TOEPLITZ OPERATORS

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1. Introduction to Toeplitz Operators

Otto Toeplitz lived from 1881-1940 in Goettingen, and it was pretty rough there, so he eventually went to Palestine and eventually contracted tuberculosis and died. He developed this theory when common operator theory notions were very new.

Toeplitz studied infinite matrices with NW-SE diagonals constant.

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Modern formulation: Let S^1 be the unit circle in $\mathbb{C}$.

$$L^2\left(S^1\right) = \left\{ f : S^1 \to \mathbb{C} : \frac{1}{2\pi}\int_0^{2\pi} \left| f\left(e^{i\theta}\right)\right|^2 d\theta < \infty \right\}$$

The set $\left\{e^{in\theta} : n \in \mathbb{Z}\right\}$ is an orthonormal basis with respect to the L^2 inner product. Given $\phi \in L^\infty\left(S^1\right)$, define $M_\phi : L^2\left(S^1\right) :\to L^2\left(S^1\right)$ by $M_\phi f = \phi f$.

Easy results:

(1) $M_\phi^* = M_{\overline{\phi}}$;

(2) $M_{\phi+\psi} = M_\phi + M_\psi$;

(3) $M_\phi M_\psi = M_{\phi\psi}$;

(4) M_ϕ is invertible iff $\phi \in L^\infty\left(S^1\right)$ is invertible.

Hardy space:

$$\begin{aligned} H^2 = H^2\left(S^1\right) &= \text{Hilbert subspace of } L^2(S^1) \text{ spanned by } \left\{e^{in\theta} : n \geq 0\right\} \\ &= \text{set of elements of } L^2\left(S^1\right) \text{ that have an analytic extension to } \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}. \end{aligned}$$

Let $P : L^2\left(S^1\right) \to H^2$ be the orthogonal projection:

$$P\left(\sum_{n\in\mathbb{Z}} c_n e^{in\theta}\right) = \sum_{n=0}^{\infty} c_n e^{in\theta}.$$

For each ϕ in $L^\infty(S^1)$, define the *Toeplitz operator* $T_\phi : H^2 \to H^2$ by the formula $T_\phi = PM_\phi$.

Question: What is the matrix for T_ϕ with respect to the orthonormal basis $\{e^{in\theta} : n \geq 0\}$?

$$
\begin{aligned}
a_{mn} &= \left\langle T_\phi e^{in\theta}, e^{im\theta} \right\rangle = \left\langle PM_\phi e^{in\theta}, e^{im\theta} \right\rangle \\
&= \left\langle M_\phi e^{in\theta}, Pe^{im\theta} \right\rangle = \left\langle M_\phi e^{in\theta}, e^{im\theta} \right\rangle \\
&= \frac{1}{2\pi} \int_0^{2\pi} \phi\left(e^{i\theta}\right) e^{in\theta} e^{-im\theta} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \phi\left(e^{i\theta}\right) e^{i(n-m)\theta} d\theta \\
&= \widehat{\phi}\,(n-m)\,,
\end{aligned}
$$

so (a_{mn}) is the same type of matrix that Toeplitz studied.

The function ϕ is called the *symbol* of T_ϕ.

Question: How do operator-theoretic properties of T_ϕ relate to function-theoretic properties of ϕ?

Answer: It's complicated.

Easy results:

(1) $T_\phi^* = T_{\overline{\phi}}$.
(2) $T_{\phi+\psi} = T_\phi + T_\psi$
(3) $T_{\phi\psi} \neq T_\phi T_\psi$ in general
(4) T_ϕ invertible implies that ϕ is invertible.

In general, the converse to this last statement is false.

Example 1.1. *$T_z = T_{e^{i\theta}}$. Then $T_z(z^n) = z^{n+1}$ for $n \geq 0$. So $1 = z^0 \notin range(T_z)$, so T_z is not invertible. The operator T_z is the unilateral shift operator.*

From now on, we only consider continuous symbols.

