Examples:

1. \( A = \{ f \in C_b(X) : f - \lambda \in C_0(X) \text{ for some } \lambda \in \mathbb{C} \} \)
   \( Y = X^+ \), the one-point compactification of \( X \)

2. \( A = C_b(X) \)
   \( Y = \beta X \), the Stone-Čech compactification of \( X \)

3. \( X = \mathbb{R} \), \( A = \{ f : \mathbb{R} \to \mathbb{C}, \lim_{x \to \infty} f(x), \lim_{x \to -\infty} f(x) \text{ exist} \} \)
   \( Y = [-\infty, \infty] \)

Theorem: Let \( X \) be paracompact and locally compact Hausdorff, \( f \) take \( ECXX \). Let \( \overline{X} \) be a compactification of \( X \), \( \delta X = \overline{X} \setminus X \).

TFAE:
(a) the closure $\overline{E}$ of $E$ in $\overline{X \times X}$ intersects the complement of $X \times X$ only in the diagonal $\Delta_{\overline{X}} = \{(w,w) : w \in \partial X\}$

(b) $E$ is proper (i.e., $E(K) \& E^{-1}(K)$ are relatively compact when $K$ is relatively compact), and for any net $\{ (x_\alpha, y_\alpha) \}$ in $E$, if $\lim_{\alpha} x_\alpha = w$, then $\lim_{\alpha} y_\alpha = w$ as well.

(c) $E$ is proper, and for every $w \in \overline{X}$ and every nbhd $V$ of $w$ in $\overline{X}$, there is a nbhd $U \subseteq V$ of $w$ in $\overline{X}$ with the property that $E \cap (U \times (X \setminus V)) = \emptyset$.

Furthermore, the collection of sets $E$ satisfying these equivalent conditions are the entourages for a proper connected coarse structure on $X$. 
Definition: This coarse structure on $X$ is the topological coarse structure or the continuously controlled coarse structure on $X$ associated to the compactification $\overline{X}$.

Examples:

1. $X^+$: $\mathcal{E} =$ collection of all proper subsets of $X \times X$ (indiscrète coarse structure)

2. $\beta X$: $\mathcal{E} =$ collection of all subsets of $X \times X$ with only finitely many points of the diagonal in $X \times X$ (descrete coarse structure)
Proposition: Let $X, Y$ be locally compact Hausdorff spaces with second countable compactifications $\overline{X}, \overline{Y}$. A continuous and proper map $f: X \to Y$ is coarse (w.r.t. $\overline{X}, \overline{Y}$) if and only if $f$ extends to a continuous map $\overline{f}: \overline{X} \to \overline{Y}$.

Remark: If $\overline{f}: \overline{X} \to \overline{Y}$ exists, then $f: X \to Y$ is coarse without the hypothesis of second countability of $\overline{X}, \overline{Y}$, but the converse requires this hypothesis.

Question: Suppose $E$ is a coarse structure on a locally compact paracompact Hausdorff space $X$. Under what conditions is $E$ the topological coarse structure associated to a compactification of $X$?
Partial Answer: Recall that a coarse structure on a para compact Hausdorff space $X$ is proper if

(i) there is a controlled nbhd of the diagonal;

(ii) every bounded subset of $X$ has compact closure. (i.e. $B \times B$ is an entourage)

($X$ is necessarily locally compact)

Let $X$ be a para compact Hausdorff space equipped with a proper coarse structure. For each $f: X \to C$ that is bounded and continuous, define $df: X \times X \to C$ by the formula

$$df(x, y) = f(x) - f(y).$$

We say $f$ is a Higson function if for each entourage $E$, the restriction $df$ to $E$ vanishes at infinity.
Proposition: The Higson functions on a proper coarse space form a unital C*-subalgebra of $C_0(X)$.

Proof: The only non-trivial point to check is closure under multiplication, which follows easily from the identity

$$dfg)(x,y) = df(x,y)g(x) + fg(y)dg(x,y).$$

We let $C_h(X)$ denote the collection of Higson functions on $X$, and we denote the corresponding compactification by $hX$. This is the Higson compactification associated to the proper coarse structure on $X$, and the set $hX \backslash X$ is called the Higson corona; it is denoted $\nu X$. 

Definition: Let $X$ be a proper coarse space. A **coarse compactification** of $X$ is a compactification whose top. coarse structure is coarser (i.e., has more entourages) than the original coarse structure on $X$.

Example: The one-point compactification of $X$ is always a coarse compactification.

Proposition: The Higson compactification $hX$ of $X$ is a coarse compactification that is universal in the following sense: given a coarse compactification $\overline{X}$ of $X$, the identity map $i: X \to \overline{X}$ extends uniquely to a continuous surjection from $hX$ to $\overline{X}$. 
The Higson compactification $\beta X$ can only be defined when $X$ is a proper coarse space. However, the Higson corona $vX$ can be defined for any coarse space. Here's how.

**Definition:** Let $X$ be a coarse space and $f: X \to \mathbb{C}$ a function. We say $f$ tends to $0$ at infinity if for every $\varepsilon > 0$ there exists a bounded set $B$ in $X$ such that $|f(x)| < \varepsilon$ for $x \in X \setminus B$. We say a function $f: X \times X \to \mathbb{C}$ tends to $0$ at infinity if for every $\varepsilon > 0$ there exists an entourage $E$ such that $|f(x,y)| < \varepsilon$ for all $(x,y) \in (X \times X) \wedge E$. 
Let
\[ B_h(X) = \{ f : X \to C \text{ bounded : } df \text{ tends to 0 at } \infty \} \]
\[ B_0(X) = \{ f : X \to C \text{ bounded : } f \text{ tends to 0 at } \infty \} \]

Then \( B_0(X) \) is an ideal in \( B_h(X) \), and

\[ C_0(X) = C_h(X) \cap B_0(X) \]
\[ B_h(X) = C_h(X) + B_0(X) \]

Now, if \( X \) is a proper coarse space, then

\[ C(vX) = \frac{C_h(X)}{C_0(X)} \]

But by the 2nd Isomorphism Theorem from \( C(S^2) \),

\[ \frac{C_h(X)}{C_0(X)} = \frac{C_h(X)}{B_0(X) \cap C_h(X)} = \frac{B_0(X) + C_h(X)}{B_0(X)} = \frac{B_h(X)}{B_0(X)} \]

Thus we can define \( vX \) as the maximal ideal space of the commutative \( C^*(S^2) \) \( \frac{B_h(X)}{B_0(X)} \).
Proposition: Let $X, Y$ be coarse spaces. A coarse map $\phi: X \to Y$ extends to a continuous map $v\phi: vX \to vY$. Moreover, if $X$ and $\bar{Y}$ are close, then $v\phi = v\bar{Y}$.

Corollary: If $X, Y$ are coarsely equivalent, then $vX$ and $vY$ are homeomorphic.

Note that we have two constructions here.

1. The Higson construction takes a proper coarse structure $\ast$ and associates to it a compactification.

2. The continuous control construction takes a compactification $\ast$ and associates to it a coarse structure.

Let $\ast$ denote the first construction and $\ast'$ the second.

Question: To what extent are these inverses to one another?
Answer: These are not inverses in general, but we do have the following two results:

Proposition: Let \((X, d)\) be a proper metric space. Then the bounded coarse structure on \(X\) is the topological coarse structure associated to its Higson compactification.

Proposition: Suppose \(X\) is a locally compact Hausdorff space that is equipped with the top. coarse structure assoc. to a 2nd countable compactification \(\overline{X}\). Then the Higson compactification \(\mathcal{h}X\) of \(X\) coincides with \(\overline{X}\).