DIFFERENTIABLE STRUCTURES

1. Classification of Manifolds

1.1. **Geometric topology.** The goals of this topic are

(1) Classify all spaces - ludicrously impossible.
(2) Classify manifolds - only provably impossible (because any finitely presented group is the fundamental group of a 4-manifold, and finitely presented groups are not able to be classified).
(3) Classify manifolds with a given homotopy type (sort of worked out in 1960's by Wall and Browder if $\pi_1 = 0$. There is something called the Browder-Novikov-Sullivan-Wall surgery (structure) sequence:

$$\to L\left(\pi_1\left(X\right)\right) \to \mathcal{S}_0\left(X\right) \to \left[X, \mathcal{G}/\mathcal{O}\right]$$

LHS=algebra, RHS=homotopy theory, $\mathcal{S}_0\left(X\right)$ is the set of smooth manifolds h.e. to X up to some equivalence relation.

Classify means either

(1) up to homeomorphism
(2) up to PL homeomorphism
(3) up to diffeomorphism

A manifold is smooth iff the tangent bundle is smooth. These are classified by the orthogonal group $\mathcal{O}$. There are analogues of this for the other types ($\mathcal{TOP}$, $\mathcal{PL}$). Given a space X, then homotopy classes of maps

$$\left[X, \mathcal{TOP}/\mathcal{PL}\right]$$

determine if X is PL-izable. It turns out $\mathcal{TOP}/\mathcal{PL} =$ a $K\left(\mathbb{Z}_2, 3\right)$. It is a harder question to look at if X has a smooth structure.

1.2. **Classifying homotopy spheres.** X is a homotopy sphere if it is a topological manifold that is h.e. to a sphere. It turns out that if $\pi_1\left(X\right) = 0$ and $H_*\left(X\right) = H_*\left(S^n\right)$, then it is a homotopy sphere. The Poincare Conjecture is that every homotopy n-sphere is S^n. We can ask this in several categories : Diff, PL, Top:

In dimensions 0, 1, all is true. In dimension 2, the Top category is taken care of by the classification of surfaces. It turns out that every topological surface has a unique differentiable and PL structure (due to Rado). In dimension 3, Perelman proved that the answer is yes in the Diff category, which implies (Moise) that the same is true in the PL and Top categories. In dimension 4, the Top version is due to Freedman. The PL and Diff categories are unsolved in dimension 4. For $n > 4$, Smale proved that the P.C. is true in the Top and

PL cases, and exotic structures exist on S^n, so the smooth P.C. is false in general. There are

1 diff structures on S^5
1 diff structures on S^6
28 diff structures on S^7
2 diff structures on S^8
8 diff structures on S^9
6 diff structures on S^{10}
992 diff structures on S^{11}
1 diff structures on S^{12}
3 diff structures on S^{13}
2 diff structures on S^{14}
16256 diff structures on S^{15}
etc.

What does it mean to have different differentiable structures? Let $\phi_i : V_i \subset \mathbb{R}^n \to U_i \subset M$ be the charts, and we require the transition functions $\phi_j^{-1}\phi_i$ to be smooth (differentiable).

Example of a different differentiable structure on $\mathbb{R}^2$:

Map $\mathbb{R}^2$ to $\mathbb{R}^2$ by $(x, y) \mapsto (x, y + \operatorname{sgn}(y)\,|x|)$, using the pullback structure. But note that this smooth manifold is actually diffeomorphic to $\mathbb{R}^2$.

2. Milnor's proof that there exist exotic spheres

Let M be a smooth, orientable, closed 7-manifold such that $H_*(M) \cong H_*(S^7)$. We need an invariant to tell differentiable structures apart. The point is to look at tangent bundles that are not isomorphic.

Work of Thom shows that every $M = \partial B$ for some smooth B, smoothly compatible with M. Let μ be the generator of $H_7(M)$, and let ν be the generator of $H_8(B, M)$, such that $\partial\nu = \mu$. Note that $H_4(B) \cong H_4(B, M)$ and $H^4(B) \cong H^4(B, M)$ by the long exact sequence. Note that $H^4(B)$ and $H^4(B, M)$ are Poincare dual, and in fact there is a quadratic form on $H^4(B, M)$:

$$\alpha, \beta \to \langle \nu, \alpha \cup \beta \rangle$$

Let $\tau(B)$ be the signature of this form.

