1}^{\infty} \sum_{\text{inconjugate } k \in G/G_p} \int_F K(\cosh d(x,kp^n x^{-1}x)) \, dx
\]

\[
= \text{area}(M) K(1) + \sum_{n=1}^{\infty} \sum_{\text{inconjugate } k \in G/G_p} \int_F K(\cosh d(x,p^n x)) \, dx
\]

\[
= \text{area}(M) K(1) + \sum_{n=1}^{\infty} \sum_{\text{inconjugate } p \text{ primitive } k \in G/G_p} \int_{F_p} K(\cosh d(x,p^n x)) \, dx
\]

where \( F_p \) is a fundamental domain for \( G_p \). Then \( p \) is conjugate to some magnification \( x \mapsto m^2x \). Then \( F_p = \{x_1 \in \mathbb{R} : 1 \leq x_2 \leq m^2\} \).

The formula follows from a direct calculation.  \( \square \)
In the particular case where $K_H(t)$ is the fundamental solution of the heat equation on $\mathbb{H}$. Then $K_H(t) = \exp(-t\Delta)$. Then

$$\text{tr} \left( \exp \left( -t\Delta \right) \right) = \sum_{n=0}^{\infty} \exp(-t\gamma_n)$$

$$= \text{area}(M) \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_{0}^{\infty} \frac{be^{-b^2/4t}}{\sinh \left( \frac{1}{2}b \right)} db$$

$$+ \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\text{inconjugate primitive } p} \frac{\ell(p)}{\sinh \left( \frac{1}{2} \ell(p') \right)} \frac{e^{-t/4}}{(4\pi t)^{1/2}} e^{-|\ell(p')|^2/4t}.$$