SHEAF THEORY

1. Presheaves

Definition 1.1. *A* presheaf *on a space X (any top. space) is a contravariant functor from open sets on X to a category (usually Ab= category of abelian groups).*

That is, given U open $\subseteq X$, you have an abelian group $A(U)$, and if $V \subseteq U$, then you have a restriction map $r_{V,U} : A(U) \to A(V)$. Then $r_{U,U} = \mathbf{1}$, $r_{W,V} r_{V,U} = r_{W,U}$ etc.

Example 1.2. *Let $A(U) = G$, a fixed abelian group $\forall U$, $r_{U,V} = \mathbf{1}$ (constant presheaf)*

Example 1.3. *Let $A(U) = G$, a fixed abelian group $\forall U$, $r_{U,V} = \mathbf{0}$ (Texas A&M presheaf)*

Example 1.4. *Let $A(X) = G$, $A(U) = 0$ $\forall$proper $U \subseteq X$ (TCC presheaf)*

Example 1.5. *Skyscraper presheaf: $x \in X$; $A(U) = \begin{cases} 0 & \text{if } x \notin U \\ G & \text{if } x \in U \end{cases}$*

Example 1.6. *$X = M$, $A(U) = \{$real analytic fcns on $U\}$*

Example 1.7. *adapt above using words like continuous, holomorphic, polynomial, differential forms, kth cohomology group $H^k(U)$,*

Example 1.8. *or $H_k^\infty(U)$ (closed support homology). For example, H_k^∞ (open disk in $\mathbb{R}^2$) $= \begin{cases} \mathbb{Z} & k = 2 \\ 0 & \text{otherwise} \end{cases}$*
restriction maps: cut things off

Note: Poincare duality for closed support: $H^*(M) = H_{n-*}^\infty(M)$ (M possibly noncompact)
Poincare duality for compact support: $H_c^*(M) = H_{n-*}^c(M)$

Example 1.9. *Orientation presheaf: on an n-manifold, $H_n^\infty(U)$*

2. Sheaves

Definition 2.1. *A* sheaf *$\mathcal{A}$ is a space together with a projection $\pi : \mathcal{A} \to X$ such that*

1. *$\mathcal{A}_x := \pi^{-1}(x)$ is an abelian group (or could generalize to other categories) $\forall x \in X$ (stalk at x)*
2. *π is a local homeomorphism ($\Rightarrow \pi^{-1}(x)$ is discrete)*
3. *group operations are continuous, ie $\{(a,b) \in \mathcal{A} \times \mathcal{A} \mid \pi(a) = \pi(b)\} \to \mathcal{A}$ given by $(a,b) \mapsto ab^{-1}$ is continuous.*

Note : often not Hausdorff. (eg line with two origins (which is a skyscraper sheaf), or over $\mathbb{R}$ you could have $\mathbb{Z}$ above all nonpositive x and the 0 group for positive x).

Example 2.2. *Constant sheaf: $X \times \mathbb{Z}$.*

Example 2.3. *Twisted constant sheaf: $S^1 \times_T \mathbb{Z}$ ($1 \mapsto -1$ upon identification on ends of $[0, 2\pi]$.)*

Example 2.4. *$X = \mathbb{R}$, $\mathcal{A}_x = \mathbb{Z}$ if $x \in (0,1)$, $= \{0\}$ otherwise.*

Example 2.5. *$X = \mathbb{R}$, $\mathcal{A}_x = \mathbb{Z}$ if $x \in [0,1]$, $= \{0\}$ otherwise. Note that total space must have nonHausdorff topology (above $x = 0$, $x = 1$).*

Example 2.6. *Skyscraper at origin: $X = \mathbb{R}$, $\mathcal{A}_x = \mathbb{Z}$ if $x = 0$, $= \{0\}$ otherwise.*

Example 2.7. *Everywhere skyscraper: $\prod_{x \in \mathbb{R}} \mathbb{Z}_x$. Use your imagination. Extremely nonHausdorff — note stalks all the same as in $\mathbb{R} \times \mathbb{Z}$.*

Example 2.8. *Sheaf of germs of smooth (pick your favorite adjective) functions.*
Stalk at $x = \{$germs of functions at $x\}$.

Note : by definition,

germ of f at $x = \varinjlim_{x \in U} (f|_U)$

germ of $f(x) = 0$ at $x = 0$ is not the same as the germ of $f(x) = x$ at $x = 0$, but it does have the same germ as that of $f(x) = \begin{cases} 0 & \text{if } -1 \le x \le 1 \\ x - 1 & \text{if } x > 1 \\ x + 1 & \text{if } x < 1 \end{cases}$

3. Sheafification

There exists a functor Γ:{Sheaves } $\to$ {Presheaves} (take sections).

$\mathcal{A} \mapsto A(U) = \Gamma(U; \mathcal{A})$

The functor is not onto.

The functor sh:{Presheaves} $\to$ {Sheaves } is *Sheafification*

This functor projects to conjunctive mono-presheaves (category equivalence).

$\Gamma(sh(A)) \neq A$, but $sh(\Gamma(\mathcal{A})) = \mathcal{A}$, however $\Gamma(sh(A)) = A$ if A is conjunctive, mono.

Suppose we have a sheaf $\mathcal{A}$, take $s \in \Gamma(X, \mathcal{A})$ .Suppose that $s_x = 0$ $\forall x$. Then $s = 0$. However this is not true for every presheaf. Recall the TCC presheaf ($A(X) = G$, $A(U) = 0$ for proper $U \subseteq X$).

