A real inner product on a real vector space $V$ (which could be infinite dimensional) is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties:

- (Symmetry) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$.
- (Linearity) $\langle av + bw, z \rangle = a \langle v, z \rangle + b \langle w, z \rangle$ for all $a, b \in \mathbb{R}$ and all $v, w, z \in V$.
- (Positivity) $\langle v, v \rangle \geq 0$ for all $v \in V$, with equality only if $v = 0$.

Similarly, a Hermitian (or complex) inner product on a complex vector space $V$ (which could be infinite dimensional) is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following properties:

- (Symmetry) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$. (Here, $\overline{\cdot}$ means the complex conjugate.)
- (Linearity) $\langle av + bw, z \rangle = a \langle v, z \rangle + b \langle w, z \rangle$ for all $a, b \in \mathbb{C}$ and all $v, w, z \in V$.
- (Positivity) $\langle v, v \rangle \geq 0$ for all $v \in V$, with equality only if $v = 0$.

A vector space $(V, \langle \cdot, \cdot \rangle)$ with real or complex inner product is called an inner product space.

A subset $\{e_1, e_2, \ldots\}$ of an inner product space $V$ is called orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This set is called a basis (or complete orthonormal basis) if it spans $V$. The symbol $\delta_{ij}$ is called the Kronecker delta symbol.

Any inner product space $(V, \langle \cdot, \cdot \rangle)$ comes equipped with a norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ called the inner product norm. From this one may make the vector space into a topological metric space using the distance function $d(v, w) = \|v - w\|$. We say that the inner product space is complete if all Cauchy sequences converge (with respect to that distance and norm). Equivalently, $(V, \langle \cdot, \cdot \rangle)$ is complete if

$$\sum_{k=0}^{\infty} \|v_k\| < \infty$$

implies always that $\sum_{k=0}^{\infty} v_k$ converges in $V$. A complete inner product space is called a Hilbert space. We say that a Hilbert space is separable if there exists a (countable) complete orthonormal basis. It turns out that all separable infinite-dimensional (real or complex) Hilbert spaces are isomorphic (say, to $l^2$; see below).

Important examples of Hilbert spaces are as follows:

- $\mathbb{R}^n$ or $\mathbb{C}^n$ with $\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$.
- $l^2(\mathbb{R})$ or $l^2(\mathbb{C})$, which is the space of sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{j=1}^{\infty} |x_j|^2 < \infty$ with inner product.
\[ \langle x, y \rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}. \]

Then \( \{(1,0,0,\ldots),(0,1,0,0,\ldots),(0,0,1,0,0,\ldots),\ldots\} \) is a complete orthonormal basis of \( \ell^2(\mathbb{R}) \) (or \( \ell^2(\mathbb{C}) \)).

- \( L^2(S^1) \), the space of square-integrable functions on the unit circle. We have:

\[
L^2(S^1) = \left\{ f : S^1 \rightarrow \mathbb{C} : \int_{S^1} |f|^2 < \infty \right\}
= \left\{ g : \mathbb{R} \rightarrow \mathbb{C} : \int_0^{2\pi} |g(x)|^2 \, dx < \infty \text{ and } g(x+2\pi) = g(x) \right\}
\]

with inner product

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx.
\]

One can check that the product given above satisfies the definition of Hermitian inner product. One technical detail is that you have to say that two functions in \( L^2(S^1) \) are considered to be the same if they are equal except on a set of measure zero. We have to do this, because otherwise the positivity property would not be satisfied.

An example of an orthogonal set in \( L^2(S^1) \) is \( \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\} \). We can verify that

\[
\int_{-\pi}^{\pi} 1 \cdot \cos nx \, dx = \int_{-\pi}^{\pi} 1 \cdot \sin nx \, dx = 0 \text{ if } n \in \mathbb{Z},
\]

\[
\int_{-\pi}^{\pi} \sin mx \cdot \cos nx \, dx = 0 \text{ for } n, m \in \mathbb{Z}
\]

\[
\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = \left\{ \begin{array}{ll} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \in \mathbb{N} \end{array} \right.
\]

\[
\int_{-\pi}^{\pi} 1 \, dx = 2\pi.
\]

