Definition: A metric space is proper if closed and bounded sets are compact.

Compare this with the definition of a proper map in topology: if $X,Y$ are top spaces, a function $f: X \to Y$ is proper if $f^{-1}(K)$ is compact in $X$ for every compact subset $K$ of $Y$. A metric space $X$ is proper if $\forall y \in Y$ the function $f: X \to \mathbb{R}$ defined by $f(x) = d(x_0, x)$ for some fixed $x_0 \in X$ is a proper function.

The following theorem is a generalization of the Hopf-Rinow theorem from differential geometry:
Theorem: Every complete locally compact length space is proper. Conversely, every proper length space is complete and locally compact.
Furthermore, all such spaces are geodesic spaces.

Definition: Let \((X, d_X), (Y, d_Y)\) be metric spaces. A map \(f: X \to Y\) is \textbf{Lipschitz} if there is a constant \(C\) such that
\[
d_Y(f(x), f(x')) \leq C d_X(x, x')
\]
for all \(x, x' \in X\).

Important example: Every smooth map between compact Riemannian manifolds is Lipschitz.
Definition: A homeomorphism $\phi: (X, d_X) \to (Y, d_Y)$ such that both $\phi$ and $\phi^{-1}$ are Lipschitz is called a bi-Lipschitz homeomorphism.

Theorem (Sullivan): Two smooth manifolds of dimension other than four (!) are bi-Lipschitz homeomorphic if and only if they are (topologically) homeomorphic.

Counterexamples in dimension 4.

Take

Definition: $f: (X, d_X) \to (Y, d_Y)$, $x_0 \in X$, $a > 0$.
The distortion of $f$ at $x_0$ is defined to be

$$D_f(x_0 ; a) = \sup \{ d_Y(f(x), f(x_0)) : d_X(x, x_0) = a \}$$

$$= \inf \{ d_Y(f(x), f(x_0)) : d_X(x, x_0) = a \}$$

Roughly speaking, $D_f(x_0 ; a)$ is a measure of how far the image $f(S(x_0 ; a))$ of the sphere of radius $a$ centered at $x_0$ deviates from being a sphere in $Y$ centered at $f(x_0)$. 
Definition: We say $f: X \rightarrow Y$ is $K$-quasi conformal if there exists a constant $K$ such that
\[
\limsup_{a \rightarrow 0} D_f(x; a) \leq K
\]
for all $x \in X$. We say $f$ is quasi conformal if it is $K$-quasi conformal for some $K$.

Examples: Every bi-Lipschitz homeomorphism is quasi conformal.

- A conformal map in the sense of Riemannian geometry is $1$-quasi conformal.
Section 1.3

Recall the following theorem from point-set topology:

Theorem: Let \((X,d)\) be a metric space, and define \(\tilde{d} : X \times X \to [0,1]\) by \(\tilde{d}(x,y) = \min \{d(x,y), 1\}\). Then \(\tilde{d}\) is a metric on \(X\), and \((X,d)\) and \((X,\tilde{d})\) are homeomorphic.

In other words, the topology of \((X,d)\) only depends on the small-scale structure of \(d\). In coarse geometry, we consider the large-scale structure of \(d\).
Definitions: Let $X, Y$ be metric spaces and let $f: X \to Y$ be a function, not necessarily continuous.

(a) $f$ is (metrically) proper if $f^{-1}(B)$ is a bounded subset of $X$ for every bounded subset $B$ of $Y$.

(b) $f$ is (uniformly) bornologous if for every $R > 0$ and $S > 0$ such that

$$d(x, x') < R \Rightarrow d_y(f(x), f(x')) < S.$$ 

(c) $f$ is coarse if it is both proper and bornologous.
Remarks:

1. Metric properness is not generally the same as topological properness, but the two concepts coincide if \( X, Y \) are proper metric spaces and \( f \) is continuous. Henceforth we shall use "proper" to mean "metrically proper."

2. Note that the definition of bornologous is sort of the converse to the definition of continuity!

Example with \( X = Y = \mathbb{N} \).

1. \( f(n) = 14n + 8 \) coarse
2. \( g(n) = 1 \) not proper
3. \( h(n) = n^2 \) not bornologous.
Definition: Let $X, Y$ be metric spaces. We say $f : X \to Y$ is **large-scale Lipschitz** if there exist positive constants $c, A$ such that

$$d(f(x), f(\tilde{x})) \leq c \cdot d(x, \tilde{x}) + A$$

for all $x, \tilde{x} \in X$.

A large-scale Lipschitz map is bornologous,

but the converse is not generally true. However, we do have the following result:

Proposition: Let $X$ be a length space and $Y$ any metric space, let $f : X \to Y$ be a map. Then:

(a) $f$ is large-scale Lipschitz;

(b) $f$ is bornologous;

(c) there exist $R, S > 0$ such that

$$d(x, \tilde{x}) \leq R \Rightarrow d(f(x), f(\tilde{x})) < S$$

for all $x, \tilde{x} \in X$. 
Definition: Maps $f, \tilde{f}: X \to Y$ are close if there exists a constant $M > 0$ such that
$$d(f(x), \tilde{f}(x)) \leq M$$
for all $x, \tilde{x} \in X$.

Proposition: $f, \tilde{f}$ are close if and only if there exists a coarse map
$$F: X \times [0,1] \to Y$$
such that $F(x,0) = f(x)$, $F(x,1) = \tilde{f}(x)$ $\forall x \in X$.

For this reason, "close" is sometimes called "bornotopic".

Definition: Metric spaces $X, Y$ are coarsely equivalent if there exist coarse maps $f: X \to Y$, $g: Y \to X$ such that $g \circ f$ is close to $\text{id}_X$ and $f \circ g$ is close to $\text{id}_Y$.

Remark: Coarse equivalence is also called "bornotopy equivalence."
Proposition: \( \mathbb{Z} \) and \( \mathbb{R} \) are coarsely equivalent.

Proof: Let \( f: \mathbb{Z} \to \mathbb{R} \) be the inclusion map and define \( g: \mathbb{R} \to \mathbb{Z} \) by \( g(x) = \lfloor x \rfloor \). Then \( g \circ f = \text{id}_\mathbb{Z} \), and \( f \circ g(x) = \lfloor x \rfloor \) is close to \( \text{id}_\mathbb{R} \); in fact, we can take \( M = 1 \).

Remark: We will see later that \( \mathbb{R} \) and \( \mathbb{R}^2 \) are not coarsely equivalent.

Next time:

Let \( \Gamma \) be a discrete group, let \( S \) be a set of generators of \( \Gamma \). For each \( \gamma \in \Gamma \), let \( \| \gamma \| \) denote the smallest integer \( n \) such that

\[ \gamma = s_1 s_2 \cdots s_n, \quad s_i \in S \text{ or } s_i^{-1} \in S \text{ for } i \leq n. \]

It is easy to see that

\[ \| \gamma \gamma^{-1} \| = \| \gamma \| + \| \gamma^{-1} \|. \]