2. Fredholm operators and the index

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space. Let $S : \mathcal{H} \to \mathcal{H}$ be a linear map, and define

$$\|S\| = \sup\left\{ \frac{\|Sv\|}{\|v\|} : v \neq 0 \right\}.$$

Define $\mathcal{B}(\mathcal{H}) = \{S : \mathcal{H} \to \mathcal{H} \text{ linear such that } \|S\| < \infty\}$. It is easy to show that $\mathcal{B}(\mathcal{H})$ is an algebra under addition and composition.

Let $\mathcal{F}$ be the set of finite rank operators on $\mathcal{H}$; i.e. the set of bounded operators on $\mathcal{H}$ that have finite-dimensional range. The set $\mathcal{F}$ is an ideal in $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{K} = \overline{\mathcal{F}}$. This is the ideal of *compact* operators.

Alternate definition: K is compact iff the image of the unit ball in $\mathcal{H}$ is compact (in the norm topology).

Remark: if $\mathcal{H}$ is separable, then $\mathcal{K}$ is the only topologically closed nontrivial proper ideal in $\mathcal{B}(\mathcal{H})$. The quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}$ is called the *Calkin algebra*, and the obvious function $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}$ is a quotient map in both the algebraic and topological senses.

If $S \in \mathcal{B}(\mathcal{H})$ is invertible modulo $\mathcal{K}(\mathcal{H})$ (i.e., $\pi(S)$ is invertible in $\mathcal{B}(\mathcal{H})/\mathcal{K}$), we say S is a *Fredholm* operator.

Theorem 2.1. *(Atkinson's Theorem) S is Fredholm iff*

(1) $\ker S$ *is finite dimensional*

(2) $\ker S^*$ *is finite dimensional*

(3) $\operatorname{ran} S$ *is (topologically) closed.*

If S is Fredholm, we define its *index* to be $\operatorname{index} S = \dim\ker S - \dim\ker S^*$.

Properties of index:

(1) If $\{S_t\}$ is a path of Fredholm operators, then $\operatorname{index} S_0 = \operatorname{index} S_1$. [Note $\dim\ker S_t$ and $\dim\ker S_t^*$ may change as a function of t.] In fact, the converse is also true.

(2) $\operatorname{index}(S+K) = \operatorname{index} S$ for all K compact.

(3) $\operatorname{index}(RS) = \operatorname{index} R + \operatorname{index} S$.

Questions: Which Toeplitz operators are Fredholm? If T_ϕ is Fredholm, can we compute its index from its symbol ϕ?

Proposition 2.2. *If ϕ and ψ are in $C(S^1)$, then $T_\phi T_\psi - T_{\phi\psi}$ is compact.*

Idea of proof: For all integers m and n, the operator $T_{z^n}T_{z^m} - T_{z^n z^m}$ is a finite rank operator.

Example: Look at $T_{z^2}T_{z^{-3}} - T_{z^2 z^{-3}}$. For $k \geq 0$,

$$T_{z^2 z^{-3}}(z^k) = T_{z^{-1}}(z^k) = \begin{cases} 0 & k = 0 \\ z^{k-1} & k \geq 1 \end{cases}$$

and

$$(T_{z^2}T_{z^{-3}})(z^k) = \begin{cases} 0 & k < 3 \\ T_{z^2}(z^{k-3}) & k \geq 3 \end{cases} = \begin{cases} 0 & k < 3 \\ z^{k-1} & k \geq 3 \end{cases}.$$

Thus

$$(T_{z^2}T_{z^{-3}} - T_{z^2 z^{-3}})(z^k) = \begin{cases} 0 & k = 0, 3, 4, 5, \dots \\ -z^{k-1} & k = 1, 2 \end{cases},$$

whence $T_{z^2}T_{z^{-3}} - T_{z^2 z^{-3}}$ has rank 2.

Back to idea of proof: If ϕ and ψ are Laurent polynomials, then $T_\phi T_\psi - T_{\phi\psi}$ is finite rank, and an approximation argument then shows that $T_\phi T_\psi - T_{\phi\psi}$ is compact for general continuous ϕ and ψ.

The *Toeplitz algebra* is $\mathcal{T} = \{T_\phi + K : \phi \in C(S^1), K \in \mathcal{K}\}$.