Then

\[
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(x), \frac{1}{\sqrt{\pi}} \sin(x), \frac{1}{\sqrt{\pi}} \cos(2x), \frac{1}{\sqrt{\pi}} \sin(2x), \ldots \right\}
\]

is an orthonormal set in \( L^2(S^1) \). By a very deep and difficult theorem, this set is a complete orthonormal basis. The hard part (and only remaining part) is to show that it spans \( L^2(S^1) \). This is equivalent to showing that if \( h(x) \) is in \( L^2(S^1) \) such that \( \langle h, \alpha \rangle = 0 \) for every element \( \alpha \) in the orthonormal set, then \( h(x) = 0 \) in \( L^2(S^1) \). Another commonly used orthonormal basis for \( L^2(S^1) \) is \( \left\{ \frac{1}{\sqrt{2\pi}} e^{i n \theta} \right\}_{n \in \mathbb{Z}} \).

Note that given any orthonormal basis \( \{e_1, e_2, \ldots\} \) of a Hilbert space \( V \) and any vector \( v \in V \),
that is the sum on the right converges to $v$ (using the inner product norm). Note that $\langle v, e_j \rangle e_j$ is the projection of $v$ onto $e_j$. Sometimes the number $\langle v, e_j \rangle$ is called the $j^{th}$ Fourier coefficient. We can take the $L^2$ norm of both sides of the equation to get Parseval’s equality (or Parseval Identity):

\[
\langle v, v \rangle = \sum_{j=1}^{\infty} (\langle v, e_j \rangle e_j, \sum_{k=1}^{\infty} \langle v, e_k \rangle e_k) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle v, e_j \rangle \langle v, e_k \rangle \langle e_j, e_k \rangle = \sum_{j=1}^{\infty} |\langle v, e_j \rangle|^2.
\]

(One has to check that you are allowed to move the infinite sum out of the inner product; this works by taking limits of finite sums.) In the case where you have an orthonormal set $\{b_j\}$ that is not necessarily a basis, you get Bessel’s inequality:

\[
\langle v, v \rangle \leq \sum_{j} |\langle v, b_j \rangle|^2.
\]

Another (real or complex) Hilbert space fact that will be useful is the Cauchy-Schwarz inequality:

\[
|\langle v, w \rangle| \leq \|v\| \|w\|.
\]

This can be quite interesting when applying to various inner products. For example, applying to the $L^2$ inner product: if $f$ is an $L^2$ function, then

\[
|\langle f, 1 \rangle| = \left| \int_{-\pi}^{\pi} f(x) \, dx \right|, \text{ so } \left| \int_{-\pi}^{\pi} f(x) \, dx \right| \leq 2\pi \sqrt{\int_{-\pi}^{\pi} (f(x))^2 \, dx}.
\]

Similarly,

\[
\left| \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \right| \leq \sqrt{\pi} \sqrt{\int_{-\pi}^{\pi} (f(x))^2 \, dx}.
\]

**Fourier Series**

The very specific application of the Hilbert space facts to $L^2(S^1)$ with basis either
\[
\left\{ e_n = \frac{1}{\sqrt{2\pi}} e^{in\theta} \right\}_{n \in \mathbb{Z}} \quad \text{or} \quad \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \cos(x), \frac{1}{\sqrt{2\pi}} \sin(x), \frac{1}{\sqrt{2\pi}} \cos(2x), \frac{1}{\sqrt{2\pi}} \sin(2x), \ldots \right\}
\]

yields the study of Fourier series.

The first fact is that any function in \( L^2(S^1) \) can be represented by

\[
f = \sum_{n \in \mathbb{Z}} c_n e_n,
\]

where

\[
c_n = \langle f, e_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \exp(-inx) \, dx
\]
is the \( n \)th Fourier coefficient. Furthermore, the Parseval identity shows that

\[
\int_{-\pi}^{\pi} (f(x))^2 \, dx = \sum_{n \in \mathbb{Z}} |c_n|^2.
\]

Similar results for the other basis are:

\[
f(x) = a_0 + \sum_{j=1}^{\infty} a_j \cos(jx) + \sum_{k=1}^{\infty} b_j \sin(jx),
\]

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(jx) \, dx \quad \text{for } j \geq 1,
\]

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx.
\]