Let $p_1 \in H^4(B^8)$ be the first Pontryagin class (second Chern class of complexification of tangent bundle) of TB. Let

$$q(B) = \left\langle \nu, p_1^2 \right\rangle .$$

Definition: Let $\lambda(M)$ be the smooth invariant (now called *Milnor invariant*) defined as

$$\lambda(M) = 2q(B) - \tau(B) \operatorname{mod} 7.$$

Theorem 2.1. *$\lambda(M)$ does not depend on the choice of B.*

Corollary 2.2. *If $\lambda(M) \neq 0 \operatorname{mod} 7$, then $M \not\cong S^7$.*

Suppose $\partial B_1 = \partial B_2 = M$. Let C be the double; $C = B_1 \cup_M (-B_2)$. We have $\nu = \nu_1 - \nu_2$.

Theorem 2.3. *(Hirzebruch Signature Theorem)*

$$\tau(C) = \left\langle \nu, \frac{1}{45}\left(7p_2(C) - p_1^2(C)\right)\right\rangle.$$

This implies $45\tau(C) + q(C) = 7\langle \nu, p_2(C)\rangle \equiv 0 \bmod 7$, which implies

$$2q(C) - \tau(C) \equiv 0 \bmod 7.$$

Lemma 2.4. *$\tau(C) = \tau(B_1) - \tau(B_2)$, $q(C) = q(B_1) - q(B_2)$.*

This is Novikov additivity: If $(M_1^{4m}, \partial M_1)$. There is a nondegenerate pairing over the rationals:

$$\mathrm{Im}\left(H_{2m}(M) \to H_{2m}(M, \partial M)\right).$$

Let σ_1 be the signature on Im :

$$\sigma(M) = \sigma(M_1) + \sigma(M_2).$$

Proof of Lemma: We have the commutative diagram, in which everything splits in the obvious way:

$$\begin{array}{ccc} H^4(B_1, M) \oplus H^4(B_2, M) & \leftarrow & H^4(C, M) \\ \downarrow & & \downarrow \\ H^4(B_1) \oplus H^4(B_2) & \leftarrow & H^4(C) \end{array}$$

Now the proof of the Theorem above is complete.

Now we have an invariant. We now need some 7 manifolds. Look at unit sphere bundles of oriented 4-d vector bundles over S^4. It is sufficient to look at the action of $SO(4)$ on the equator. In fact,

$$\left\{\mathbb{R}^4 \text{ bundles over } S^4\right\} \cong \pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

The map going the other way is: a pair of integers (h, j) maps to $[f_{hj}]$, where

$$\begin{aligned} f_{hj} &: S^3 \to SO(4) \\ f_{hj}(u)v &= u^h v u^j \end{aligned}$$

for $v \in \mathbb{R}^4 = \mathbb{H}$. (quaternion multiplication) Let ξ_{hj} be the corresponding vector bundle, and let B_{hj} be the ball bundle, and let E_{hj} be the corresponding sphere bundle. Our goal is to compute $p_1(\xi_{hj}) \in H^4(S^4) = \mathbb{Z}$ and $e(\xi_{hj}) \in H^4(S^4) = \mathbb{Z}$.

Lemma 2.5. *p_1 and e are linear in h and j.*

Proof. Look at the picture. True for any characteristic class. □

Claim 2.6. *p_1 is antisymmetric in h and j, e is symmetric in h, j.*

Proof. If $F \to S^4$ is a vector bundle, then let $\overline{F}$ denote the bundle with reversed orientation. $p_1(\xi_{h,j}) = p_1\left(\overline{\xi_{h,j}}\right)$ since this is found by tensoring with $\mathbb{C}$. However, $e(\xi_{h,j}) = -e\left(\overline{\xi_{h,j}}\right)$. But what is $\overline{\xi_{h,j}}$ as a $\xi_{x,y}$? On S^3 = unit quaternions in the fiber, reversing orientation can be achieved by $g(v) = v^{-1}$. Next, we are trying to glue $D^4 \times \mathbb{R}^4 \cup D^4 \times \mathbb{R}^4$ on the equator.

so our new gluing function restricted to S^4 is

$\overline{f_{hj}(u)}\, v = \left(u^h v^{-1} u^j\right)^{-1} = u^{-j} v u^{-h} = f_{-j,-h}(u)\, v$,

So $\overline{\xi_{h,j}} = \xi_{-j,-h}$.

