Introduction to Fourier series

by Ken Richardson

Hilbert spaces

A **real inner product** on a real vector space V (which could be infinite dimensional) is a map $\langle \bullet , \bullet \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties:

- (**Symmetry**) $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$.
- (**Linearity**) $\langle av + bw, z \rangle = a\langle v, z \rangle + b\langle w, z \rangle$ for all $a, b \in \mathbb{R}$ and all $v, w, z \in V$.
- (**Positivity**) $\langle v, v \rangle \geq 0$ for all $v \in V$, with equality only if $v = 0$.

Similarly, a **Hermitian** (**or complex**) **inner product** on a complex vector space V (which could be infinite dimensional) is a map $\langle \bullet , \bullet \rangle : V \times V \to \mathbb{C}$ that satisfies the following properties:

- (**Symmetry**) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$. (Here, $\overline{z}$ means the complex conjugate.)
- (**Linearity**) $\langle av + bw, z \rangle = a\langle v, z \rangle + b\langle w, z \rangle$ for all $a, b \in \mathbb{C}$ and all $v, w, z \in V$.
- (**Positivity**) $\langle v, v \rangle \geq 0$ for all $v \in V$, with equality only if $v = 0$.

A vector space $(V, \langle \bullet, \bullet \rangle)$ with real or complex inner product is called an **inner product space**.

A subset $\{e_1, e_2, \ldots\}$ of an inner product space V is called **orthonormal** if

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

This set is called a **basis** (**or complete orthonormal basis**) if it spans V. The symbol δ_{ij} is called the **Kronecker delta symbol**.

Any inner product space $(V, \langle \bullet, \bullet \rangle)$ comes equipped with a norm $\|\bullet\| = \langle \bullet, \bullet \rangle^{1/2}$ called the inner product norm. From this one may make the vector space into a topological metric space using the distance function $d(v, w) = \|v - w\|$. We say that the inner product space is **complete** if all Cauchy sequences converge (with respect to that distance and norm). Equivalently, $(V, \langle \bullet, \bullet \rangle)$ is complete if

$$\sum_{k=0}^{\infty} \|v_k\| < \infty$$

implies always that $\sum_{k=0}^{\infty} v_k$ converges in V. A complete inner product space is called a **Hilbert space**. We say that a Hilbert space is **separable** if there exists a (countable) complete orthonormal basis. It turns out that all separable infinite-dimensional (real or complex) Hilbert spaces are isomorphic (say, to ℓ^2 ; see below).

Important **examples** of Hilbert spaces are as follows:

- $\mathbb{R}^n$ or $\mathbb{C}^n$ with $\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$.
- $\ell^2(\mathbb{R})$ or $\ell^2(\mathbb{C})$, which is the space of sequences $x = (x_1, x_2, \ldots)$ such that $\sum_{j=1}^{\infty} |x_j|^2 < \infty$ with inner product

$$\langle x,y\rangle = \sum_{j=1}^{\infty} x_j\overline{y_j}\ .$$

Then $\{(1,0,0,\dots),(0,1,0,0,\dots),(0,0,1,0,0,\dots),\dots\}$ is a complete orthonormal basis of $\ell^2(\mathbb{R})$ (or $\ell^2(\mathbb{C})$).

- $L^2(S^1)$, the space of square-integrable functions on the unit circle. We have:

$$\begin{aligned} L^2(S^1) &= \left\{f : S^1 \to \mathbb{C} : \int_{S^1} |f|^2 < \infty\right\} \\ &= \left\{g : \mathbb{R} \to \mathbb{C} : \int_0^{2\pi} |g(x)|^2\, dx < \infty \text{ and } g(x+2\pi) = g(x)\right\} \end{aligned}$$

with inner product

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

One can check that the product given above satisfies the definition of Hermitian inner product. One technical detail is that you have to say that two functions in $L^2(S^1)$ are considered to be the same if they are equal except on a set of measure zero. We have to do this, because otherwise the positivity property would not be satisfied.