Remark: $\mathcal{T}$ is the universal C^*-algebra generated by a non unitary isometry (Coburn's Theorem).

Theorem 2.3. *There exists a short exact sequence*
$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0,$$
where the map $\sigma : \mathcal{T} \longrightarrow C(S^1)$ is defined by $\sigma(T_\phi + K) = \phi$.

Remark: We also have a linear splitting $\xi : C(S^1) \longrightarrow \mathcal{T}$ given by $\xi(\phi) = T_\phi$, but ξ is not multiplicative.

Corollary 2.4. *An element T of the Toeplitz algebra is Fredholm if and only if its symbol $\sigma(T)$ is invertible; i.e., nowhere vanishing on the circle.*

Theorem 2.5. *If T in $\mathcal{T}$ is Fredholm, then*
$$\operatorname{Index} T = -(\text{winding number } of\, \sigma(T)) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{d\sigma(T)}{\sigma(T)}.$$

Proof: Because of Property (2) of the index, we need only prove the result for honest Toeplitz operators T_ϕ with ϕ nowhere vanishing on the circle. Next, every nonvanishing function on S^1 can be homotoped to z^n for some n, so by Property (1) of the index and homotopy invariance of winding number, it is enough to prove the theorem for Toeplitz operators T_{z^n}. And because of Property (3) of the index (along with the obvious property $\operatorname{Index}(S^*) = -\operatorname{Index} S$ for all Fredholm operators S) and the algebraic properties of winding number, it suffices to establish the result for T_z. As we saw earlier, T_z is injective, so $\dim\ker T_z = 0$. It is easy to show that the kernel of $T_z^* = T_{z^{-1}}$ is spanned by the constant function 1, and hence $\dim\ker T_z^* = 1$. Therefore
$$\operatorname{Index} T_z = 0 - 1 = -1 = -(\text{winding number of } z).$$

If S in $\mathcal{B}(\mathcal{H})$ is invertible, then S^* is also invertible, whence S is a Fredholm operator of index 0. The converse of this result is not generally true, but surprisingly it is for Toeplitz operators (but not for arbitrary elements of $\mathcal{T}$).

Lemma 2.6 (F. and M. Riesz)**.** *If $f \in H^2$ is not the zero function, then f in nonzero almost everywhere.*

Proposition 2.7 (Coburn)**.** *Suppose $\phi \in C(S^1)$ is nowhere vanishing. Then either* $\ker T_\phi = \{0\}$ *or* $\ker T_\phi^* = \{0\}$.

Proof: Suppose there exist f and g are nonzero functions (a.e.) such that $T_\phi f = 0$ and $T_{\overline{\phi}} g = 0$. Then $P(\phi f) = 0$ and $P(\overline{\phi} g) = 0$, whence $\overline{\phi f}$ and $\phi\overline{g}$ are in H^2. Therefore $\phi f \overline{g}$ and $\overline{\phi f} g = \overline{\phi f \overline{g}}$ are in H^1, which implies that $\phi f\overline{g}$ equals a real constant almost everywhere. But, note that (surprisingly!)
$$\frac{1}{2\pi}\int \phi f \overline{g} = \left(\frac{1}{2\pi}\int f\right)\left(\frac{1}{2\pi}\int \phi\overline{g}\right) = 0,$$
so $\phi f\overline{g}$ is zero a.e., a contradiction.

Corollary 2.8. *T_ϕ is invertible if and only if ϕ is nowhere vanishing and* $\operatorname{Index} T_\phi = 0$.

Warning: If T_ϕ is invertible, it is not generally true that $(T_\phi)^{-1} = T_{\phi^{-1}}$.

3. Generalizations

Easy generalization: Suppose Φ is in $\mathrm{M}(n, C(S^1))$. Then we have a matrix multiplication operator

$$M_\Phi : \left(L^2(S^1)\right)^n \longrightarrow \left(L^2(S^1)\right)^n, \qquad M_\Phi(F) = \Phi F$$

and Toeplitz operator

$$T_\Phi : \left(H^2\right)^n \longrightarrow \left(H^2\right)^n, \qquad T_\Phi(F) = PM_\Phi,$$

where $P : (L^2(S^1))^n \longrightarrow (H^2)^n$ is the obvious map.