The Parseval identity is

\[
\int_{-\pi}^{\pi} (f(x))^2 \, dx = 2\pi |a_0|^2 + \pi \sum_{j=1}^{\infty} (|a_j|^2 + |b_j|^2).
\]

The convergence of the Fourier series is with respect to the Hilbert space norm. For example, this means for instance that for any \( L^2 \) function \( f \),

\[
\left\| f - \sum_{n=-N}^{N} c_n e_n \right\| \to 0
\]
as \( N \) goes to infinity, with \( \|g\| = \sqrt{\int_{-\pi}^{\pi} (g(x))^2 \, dx} \). But it turns out that if we assume additional conditions on \( f \), other types of convergence are also manifested. Here are the two important theorems:

**Theorem** Suppose that \( f \in L^2(S^1) \) is piecewise \( C^1 \) (continuously differentiable) and continuous. Then the Fourier series of \( f \) converges pointwise and uniformly to \( f \).

The phrase “converges pointwise” means that for each fixed \( x \in [-\pi, \pi] \), if
\[ F_N(x) = a_0 + \sum_{j=1}^{N} a_j \cos(jx) + \sum_{k=1}^{N} b_j \sin(jx), \]

then \( \lim_{N \to \infty} F_N(x) = f(x) \). (Similarly for the \( e_n \) basis.). The phrase “converges uniformly” means that given \( \varepsilon > 0 \), there exists \( M > 0 \) such that for all \( N \geq M \) and all \( x \in [-\pi, \pi] \),

\[ |F_N(x) - f(x)| < \varepsilon. \]

The next theorem deals with the case where there are jump discontinuities:

**Theorem** Suppose that \( f \in L^2(S^1) \) is piecewise \( C^1 \) and has a finite number of jump discontinuities. Then the Fourier series of \( f \) converges to

\[ \overline{f}(x) = \frac{1}{2} \left( \lim_{y \to x^-} f(y) + \lim_{y \to x^+} f(y) \right). \]

This theorem will be demonstrated in the examples. At all points \( x \) where \( f \) is continuous, note that the Fourier series converges to \( f(x) \). Notice also that the word “uniform” is removed from the conclusion. In fact, a very interesting situation occurs when there is a point of discontinuity of \( f \), where the Fourier series converges to the average of the left and right limits. As the number \( N \) increases, there are \( x \)-values very close to the point of discontinuity where \( F_N(x) \) differs from \( f(x) \) by an amount that does not decrease and in fact converges to the number

\[ a(0.08949\ldots), \]

where \( a \) is the gap between the left and right limits, and the number 0.08949\ldots is called the Wilbraham–Gibbs constant. The exact formula for the constant is

\[ \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin(x)}{x} \, dx - \frac{1}{2}. \]

This situation is called the **Gibbs Phenomenon**. We will see this phenomenon in the examples.

**Examples and Applications**

**Square Wave**

Define

\[ f(x) = \begin{cases} 
-1 & \text{if } -\pi \leq x \leq 0 \\
1 & \text{if } 0 \leq x < \pi
\end{cases} \]

We now compute the Fourier series:
\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0, \]
\[ a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(jx) \, dx = 0 \text{ for } j \geq 1 \]
\[ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx \]
\[ = \frac{1}{\pi} \left( -\int_{-\pi}^{0} \sin(kx) \, dx + \int_{0}^{\pi} \sin(kx) \, dx \right) \]
\[ = \frac{1}{\pi} \left( \frac{1}{k} \cos(kx) \bigg|_{-\pi}^{0} - \frac{1}{k} \cos(kx) \bigg|_{0}^{\pi} \right) \]
\[ = \frac{2}{\pi k} (1 - (-1)^k) = \begin{cases} \frac{4}{\pi k} & k \text{ odd} \\ 0 & \text{otherwise} \end{cases} \]

Thus we have
\[ f(x) = \sum_{k \text{ odd}} \frac{4}{k\pi} \sin(kx) \]
\[ = \frac{4}{\pi} \left( \sin(x) + \frac{1}{3} \sin(3x) + \ldots \right). \]

Plugging in \( x = \frac{\pi}{2} \), we get
\[ 1 = \frac{4}{\pi} \left( 1 - \frac{1}{3} + \frac{1}{5} - \ldots \right), \text{ or} \]
\[ \pi = \left( 4 - \frac{4}{3} + \frac{4}{5} - \ldots \right). \]

(A famous formula for \( \pi \)!) Next, consider Parseval’s Identity:
\[ \int_{-\pi}^{\pi} (f(x))^2 \, dx = \pi \sum_{k \text{ odd}} \left( \frac{4}{k\pi} \right)^2, \text{ or} \]
\[ 2\pi = \frac{16}{\pi} \left( 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \right), \]
or
\[ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}. \]

(another famous formula!)