An example of an orthogonal set in $L^2(S^1)$ is $\{1,\cos(x),\sin(x),\cos(2x),\sin(2x),\dots\}$. We can verify that

$$\int_{-\pi}^{\pi} 1\cdot\cos nx\, dx = \int_{-\pi}^{\pi} 1\cdot\sin nx\, dx = 0 \text{ if } n \in \mathbb{Z},$$

$$\int_{-\pi}^{\pi} \sin mx\cdot\cos nx\, dx = 0 \text{ for } n,m \in \mathbb{Z}$$

$$\int_{-\pi}^{\pi} \cos mx\cdot\cos nx\, dx = \int_{-\pi}^{\pi} \sin mx\cdot\sin nx\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \in \mathbb{N} \end{cases}$$

$$\int_{-\pi}^{\pi} 1\, dx = 2\pi.$$

Then

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\cos(2x), \frac{1}{\sqrt{\pi}}\sin(2x),\dots\right\}$$

is an orthonormal set in $L^2(S^1)$. By a very deep and difficult theorem, this set is a complete orthonormal basis. The hard part (and only remaining part) is to show is that it spans $L^2(S^1)$. This is equivalent to showing that if $h(x)$ is in $L^2(S^1)$ such that $\langle h,\alpha\rangle = 0$ for every element α in the orthonormal set, then $h(x) = 0$ in $L^2(S^1)$. Another commonly used orthonormal basis for $L^2(S^1)$ is $\left\{\frac{1}{\sqrt{2\pi}}e^{in\theta}\right\}_{n\in\mathbb{Z}}$.

Note that given any orthonormal basis $\{e_1,e_2,\dots\}$ of a Hilbert space V and any vector $v \in V$,

$$v = \sum_{j=1}^{\infty} \langle v, e_j \rangle e_j \ ,$$

that is the sum on the right converges to v (using the inner product norm). Note that $\langle v, e_j \rangle e_j$ is the projection of v onto e_j. Sometimes the number $\langle v, e_j \rangle$ is called the j^{th} **Fourier coefficient**. We can take the L^2 norm of both sides of the equation to get **Parseval's equality** (**or Parseval Identity**):

$$\begin{aligned}
\langle v, v \rangle &= \left\langle \sum_{j=1}^{\infty} \langle v, e_j \rangle e_j, \sum_{k=1}^{\infty} \langle v, e_k \rangle e_k \right\rangle \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle v, e_j \rangle \overline{\langle v, e_k \rangle} \langle e_j, e_k \rangle \\
&= \sum_{j=1}^{\infty} |\langle v, e_j \rangle|^2.
\end{aligned}$$

(One has to check that you are allowed to move the infinite sum out of the inner product; this works by taking limits of finite sums.) In the case where you have an orthonormal set $\{b_j\}$ that is not necessarily a basis, you get **Bessel's inequality**:

$$\langle v, v \rangle \leq \sum_{j} |\langle v, b_j \rangle|^2.$$

Another (real or complex) Hilbert space fact that will be useful is the **Cauchy-Schwarz inequality**:

$$|\langle v, w \rangle| \leq \|v\| \ \|w\|.$$

This can be quite interesting when applying to various inner products. For example, applying to the L^2 inner product: if f is an L^2 function, then

$$|\langle f, 1 \rangle| = \left| \int_{-\pi}^{\pi} f(x)\ dx \right|, \text{ so}$$

$$\left| \int_{-\pi}^{\pi} f(x)\ dx \right| \leq \sqrt{2\pi} \sqrt{\int_{-\pi}^{\pi} (f(x))^2\ dx}\,.$$

Similarly,

$$\left| \int_{-\pi}^{\pi} f(x) \cos(nx)\ dx \right| \leq \sqrt{\pi} \sqrt{\int_{-\pi}^{\pi} (f(x))^2\ dx}\,.$$

Fourier Series

The very specific application of the Hilbert space facts to $L^2(S^1)$ with basis either

$\left\{ e_n = \frac{1}{\sqrt{2\pi}} e^{in\theta} \right\}_{n\in\mathbb{Z}}$ or $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(x), \frac{1}{\sqrt{\pi}}\cos(2x), \frac{1}{\sqrt{\pi}}\sin(2x), \ldots \right\}$ yields the study of Fourier series.