There exists a short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes \mathrm{M}(n, \mathbb{C}) \longrightarrow \mathcal{T} \otimes \mathrm{M}(n, \mathbb{C}) \longrightarrow C(S^1) \otimes \mathrm{M}(n, \mathbb{C}) \longrightarrow 0,$$

or

$$0 \longrightarrow \mathrm{M}(n, K) \longrightarrow \mathrm{M}(n, \mathcal{T}) \longrightarrow \mathrm{M}(n, C(S^1)) \longrightarrow 0.$$

Theorem 3.1. *T_Φ is Fredholm if and only if $\Phi(z)$ is invertible for every z in S^1, and in this case, the index of T_Φ equals the negative of the winding number of* $\det \Phi$.

Remark: In this generality, it is not true that $\operatorname{Index} T_\Phi = 0$ implies that T_Φ is invertible.

Back to scalar (non-matrix) case for a minute. Consider $D = \frac{1}{i}\frac{d}{d\theta}$ as an (unbounded) differential operator on $L^2(S^1)$. The functions $e^{in\theta}$ for a complete set of eigenvectors for D: $D(e^{in\theta}) = ne^{in\theta}$. Note that P is the projection onto the span of the eigenspaces associated to nonnegative eigenvalues of D; in other words, P is the *positive spectral projection* of D.

Crazy generalization:

Let X be a topological space. A *complex vector space* is a topological space E and a continuous surjection $p : E \longrightarrow X$ such that for each x in X there is a neighborhood U such that $p^{-1}(U) \cong U \times \mathbb{C}^n$ for some natural number n. The set $E_x = p^{-1}(x)$ is called the *fiber* of E over x.

A *section* of E is a continuous function $s : X \longrightarrow E$ with the property that $s(x) \in E_x$ for each x in X, or, in other words, $p(s(x)) = x$ for all x in X. We call E a *Hermitian* vector bundle if each E_x has a complex inner product on it and these inner products vary continuously.

Now suppose X is a smooth manifold and that E is a smooth Hermitian vector bundle over X. Let D be a self-adjoint *elliptic* differential operator acting on sections of a smooth Hermitian complex vector bundle E over a smooth manifold X; "elliptic" roughly means that D differentiates in all directions.

Given ϕ in $C(X)$, we have a multiplication operator $M_\phi : L^2(X,E) \longrightarrow L^2(X,E)$ and a Toeplitz operator $T_\phi = PM_\phi$, where P is the positive spectral projection of D. More generally, we can form Toeplitz operators with matrix-valued symbols.

Theorem 3.2 (Baum-Douglas)**.** [Odd Atiyah-Singer Index Theorem] *T_Φ is Fredholm if and only if Φ is in* $\mathrm{GL}(n, C(M))$*, and in this case there is a formula for the index involving F, topological invariants of M and E, and the* principal symbol *of D.*

Remark 1: In general we only get nonzero indices here if M is odd-dimensional.

Remark 2: The formula mentioned in the theorem can be expressed as an integral over M, with one factor in the integrand being the *Chern character of* Φ*:*

$$\mathrm{ch}(\Phi) = \sum_{n=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \operatorname{tr} \left(\Phi^{-1}\, d\Phi\right)^{2k+1}$$

Note that the integrand in the winding number formula is one term in this sum.

4. Traces, Determinants, and Toeplitz Operators

Given a finite matrix with complex entries, we can compute its trace and determinant in terms of those entries, and we know that these numbers are equal to the sum and the product of the eigenvalues of that matrix. We can try to do the same thing with operators on a separable infinite-dimensional Hilbert space $\mathcal{H}$. Take S in $\mathcal{B}(\mathcal{H})$, let $\{e_n\}$ be an orthonormal basis for a Hilbert space and try to compute

$$\sum \langle Se_n, e_n \rangle.$$

Obvious problem: this sum may not converge.

Not-so-obvious problem: even if this sum converges, it might converge to something different if we change our orthonormal basis – or it might not converge at all!