So $p_1(\xi_{-j,-h}) = p_1(\xi_{h,j})$, $e(\xi_{-j,-h}) = -e(\xi_{h,j})$, and linearity completes the result. □

Thus,

$$\begin{aligned} p_1(\xi_{h,j}) &= \pm C_1(h-j) \\ e(\xi_{h,j}) &= C_2(h+j). \end{aligned}$$

We test on examples:

Claim 2.7. *$B_{1,0} = \mathbb{HP}^2 - D^8$.*

Proof. By analogy, $\mathbb{CP}^2 = \mathbb{CP}^1 \cup D^4 = S^2 \cup D^4 =$ normal bundle of $\mathbb{CP}^1 \cup_{S^3} D^4$, so $\mathbb{HP}^2 = \mathbb{HP}^1 \cup D^8 = S^4 \cup D^8 =$ normal bundle of $\mathbb{HP}^1 \cup_{S^7} D^8 = \xi_{h,j}$. You can check that $h, j = 0, 1$ is the one. □

The inclusion $i : \mathbb{HP}^1 \hookrightarrow \mathbb{HP}^2$ induces an isomorphism on

$$H^4\left(\mathbb{HP}^2\right) \cong H^4\left(\mathbb{HP}^1\right) = H^4\left(S^4\right) \cong \mathbb{Z}.$$

Let $T\mathbb{HP}^2$= tangent bundle of $\mathbb{HP}^2$.

Now we want to compute

$$\begin{aligned} p_1(\xi_{1,0}) &= p_1\left(\xi_{1,0} \oplus TS^4\right) = p_1\left(T\mathbb{HP}^2|_{S^4}\right) \\ &= p_1\left(T\mathbb{HP}^2|_{S^4}\right) = p_1\left(i^*\left(T\mathbb{HP}^2\right)\right) \\ &= i^* p_1\left(T\mathbb{HP}^2\right) = i^* 2 = \pm 2. \end{aligned}$$

The conclusion is that

$$p_1(\xi_{h,j}) = \pm 2(h-j).$$

Gysin sequence

$$H^{i+3}\left(E_{1,0} = S^7\right) \to H^i\left(S^4\right) \stackrel{\cup e}{\to} H^{i+4}\left(S^4\right) \to H^{i+4}\left(E_{1,0} = S^7\right) \to$$

Take $i = 0$.

$$H^3\left(S^7\right) = 0 \to \mathbb{Z} \stackrel{\cup e}{\to} \mathbb{Z} \to 0 \to$$

so $e = \pm 1$.

So

$$e(\xi_{h,j}) = \pm(h+j).$$

3. Putting it all together

Let M be a homotopy 7-sphere. Suppose $M = \partial B$. Define the Milnor invariant $\lambda(M)$. Let $\tau(B)$ be the signature of the Poincare dualiy pairing on $H^4(B) \cong H^4(B, M)$. Let $q(B) = \left\langle [B], p_1(B)^2 \right\rangle$. Then

$$\lambda(M) = 2q(B) - \tau(B) \bmod 7$$

is independent of B.

Next, suppose $B_{h,j}$ is a 4-ball bundle over S^4 with $E_{h,j}$ the associated sphere bundle and $\xi_{h,j}$ the associated vector bundle, with $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$. Then the Euler class and first Pontryagin class are represented by

$$\begin{aligned} e(\xi_{h,j}) &= \pm(h+j)\nu \\ p_1(\xi_{h,j}) &= \pm 2(h-j)\nu \end{aligned}$$

in $H^4(S^4)$, with ν the fundamental cocycle of S^4.