The first fact is that any function in $L^2(S^1)$ can be represented by

$$f = \sum_{n\in\mathbb{Z}} c_n e_n \ ,$$

where

$$c_n = \langle f, e_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \exp(-inx)\, dx$$

is the n^{th} Fourier coefficient. Furthermore, the Parseval identity shows that

$$\int_{-\pi}^{\pi} (f(x))^2 dx = \sum_{n\in\mathbb{Z}} |c_n|^2.$$

Similar results for the other basis are:

$$\begin{aligned} f(x) &= a_0 + \sum_{j=1}^{\infty} a_j \cos(jx) + \sum_{k=1}^{\infty} b_j \sin(jx), \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx,\ a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(jx)\, dx \text{ for } j \geq 1, \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx. \end{aligned}$$

The Parseval identity is

$$\int_{-\pi}^{\pi} (f(x))^2 dx = 2\pi |a_0|^2 + \pi \sum_{j=1}^{\infty} \left(|a_j|^2 + |b_j|^2 \right).$$

The convergence of the Fourier series is with respect to the Hilbert space norm. For example, this means for instance that for any L^2 function *f*,

$$\left\| f - \sum_{n=-N}^{N} c_n e_n \right\| \to 0$$

as N goes to infinity, with $\|g\| = \sqrt{\int_{-\pi}^{\pi} (g(x))^2 dx}$. But it turns out that if we assume additional conditions on *f*, other types of convergence are also manifested. Here are the two important theorems:

Theorem *Suppose that $f \in L^2(S^1)$ is piecewise C^1(continuously differentiable) and continuous. Then the Fourier series of f converges pointwise and uniformly to f.*

The phrase “converges pointwise” means that for each fixed $x \in [-\pi, \pi]$, if

$$F_N(x) = a_0 + \sum_{j=1}^{N} a_j \cos(jx) + \sum_{k=1}^{N} b_j \sin(jx),$$

then $\lim\limits_{N\to\infty} F_N(x) = f(x)$. (Similarly for the e_n basis.). The phrase "converges uniformly" means that given $\varepsilon > 0$, there exists $M > 0$ such that for all $N \geq M$ and **all** $x \in [-\pi, \pi]$,

$$|F_N(x) - f(x)| < \varepsilon.$$

The next theorem deals with the case where there are jump discontinuities:

Theorem *Suppose that $f \in L^2(S^1)$ is piecewise C^1 and has a finite number of jump discontinuities. Then the Fourier series of f converges to*

$$\overline{f}(x) = \frac{1}{2}\left(\lim_{y\to x^-} f(y) + \lim_{y\to x^+} f(y)\right).$$

This theorem will be demonstrated in the examples. At all points x where f is continuous, note that the Fourier series converges to $f(x)$. Notice also that the word "uniform" is removed from the conclusion. In fact, a very interesting situation occurs when there is a point of discontinuity of f, where the Fourier series converges to the average of the left and right limits. As the number N increases, there are x-values very close to the point of discontinuity where $F_N(x)$ differs from $f(x)$ by an amount that does not decrease and in fact converges to the number

$$a(0.08949\ldots),$$

where a is the gap between the left and right limits, and the number $0.08949\ldots$ is called the Wilbraham–Gibbs constant. The exact formula for the constant is

$$\frac{1}{\pi}\int_0^{\pi} \frac{\sin(x)}{x}\,dx - \frac{1}{2}.$$

This situation is called the **Gibbs Phenomenon**. We will see this phenomenon in the examples.

