SHEAF THEORY

1. Presheaves

Definition 1.1. A presheaf on a space \( X \) (any top. space) is a contravariant functor from open sets on \( X \) to a category (usually \( \text{Ab} = \) category of abelian groups).

That is, given \( U \) open \( \subseteq X \), you have an abelian group \( A(U) \), and if \( V \subseteq U \), then you have a restriction map \( r_{V,U}: A(U) \to A(V) \). Then \( r_{U,U} = 1 \), \( r_{W,V}r_{V,U} = r_{W,U} \) etc.

Example 1.2. Let \( A(U) = G \), a fixed abelian group \( \forall U \), \( r_{U,V} = 1 \) (constant presheaf)

Example 1.3. Let \( A(U) = G \), a fixed abelian group \( \forall U \), \( r_{U,V} = 0 \) (Texas \( \text{A&}M \) presheaf)

Example 1.4. Let \( A(X) = G \), \( A(U) = 0 \) \( \forall \) proper \( U \subseteq X \) (TCC presheaf)

Example 1.5. Skyscraper presheaf: \( x \in X \); \( A(U) = \begin{cases} 0 & \text{if } x \notin U \\ Z & \text{if } x \in U \end{cases} \)

Example 1.6. \( X = M \), \( A(U) = \{\text{real analytic fcns on } U\} \)

Example 1.7. adapt above using words like continuous, holomorphic, polynomial, differential forms, \( k \)th cohomology group \( H^k(U) \),

Example 1.8. or \( H^\infty_k(U) \) (closed support homology). For example, \( H^\infty_k(\text{open disk in } \mathbb{R}^2) = \begin{cases} \mathbb{Z} & k = 2 \\ 0 & \text{otherwise} \end{cases} \)

restriction maps: cut things off

Note: Poincare duality for closed support: \( H^*(M) = H^{\infty,n-*}(M) \) (\( M \) possibly noncompact)

Poincare duality for compact support: \( H^*_c(M) = H^{n-*}_c(M) \)

Example 1.9. Orientation presheaf: on an \( n \)-manifold, \( H^\infty_n(U) \)

2. Sheaves

Definition 2.1. A sheaf \( \mathcal{A} \) is a space together with a projection \( \pi : \mathcal{A} \to X \) such that

1. \( \mathcal{A}_x := \pi^{-1}(x) \) is an abelian group (or could generalize to other categories) \( \forall x \in X \) (stalk at \( x \))

2. \( \pi \) is a local homeomorphism (\( \Rightarrow \pi^{-1}(x) \) is discrete)

3. group operations are continuous, ie \( \{(a,b) \in \mathcal{A} \times \mathcal{A} \mid \pi(a) = \pi(b)\} \to \mathcal{A} \) given by \( (a,b) \mapsto ab^{-1} \) is continuous.

Note: often not Hausdorff. (eg line with two origins (which is a skyscraper sheaf), or over \( \mathbb{R} \) you could have \( \mathbb{Z} \) above all nonpositive \( x \) and the 0 group for positive \( x \)).

Example 2.2. Constant sheaf: \( X \times \mathbb{Z} \).

Example 2.3. Twisted constant sheaf: \( S^1 \times_T \mathbb{Z} \) (\( 1 \mapsto -1 \) upon identification on ends of \([0,2\pi]\)).
Example 2.4. \( X = \mathbb{R}, \mathcal{A}_x = \mathbb{Z} \) if \( x \in (0, 1) \), \( = \{0\} \) otherwise.

Example 2.5. \( X = \mathbb{R}, \mathcal{A}_x = \mathbb{Z} \) if \( x \in [0, 1] \), \( = \{0\} \) otherwise. Note that total space must have non-Hausdorff topology (above \( x = 0, x = 1 \)).

Example 2.6. Skyscraper at origin: \( X = \mathbb{R}, \mathcal{A}_x = \mathbb{Z} \) if \( x = 0 \), \( = \{0\} \) otherwise. Note that total space must have non-Hausdorff topology (above \( x = 0 \), \( x = 1 \)).

Example 2.7. Everywhere skyscraper: \( \prod_{x \in \mathbb{R}} \mathbb{Z}_x \). Use your imagination. Extremely non-Hausdorff — note stalks all the same as in \( \mathbb{R} \times \mathbb{Z} \).

Example 2.8. Sheaf of germs of smooth (pick your favorite adjective) functions. Stalk at \( x = \{\text{germs of functions at } x\} \).

Note: by definition, germ of \( f \) at \( x = \lim_{\to x \in U} (f|_U) \) germ of \( f(x) = 0 \) at \( x = 0 \) is not the same as the germ of \( f(x) = x \) at \( x = 0 \), but it does have the same germ as that of \( f(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 1 \\ x - 1 & \text{if } x > 1 \\ x + 1 & \text{if } x < 1 \end{cases} \).

3. Sheafification

There exists a functor \( \Gamma: \{\text{Sheaves} \} \rightarrow \{\text{Presheaves} \} \) (take sections).

\[ A \mapsto A(U) = \Gamma(U; A) \]

The functor is not onto.

The functor \( sh: \{\text{Presheaves} \} \rightarrow \{\text{Sheaves} \} \) is Sheafification

This functor projects to conjunctive mono-presheaves (category equivalence).

\( \Gamma(sh(A)) \neq A \), but \( sh(\Gamma(A)) = A \), however \( \Gamma(sh(A)) = A \) if \( A \) is conjunctive, mono.

Suppose we have a sheaf \( A \), take \( s \in \Gamma(X, A) \). Suppose that \( s_x = 0 \ \forall x \). Then \( s = 0 \). However this is not true for every presheaf. Recall the TCC presheaf ( \( A(X) = G, A(U) = 0 \) for proper \( U \subseteq X \)).