Lemma 3.1. *If $h + j = 1$, then $E_{h,j}$ is homeomorphic to S^7.*

Proof. Consider the Gysin sequence

$$\to H^i\left(S^4\right) \overset{\cup e}{\to} H^{i+4}\left(S^4\right) \to H^{i+4}\left(E_{h,j}\right) \to H^{i+1}\left(S^4\right) \to$$

This easily implies $H^i\left(E_{hj}\right) = 0$ for $i \neq 0, 3, 4, 7$. Take $i = 0$.

$$H^3\left(S^7\right) = 0 \to \mathbb{Z} \overset{\cup e}{\to} \mathbb{Z} \to 0 \to$$

so $H_*\left(E_{hj}\right) = H_*\left(S^7\right)$. Also, $\pi_1\left(E_{hj}\right) = 0$ by long exact homotopy sequence. So Hurewicz/Whitehead imply E_{hj} is homotopy equivalent to S^7. (Note: a subtle point – because the groups are the same and the fact that we have a map $E_{hj} \to S^7$ by collapsing outside a nbhd of a point, can use H-W.) The Poincare conjecture implies E_{hj} is homeomorphic to S^7. □

Lemma 3.2. *Let $h - j = k$, $h + j = 1$. Then*

$$\lambda\left(E_{hj}\right) = \left(k^2 - 1\right) \operatorname{mod} 7.$$

Proof. Consider $B_{h,j}$. The signature is $\tau\left(B_{hj}\right) = 1$, because $H^4\left(B_{hj}\right) \cong H^4\left(B_{hj}, E_{hj}\right) \cong H^4\left(B_{hj}, B_{hj} \setminus S^4\right) \cong H^4\left(S^4\right) = \mathbb{Z}$, and the intesection number of S^4 with itself is the Euler characteristic of ξ_{hj}, which is $h + j = 1$. Thus, the signature is the Euler class cap the fundamental class, which is 1. Next, q is the cap product $\langle\left[B_{hj}\right], p_1^2\rangle$. Note

$$\begin{aligned} p_1\left(B_{h,j}\right) &= p_1\left(\pi^*\left(TS^4\right) \oplus \pi^*\left(\xi_{hj}\right)\right), \\ &= p_1\left(\pi^*\left(\xi_{hj}\right)\right) \\ &= \pm\pi^* 2\left(h - j\right)\nu = 2k\left(\text{generator of } H^4\left(B_{hj}\right)\right) \end{aligned}$$

where $\pi : B_{hj} \to S^4$. So

$$q\left(B_{hj}\right) = 4k^2.$$

So

$$\begin{aligned} \lambda\left(E_{hj}\right) &\equiv 2q - \tau \operatorname{mod} 7 \\ &\equiv 8k^2 - 1 \equiv \left(k^2 - 1\right) \operatorname{mod} 7. \end{aligned}$$

□

So if

$$\begin{aligned} k &= 3, \lambda\left(E_{hj}\right) = 1 \\ k &= 5, \lambda\left(E_{hj}\right) = 3 \\ k &= 7, \lambda\left(E_{hj}\right) = 6. \\ \lambda\left(S^7\right) &= 0 \end{aligned}$$

Thus there are at least four different differentiable structures on S^7!

Milnor: $\lambda\left(-M\right) = -\lambda\left(M\right)$. So if $k = 3$, E_{hj} is an example of a homotopy sphere that is not diffeomorphic to its mirror image. Moreover, if $\lambda\left(M\right)$ is 1 or 6 or 3, this manifold is not diffeo to its mirror image. So we get at least 5 exotic S^7's.

Alternate (original) proof that E_{hj} is homeomorphic to a standard S^7: S^4, two halves identified via stereographic projection. $u' = \frac{u}{\|u\|^2}$. Take two copies of $\mathbb{R}^4 \times S^3$. identify the two $\left(\mathbb{R}^4 \setminus 0\right) \times S^3$ by

$$\left(u, v\right) \mapsto \left(u', v'\right) = \left(\frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|}\right).$$

Replace u', v' by u'', v' where $u'' = u' \left(v'\right)^{-1}$. Let

$$f\left(x\right) = \frac{\mathrm{Re}\left(v\right)}{\left(1+\|u\|^{2}\right)^{1/2}} = \frac{\mathrm{Re}\left(u''\right)}{\left(1+\|u''\|^{2}\right)^{1/2}}$$

The critical points are $(0, \pm 1)$.

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