Let’s now graph the square wave with some of the Fourier approximations:
\[ y = \sum_{n=0}^{N} \frac{4}{(2n+1)\pi} \sin((2n+1)x) \]
Note the Gibbs Phenomenon:
\[
\sum_{n=0}^{N} \frac{4}{(2n+1)\pi} \sin \left( (2n + 1) \left( \frac{\pi}{2N+1} \right) \right)
\]
\[
\sum_{n=0}^{5} \frac{4}{(2n+1)\pi} \sin \left( (2n + 1) \left( \frac{\pi}{11} \right) \right) = 1.17345830790194905
\]
\[
\sum_{n=0}^{20} \frac{4}{(2n+1)\pi} \sin \left( (2n + 1) \left( \frac{\pi}{41} \right) \right) = 1.17858309426023996
\]
\[
\sum_{n=0}^{50} \frac{4}{(2n+1)\pi} \sin \left( (2n + 1) \left( \frac{\pi}{101} \right) \right) = 1.17891438974940976
\]
\[
\frac{4}{0.17891438974940976} = 0.08945719487470488
\]

Sawtooth Wave

Define
\[f(x) = x\]
for \(-\pi \leq x < \pi\) and can be thought of as being periodic. We now compute the Fourier series:
\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0,
\]
\[
a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(jx) \, dx = 0 \text{ for } j \geq 1
\]
\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) \, dx
\]
\[
= \frac{1}{\pi} \left( \frac{1}{k^2} \sin(kx) - \frac{1}{k} x \cos(kx) \right]_{-\pi}^{\pi}
\]
\[
= \frac{1}{\pi} \left( -(-1)^k \pi - (-1)^k (-\pi) \right) = \frac{2(-1)^{k+1}}{k}
\]

Then
\[ x = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx). \]

Plugging in \( x = \frac{\pi}{2} \), we get
\[ \frac{\pi}{2} = 2 \left(1 - \frac{1}{3} + \frac{1}{5} - \ldots \right), \]
which yields the formula for \( \pi \) found in the previous section. Parseval’s identity yields:
\[ \int_{-\pi}^{\pi} x^2 \, dx = \pi \sum_{k=1}^{\infty} \frac{4}{k^2}, \text{ or} \]
\[ \frac{2\pi^3}{3} = 4\pi \sum_{k=1}^{\infty} \frac{1}{k^2}, \]
so that
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4\pi} \frac{2\pi^3}{3} = \frac{\pi^2}{6}. \]
(Another famous formula!)

We now plot the Fourier approximations to the saw tooth wave:
\[ y = \sum_{k=1}^{N} \frac{2(-1)^{k+1}}{k} \sin(kx) \]

Let’s check out the Gibbs Phenomenon:
\[ y = \sum_{k=1}^{5} \frac{2(-1)^{k+1}}{k} \sin(k \left( \frac{N\pi}{N+1} \right)) = 2.16570477234332464 \]
\[ y = \sum_{k=1}^{20} \frac{2(-1)^{k+1}}{k} \sin(k \left( \frac{5\pi}{6} \right)) = 3. 16570477234332464 \]
\[ y = \sum_{k=1}^{5} \frac{2(-1)^{k+1}}{k} \sin(k \left( \frac{20\pi}{21} \right)) = 3. 5530869812998034 \]
\[ y = \sum_{k=1}^{50} \frac{2(-1)^{k+1}}{k} \sin \left( k \left( \frac{50\pi}{51} \right) \right) = 3.64207293630163028 \]

\[ y = \sum_{k=1}^{200} \frac{2(-1)^{k+1}}{k} \sin \left( k \left( \frac{200\pi}{201} \right) \right) = 3.68823132970234016 \]

\[
3.68823132970234016 - \frac{0.562268490309511067}{0.562268490309511067} = 0.562268490309511067
\]