Definition 3.1. *(Sheafification) Given a presheaf A, the stalk over x is $\pi^{-1}(x) = \varinjlim_{x \in U} A(U)$, with the following topology. Given any $s_x \in \pi^{-1}(x)$, s_x is represented by some $s \in A(V)$, where $x \in V$. A basis element is the collection of images of s under the direct limit in other $y \in V$ close to x.*

Definition 3.2. *A is a* mono *presheaf if for any element $s \in A(X)$ and any $x \in X$, $s_x = 0$, then $s = 0 \in A(X)$.*

Definition 3.3. *A is* conjunctive *if given open sets U_j and sections $s_j \in A(U_j)$ such that $s_j|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}$, then there exists $s \in A(\bigcup U_i)$ such that $s|_{U_i} = s_i$.*

4. Sheaf Cohomology

4.1. **Derived functors.** Tor (related to tensor product $\otimes B$)

Ext (related to $Hom(\cdot, B)$)

Let A and B be abelian groups. Then consider $\otimes B$ as a functor.

Consider an exact sequence of groups (called a free resolution - all groups are free except A)

$$\begin{array}{rcl} ... & \rightarrow & F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0 \\ \text{Let } ... & \rightarrow & F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0 = F_* \end{array}$$

Now cut out the A: $H_0(F_*) = A$, $H_i(F_*) = 0$ for $i > 0$.

Next, the left derived functor of $A \otimes B$ is $H_i(F_* \otimes B) = Tor_i(A, B)$. Tensor is only right exact, so you don't get trivial torsion. Note that $H_0(F_* \otimes B) = Tor_0(A, B) = A \otimes B$. Note that this is independent of choice of resolution.

Example: $Tor(\mathbb{Z}_2, \mathbb{Z}_2)$:

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}_2$$

So

$$H_i(F_* \otimes \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i = 1, 0 \\ 0 & \text{otherwise} \end{cases}$$

In general, $Tor(\mathbb{Z}_a, \mathbb{Z}_b) = \mathbb{Z} / \gcd(a, b)$.

You could use projective modules (direct summand of a free module) instead of free modules.

Tensor: right exact covariant, Hom: left exact contravariant

Suppose F is a left exact covariant functor from an abelian category to groups (eg $Hom(A, \cdot)$). Then the right derived functor $RF(A) = F(I^*)$ is defined as follows. Here I^* is an injective resolution , that is

$$A \rightarrow I^0 \rightarrow I^1 \rightarrow ...$$

is exact, and each I^j is injective, meaning this

$$\begin{array}{ccccc} 0 & \rightarrow & A & \rightarrow & B \\ & & \downarrow & (\swarrow) & \\ & & I & & \end{array}$$

Then $H^*(I^*) = H^*(A)$, and

$$R^i F(A) := H^i(F(I^*)).$$

Definition 4.1. *The class $\mathcal{J}$ of objects in the category is "adapted to F" or "F acyclic" if $R^i(F(J)) = 0$ for $i > 0$, $J \in \mathcal{J}$.*

For example, the sheaf of differential forms is $\Gamma(X; \cdot)$ acyclic. (Any soft, flabby, or fine sheaf works.)

Theorem 4.2. *If we have a resolution J^* of A by objects in a class $\mathcal{J}$ adapted to F, then*

$$R^i F(A) = H^i(F(I^*)) = H^i(F(J^*)).$$

A morphism of sheaves is

$$\begin{array}{ccccc} & f: & \mathcal{A} & \rightarrow & \mathcal{B} \\ \mathcal{A} & & \searrow & & \swarrow \\ & & & X & \end{array}$$

such that each $f_x : \mathcal{A}_x \rightarrow \mathcal{B}_x$ is a homomorphism.

Given $f : \mathcal{A} \rightarrow \mathcal{B}$ (over X), there exists

$$\begin{array}{rcl} \ker f & = & \{x \in \mathcal{A} \text{ such that } f(x) = 0\} \\ co\ker f & = & \{\text{cok of stalk}\} \end{array}$$

Notice $f : \Gamma(U;\mathcal{A}) \to \Gamma(U,\mathcal{B})$ implies

$$cok\ f = Sh\left(U \to \frac{\Gamma(U;\mathcal{B})}{\operatorname{Im}(\Gamma(U,\mathcal{A}) \to \Gamma(U,\mathcal{B}))}\right)$$

Note $\Gamma(U;\cdot)$ is left exact.

Example 4.3. *Let $X = \mathbb{R}$, let $\mathcal{A} = \mathbb{Z}_{(0,1)}$, $= \mathbb{Z}_{\mathbb{R}}$, $f : \mathcal{A} \to \mathcal{B}$ inclusion. On $\mathcal{A}$, the global sections are the trivial section. We have*

$$\begin{aligned} 0 &\to \mathcal{A} \to \mathcal{B} \to cok\ f \to 0 \\ 0 &\to \Gamma(\mathbb{R};\mathcal{A}) \to \Gamma(\mathbb{R};\mathcal{B}) \to \Gamma(\mathbb{R}; cok\ f) \ \ \textit{is} \\ 0 &\to 0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

We define the sheaf cohomology as

$$H^i(X;\mathcal{A}) := R^i\Gamma(X;\mathcal{A}) = H^i(\Gamma(X;I^*))$$

Example 4.4. *Let M be a manifold. Let $\mathbb{R}$ be the constant sheaf. Resolution of sheaves (exact because exact at stalks because of Poincaré Lemma):*

$$\mathbb{R} \to \Omega^0(M) \to \Omega^1(M) \to \ldots$$

Everything beyond the first is soft. Then

$$\begin{aligned} H^i(M;\mathbb{R}) &= H^i(\Gamma(M,\Omega^*)) \\ &= H^i_{dR}(M). \end{aligned}$$

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