Examples and Applications

Square Wave

Define

$$f(x) = \begin{cases} -1 & \text{if } -\pi \leq x \leq 0 \\ 1 & \text{if } 0 \leq x < \pi \end{cases}$$

We now compute the Fourier series:

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0, \\
a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \cos(jx)\,dx = 0 \text{ for } j \geq 1 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin(kx)\,dx \\
&= \frac{1}{\pi}\left(-\int_{-\pi}^{0} \sin(kx)\,dx + \int_{0}^{\pi} \sin(kx)\,dx\right) \\
&= \frac{1}{\pi}\left(\frac{1}{k}\cos(kx)\Big|_{-\pi}^{0} - \frac{1}{k}\cos(kx)\Big|_{0}^{\pi}\right) \\
&= \frac{2}{\pi k}\left(1-(-1)^k\right) = \begin{cases} \frac{4}{\pi k} & k \text{ odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
f(x) &= \sum_{k \text{ odd}} \frac{4}{k\pi}\sin(kx) \\
&= \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \ldots\right).
\end{aligned}
$$

Plugging in $x = \frac{\pi}{2}$, we get

$$
\begin{aligned}
1 &= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right), \text{or} \\
\pi &= \left(4 - \frac{4}{3} + \frac{4}{5} - \ldots\right).
\end{aligned}
$$

(A famous formula for π !) Next, consider Parseval's Identity:

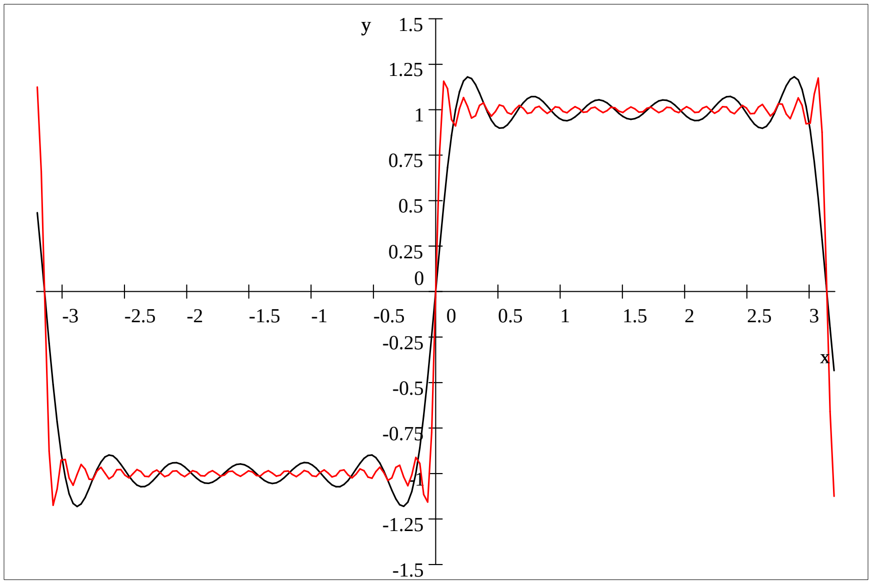
$$
\begin{aligned}
\int_{-\pi}^{\pi} (f(x))^2\,dx &= \pi \sum_{k \text{ odd}} \left(\frac{4}{k\pi}\right)^2, \text{ or} \\
2\pi &= \frac{16}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right),
\end{aligned}
$$

or

$$
1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}.
$$

(another famous formula!)

Let's now graph the square wave with some of the Fourier approximations: $y = \sum_{n=0}^{N} \frac{4}{(2n+1)\pi}\sin((2n+1)x)$



Note the Gibbs Phenomenon:

$\sum_{n=0}^{N} \frac{4}{(2n+1)\pi} \sin\left((2n+1)\left(\frac{\pi}{2N+1}\right)\right)$

$\sum_{n=0}^{5} \frac{4}{(2n+1)\pi} \sin((2n+1)(\frac{\pi}{11})) = 1.17345830790194905$

$\sum_{n=0}^{20} \frac{4}{(2n+1)\pi} \sin((2n+1)(\frac{\pi}{41})) = 1.17858309426023996$

$\sum_{n=0}^{50} \frac{4}{(2n+1)\pi} \sin\left((2n+1)\left(\frac{\pi}{101}\right)\right) = 1.17891438974940976$

$\frac{0.17891438974940976}{2} = 0.08945719487470488$

Sawtooth Wave

Define

$$f(x) = x$$

for $-\pi \leq x < \pi$ and can be thought of as being periodic. We now compute the Fourier series:

