Proposition: Let $X$ be a coarse space.

(c) If $B$ is a bounded subset of $X$ and $E \subseteq XX$ is controlled, then $E[B]$ is bounded.

(b) If $Y, Z$ are bounded subsets of $X$ with nonempty intersection, then $Y \cup Z$ is bounded.

Given a coarse space $X$, decree that $x \sim \bar{x}$ if $\{x, \bar{x}\}$ is a bounded subset of $X$. This is an equivalence relation on $X$, and

$[x] = \text{coarsely connected component of } x \text{ in } X.$

We can extend our earlier definition on metric spaces to coarse spaces:
Definitions: Let $X, Y$ be coarse spaces, $f: X \to Y$ a map.

(a) $f$ is proper if $f^{-1}(B)$ is bounded in $X$ for every bounded subset $B$ of $Y$.

(b) $f$ is bornologous if $(fxf)(E)$ is a controlled subset of $Y \times Y$ for every controlled subset $E$ of $X \times X$.

(c) $f$ is coarse if it is proper and bornologous.

(d) $X, Y$ are coarsely equivalent if there exist coarse maps $f: X \to Y$, $g: Y \to X$ such that $gof$ is close to $id_X$ and $fog$ is close to $id_Y$.

The Coarse Category:

- Objects are the class of all coarse spaces
- Morphisms are coarse maps, with close coarse maps identified.
IF $X$ is both a topological space and a coarse space, we would like some compatibility between the two structures.

**Definition:** Let $X$ be a paracompact Hausdorff top space. A coarse structure on $X$ is proper if

(i) there is a controlled nbhd of the diagonal;

(ii) every bounded subset of $X$ has compact closure.

**Remark:** These conditions imply that $X$ is locally compact.

**Proposition:** Let $X$ be a connected top space equipped with a proper coarse structure. Then $X$ is coarsely connected. A subset of $X$ is bounded iff it has compact closure, and every controlled subset of $X \times X$ is proper (by our definition at the beginning of this chapter).
Proposition: Let \((X,d)\) be a metric space. Its bounded coarse structure is proper if and only if it is proper as a metric space; i.e., closed bounded sets are compact.

Compactifications - As viewed by an Operator Algebraist!

- A top space \(X\) is \underline{locally compact at} \(x \in X\) if there exists a nbhd of \(x\) with compact closure.
  If \(X\) is locally compact at each of its points, we say \(X\) is \underline{locally compact}.

- A \underline{compactification} of a top space \(X\) is a compact top space \(Y\) such that
  - \(X\) is a subspace of \(Y\);
  - \(\overline{X} = Y\).

- A (Hausdorff) top space \(X\) admits a compactification if and only if \(X\) is locally compact.
Let $X$ be a top space. A function $f : X \to \mathbb{C}$ vanishes at infinity if for every $\varepsilon > 0$ there exists a compact subset $K$ of $X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$.

Let $X$ be a locally compact Hausdorff space.

We let $C_0(X)$ denote the collection of continuous $\mathbb{C}$-valued functions on $X$ that vanish at infinity. $C_0(X)$ is a normed algebra with norm

$$||f|| := \sup \{|f(x)| : x \in X\},$$

and has a unit if and only if $X$ is actually compact.

Next, let $C_b(X)$ be the collection of continuous functions $f : X \to \mathbb{C}$ with the property that

$$\sup \{|f(x)| : x \in X\} < \infty.$$ 

This is a unital normed algebra that contains $C_0(X)$ as an ideal. Note that $C_b(X) = C_0(X)$ if and only if $X$ is compact. Also note $C_b(X), C_0(X)$ are closed under complex conjugation.
Now suppose $A$ is a unital normed-closed subalgebra of $C_b(X)$ that is closed under complex conjugation $t$ and contains $C_0(X)$.

By the Gelfand-Naimark Theorem,

$$A \cong C(Y)$$

where $Y$ is a compact Hausdorff space. Specifically,

$$Y = \text{space of mult. lm. functionals } Y \to \mathbb{C}$$

in the weak* topology (topology of pointwise convergence).

$Y$ is the maximal ideal space of $A$.

Observe that $X$ acts naturally inside of $Y$ as evaluation maps: $\phi_x: A \to \mathbb{C},$

$$\phi_x(f) = f(x).$$

Moreover, $\bar{X} = Y$, so $A$ determines a compactification of $X$.
Examples:

1. \( A = \{ f \in C_b(x) : f - \lambda \in C_c(x) \text{ for some } \lambda \in \mathbb{C} \} \)
   
   \( Y = X^+ \), the one-point compactification of \( X \)

2. \( A = C_b(x) \)
   
   \( Y = \beta X \), the Stone–Čech compactification of \( X \)

3. \( X = \mathbb{R}, \ A = \{ f : \mathbb{R} \to \mathbb{C}, \lim_{x \to \infty} f(x), \lim_{x \to -\infty} f(x) \text{ exist} \} \)
   
   \( Y = [-\infty, \infty] \)

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**Theorem:** Let \( X \) be paracompact and locally compact Hausdorff, and take \( EC X \times X \). Let \( \overline{X} \) be a compactification of \( X \), and let \( \partial X = \overline{X} \setminus X \).

**TFAE:**