Instead look at

$$\sum \langle \sqrt{S^*S} e_n, e_n \rangle.$$

This (possibly infinite) sum does not depend on the choice of orthonormal basis. We say S is *trace class* if the sum in finite. In this case, $\sum \langle Se_n, e_n \rangle$ converges as well and also does not depend on the choice of orthonormal basis $\{e_n\}$. We can therefore define the *trace* of S in this case to be

$$\operatorname{tr} S = \sum \langle Se_n, e_n \rangle.$$

The set of trace class operators is denoted $\mathcal{L}^1 = \mathcal{L}^1(\mathcal{H})$.

Properties of trace class operators:

(1) $\mathcal{F} \subset \mathcal{L}^1 \subset \mathcal{K}$;
(2) $\mathcal{L}^1$ is a (nonclosed) ideal in $\mathcal{B}(\mathcal{H})$;
(3) $\operatorname{tr} S =$ sum of eigenvalues of S;
(4) If K and L are trace class, then $K + L$ is trace class and $\operatorname{tr}(K + L) = \operatorname{tr} K + \operatorname{tr} L$;
(5) If K is trace class and S is any (bounded) operator, then $\operatorname{tr}(KS) = \operatorname{tr}(SK)$.

Proposition 4.1. *If ϕ, ψ are smooth functions on S^1, then $T_\phi T_\psi - T_\psi T_\phi$ is trace class. Furthermore, the trace of this commutator is invariant under trace class perturbations: if K and L are trace class, then $(T_\phi + K)(T_\psi + L) - (T_\psi + L)(T_\phi + K)$ is trace class and*

$$\operatorname{tr}\left((T_\phi + K)(T_\psi + L) - (T_\psi + L)(T_\phi + K)\right) = \operatorname{tr}(T_\phi T_\psi - T_\psi T_\phi).$$

Proof: We noted earlier that $T_\phi T_\psi - T_\psi T_\phi$ is finite rank if ϕ and ψ are Laurent polynomials; we can then do an approximation argument to establish the first statement. To prove the second statement, multiply out $(T_\phi + K)(T_\psi + L) - (T_\psi + L)(T_\phi + K)$ and then use Properties 4 and 5 above to get rid of all the terms except $T_\phi T_\psi - T_\psi T_\phi$.

The proposition suggests that we should be able to write down a formula for $\operatorname{tr}(T_\phi T_\psi - T_\psi T_\phi)$ in terms of ϕ and ψ.

Theorem 4.2 (Helton-Howe)**.** *With the hypotheses above,*

$$\operatorname{tr}(T_\phi T_\psi - T_\psi T_\phi) = \frac{1}{2\pi i}\int \phi \, d\psi = -\frac{1}{2\pi i}\int \psi \, d\phi.$$

Note that unlike the situation in finite-dimensional vector spaces, the trace of a commutator can be nonzero!

What about determinants?
Let $(\mathcal{L}^1)^+ = \{I + S : S \in \mathcal{L}^1\}$.

Lemma 4.3. *Every element A in $(\mathcal{L}^1)^+$ can be written as $\exp S$ for some S in $\mathcal{L}^1$.*

Definition: Given A in $(\mathcal{L}^1)^+$, its *determinant* is defined as follows: Write $A = \exp(S)$. Then $\det A = e^{\operatorname{tr} S}$. In other symbols, $\det(e^S) = e^{\operatorname{tr} S}$.

This agrees with the usual definition of determinant for linear maps on finite-dimensional vector spaces. The elements of $(\mathcal{L}^1)^+$ are called *determinant class operators*.

Suppose that ϕ and ψ are nonvanishing smooth functions on the circle that have winding number 1. Then T_ϕ and T_ψ are both invertible by our earlier results, and the multiplicative commutator $T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}$ is in $(\mathcal{L}^1)^+$:

$$T_\phi T_\psi T_\phi^{-1} T_\psi^{-1} = I + (T_\phi T_\psi - T_\psi T_\phi) T_\phi^{-1} T_\psi^{-1}.$$

Question: What is $\det(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1})$?