Definition 3.1. (Sheafification) Given a presheaf \( A \), the stalk over \( x \) is \( \pi^{-1}(x) = \lim_{x \in U} A(U) \), with the following topology. Given any \( s_x \in \pi^{-1}(x) \), \( s_x \) is represented by some \( s \in A(V) \), where \( x \in V \). A basis element is the collection of images of \( s \) under the direct limit in other \( y \in V \) close to \( x \).

Definition 3.2. \( A \) is a mono presheaf if for any element \( s \in A(X) \) and any \( x \in X \), \( s_x = 0 \), then \( s = 0 \in A(X) \).

Definition 3.3. \( A \) is conjunctive if given open sets \( U_j \) and sections \( s_j \in A(U_j) \) such that \( s_j|_{U_i \cap U_j} = s_i|_{U_i \cap U_j} \), then there exists \( s \in A(\bigcup U_i) \) such that \( s|_{U_i} = s_i \).

4. Sheaf Cohomology

4.1. Derived functors. Tor (related to tensor product \( \otimes B \))

\[ \text{Ext} \ (\text{related to } Hom(\cdot, B)) \]

Let \( A \) and \( B \) be abelian groups. Then consider \( \otimes B \) as a functor.
Consider an exact sequence of groups (called a free resolution - all groups are free except $A$)

$$
... \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0
$$

Let $$
... \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0 = F_*
$$

Now cut out the $A$: $H_0 (F_*) = A$, $H_i (F_*) = 0$ for $i > 0$.

Next, the left derived functor of $A \otimes B$ is $H_i (F_* \otimes B) = Tor_i (A, B)$. Tensor is only right exact, so you don’t get trivial torsion. Note that $H_0 (F_* \otimes B) = Tor_0 (A, B) = A \otimes B$. Note that this is independent of choice of resolution.

Example: $Tor (\mathbb{Z}_2, \mathbb{Z}_2)$:

$$
\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2
$$

So

$$
H_i (F_* \otimes \mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2 & i = 1, 0 \\
0 & \text{otherwise}
\end{cases}
$$

In general, $Tor (\mathbb{Z}_a, \mathbb{Z}_b) = \mathbb{Z} / \gcd (a, b)$.

You could use projective modules (direct summand of a free module) instead of free modules.

Tensor: right exact covariant, Hom: left exact contravariant

Suppose $F$ is a left exact covariant functor from an abelian category to groups (eg $Hom (A, \cdot)$). Then the right derived functor $R^i F (A) = F (I^*)$ is defined as follows. Here $I^*$ is an injective resolution, that is

$$
A \rightarrow I^0 \rightarrow I^1 \rightarrow ...
$$
is exact, and each $I^j$ is injective, meaning this

$$
\begin{align*}
0 & \rightarrow A \\
\downarrow & \\
I & \rightarrow B
\end{align*}
$$

Then $H^* (I^*) = H^* (A)$, and

$$
R^i F (A) := H^i (F (I^*)).
$$

**Definition 4.1.** The class $\mathcal{J}$ of objects in the category is “adapted to $F$” or “$F$ acyclic” if $R^i (F (J)) = 0$ for $i > 0$, $J \in \mathcal{J}$.

For example, the sheaf of differential forms is $\Gamma (X; \cdot)$ acyclic. (Any soft, flabby, or fine sheaf works.)

**Theorem 4.2.** If we have a resolution $J^*$ of $A$ by objects in a class $\mathcal{J}$ adapted to $F$, then

$$
R^i F (A) = H^i (F (I^*)) = H^i (F (J^*)).
$$

A morphism of sheaves is

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
$$
such that each $f_x : A_x \rightarrow B_x$ is a homomorphism.

Given $f : A \rightarrow B$ (over $X$), there exists

$$
\ker f = \{ x \in A \text{ such that } f (x) = 0 \}
$$

$$
\operatorname{coker} f = \{ \operatorname{cok} \text{ of stalk} \}.$$
Notice $f : \Gamma (U; \mathcal{A}) \to \Gamma (U, \mathcal{B})$ implies
\[
cok f = \text{Sh} \left( U \to \frac{\Gamma (U; \mathcal{B})}{\text{Im} (\Gamma (U, \mathcal{A}) \to \Gamma (U, \mathcal{B}))} \right)
\]

Note $\Gamma (U; \cdot)$ is left exact.

**Example 4.3.** Let $X = \mathbb{R}$, let $\mathcal{A} = \mathbb{Z}_{(0,1)} = \mathbb{Z}_{\mathbb{R}}$, $f : \mathcal{A} \to \mathcal{B}$ inclusion. On $\mathcal{A}$, the global sections are the trivial section. We have
\[
\begin{align*}
0 & \to \mathcal{A} \to \mathcal{B} \to \text{cok } f \to 0 \\
0 & \to \Gamma (\mathbb{R}; \mathcal{A}) \to \Gamma (\mathbb{R}; \mathcal{B}) \to \Gamma (\mathbb{R}; \text{cok } f) \text{ is}
\end{align*}
\]

We define the sheaf cohomology as
\[
H^i (X; \mathcal{A}) := R^i \Gamma (X; \mathcal{A}) = H^i (\Gamma (X; I^*))
\]

**Example 4.4.** Let $M$ be a manifold. Let $\mathbb{R}$ be the constant sheaf. Resolution of sheaves (exact because exact at stalks because of Poincaré Lemma):
\[
\mathbb{R} \to \Omega^0 (M) \to \Omega^1 (M) \to ...
\]

Everything beyond the first is soft. Then
\[
H^i (M; \mathbb{R}) = H^i (\Gamma (M, \Omega^*)) = H^i_{\text{dR}} (M).
\]

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129, USA

E-mail address: g.friedman@tcu.edu