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0, \\
a_j &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(jx)\,dx = 0 \text{ for } j \geq 1 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx \\
&= \frac{1}{\pi}\left(\frac{1}{k^2}\sin(kx) - \frac{1}{k}x\cos(kx)\Big|_{-\pi}^{\pi}\right) \\
&= \frac{1}{\pi}\left(-\frac{(-1)^k\pi - (-1)^k(-\pi)}{k}\right) = \frac{2(-1)^{k+1}}{k}
\end{aligned}$$

Then

$$x = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin(kx).$$

Plugging in $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots\right),$$

which yields the formula for π found in the previous section. Parseval's identity yields:

$$\int_{-\pi}^{\pi} x^2 \, dx = \pi \sum_{k=1}^{\infty} \frac{4}{k^2}, \text{ or}$$

$$\frac{2\pi^3}{3} = 4\pi \sum_{k=1}^{\infty} \frac{1}{k^2},$$

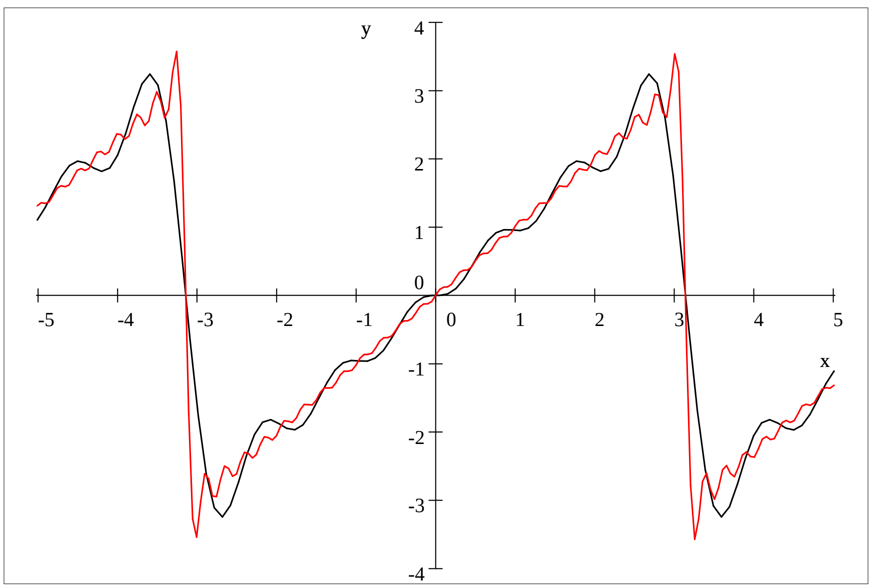
so that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4\pi} \frac{2\pi^3}{3} = \frac{\pi^2}{6}.$$

(Another famous formula!)

We now plot the Fourier approximations to the saw tooth wave:

$y = \sum_{k=1}^{N} \frac{2(-1)^{k+1}}{k} \sin(kx)$



Let's check out the Gibbs Phenomenon:

$y = \sum_{k=1}^{N} \frac{2(-1)^{k+1}}{k} \sin\left(k\left(\frac{N\pi}{N+1}\right)\right)$

$y = \sum_{k=1}^{5} \frac{2(-1)^{k+1}}{k} \sin\left(k\left(\frac{5\pi}{6}\right)\right) = 3.16570477234332464$

$y = \sum_{k=1}^{20} \frac{2(-1)^{k+1}}{k} \sin(k(\frac{20\pi}{21})) = 3.5530869812998034$

$$y = \sum_{k=1}^{50} \frac{2(-1)^{k+1}}{k} \sin\left(k\left(\frac{50\pi}{51}\right)\right) = 3.642\,072\,936\,301\,630\,28$$
$$y = \sum_{k=1}^{200} \frac{2(-1)^{k+1}}{k} \sin\left(k\left(\frac{200\pi}{201}\right)\right) = 3.688\,231\,329\,702\,340\,16$$
$$3.688\,231\,329\,702\,340\,16 - \frac{200\pi}{201} = 0.562\,268\,490\,309\,511\,067$$
$$\frac{0.562\,268\,490\,309\,511\,067}{2\pi} = .08\,948\,780\,957\,757\,613\,79$$