One property of the determinant is that $\det(ABA^{-1}B^{-1}) = 1$ if A and B are invertible determinant class operators (but not in general!). This fact along with a bit of computation implies that

$$\det(STS^{-1}T^{-1}) = \det(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1})$$

for any invertible S and T in $\mathcal{T}$ such that $\sigma(S) = \phi$ and $\sigma(T) = \psi$. This suggests that perhaps there is formula for $\det(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1})$ in terms of the symbols ϕ and ψ.

Because ϕ and ψ are nowhere zero and have winding number zero, $\log \phi$ and $\log \psi$ are defined. Write these in terms of their Fourier series:

$$\log \phi = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \qquad \log \psi = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}$$

Theorem 4.4 (Helton-Howe)**.** *If ϕ and ψ are nowhere zero and have winding number zero, then*

$$\det(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}) = \exp\left(\sum_{n=-\infty}^{\infty} n a_{-n} b_n\right)$$

Not very pretty! But there's a much prettier formula (IMHO).

Suppose that ϕ and ψ are restrictions of meromorphic functions (which we also denote ϕ and ψ) defined in a neighborhood of the closed unit disk such that neither ϕ nor ψ has zeros or poles on the unit circle. For each point z in the open unit disk $\mathbb{D}$, define

$$v(\phi, z) = \begin{cases} m & \text{if } \phi \text{ has a zero of order } m \text{ at } z \\ -m & \text{if } \phi \text{ has a pole of order } m \text{ at } z \\ 0 & \text{if } \phi \text{ has neither a zero nor a pole at } z, \end{cases}$$

and similarly define $v(\psi, z)$. The quantity

$$\lim_{w \to z} (-1)^{v(\phi,z)v(\psi,z)} \frac{\psi(w)^{v(\phi,z)}}{\phi(w)^{v(\psi,z)}}$$

is called the *tame symbol* of ϕ and ψ at z and is denoted $(\phi, \psi)_z$.

Theorem 4.5 (Carey-Pincus)**.** *If ϕ and ψ are nowhere zero and have winding number zero, then*

$$\det(T_\phi T_\psi T_\phi^{-1} T_\psi^{-1}) = \prod_{z \in \mathbb{D}} (\phi, \psi)_z^{-1}.$$

Remark: The quantity $(\phi, \psi)_z^{-1}$ equals 1 for all but finitely many z in the open unit disk, so the RHS is well defined.

If ϕ and ψ are nowhere zero but do not have winding number zero, there is a way to generalize the Carey-Pincus result. Define matrices

$$\Phi = \begin{pmatrix} \phi & 0 & 0 \\ 0 & \phi^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \psi^{-1} \end{pmatrix}.$$

Then

$$R_\phi = \begin{pmatrix} 2T_\phi - T_\phi T_{\phi^{-1}} T_\phi & T_\phi T_{\phi^{-1}} - I & 0 \\ I - T_{\phi^{-1}} T_\phi & T_{\phi^{-1}} & 0 \\ 0 & 0 & I \end{pmatrix}$$

and

$$S_\psi = \begin{pmatrix} 2T_\psi - T_\psi T_{\psi^{-1}} T_\psi & 0 & T_\psi T_{\psi^{-1}} - I \\ 0 & I & 0 \\ I - T_{\psi^{-1}} T_\psi & 0 & T_{\psi^{-1}} \end{pmatrix}$$

are invertible matrices with entries in $\mathcal{T}$, and therefore $\det(R_\phi S_\psi R_\phi^{-1} S_\psi^{-1})$ is defined. Furthermore, its value does not depend on our choices of invertible "lifts" of Φ and Ψ.

Theorem 4.6.

$$\det(R_\phi S_\psi R_\phi^{-1} S_\psi^{-1}) = \prod_{z \in \mathbb{D}} (\phi, \psi)_z^{-1}$$

We see that the determinant in the theorem above only depends on ϕ and ψ. But in fact, this determinant really only depends on the *Steinberg symbol* (different meaning of the word "symbol"!) $\{\phi, \psi\}$ in the algebraic K-theory group $K_2^{alg}(C^\infty(S^1))$.

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