1. An Example

Consider $M(n, \mathbb{C})$. Two quantities trace and determinant are invariants of matrices; can they be generalized to infinite dimensions?

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded (continuous) linear maps (operators) from $\mathcal{H}$ to $\mathcal{H}$. Suppose we try to compute the trace of $A \in \mathcal{B}(\mathcal{H})$.

$$Tr(A) = \sum_{i=1}^{\infty} \langle Ae_i, e_i \rangle,$$

where $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$. This doesn’t always converge. Let

$$L^1(\mathcal{H}) = \left\{ A \in \mathcal{B}(\mathcal{H}) : Tr\sqrt{A^*A} < \infty \right\}.$$

Then the trace is well-defined on $L^1(\mathcal{H})$ — the ideal of trace-class operators on $\mathcal{H}$. The determinant is well-defined on operators in $1 + L^1(\mathcal{H})$.

Question: Is there an “easy” way to compute det $(A)$ for $A \in 1 + L^1(\mathcal{H})$? Why could you expect that the answer to be yes ever?

Answer: Scott Nollet

The most notable example is the Fredholm determinant, the index.

2. Toeplitz Operators

Consider $\mathcal{H} = L^2(S^1)$. Let $\{z^n\}_{n \in \mathbb{Z}}$ be an o-n basis. For $f \in C(S^1)$, let

$$m_f : L^2(S^1) \rightarrow L^2(S^1)$$

be defined by $m_f(g) = fg$. Note that $m_fm_g = m_gm_f$ for all $f \in C(S^1)$.

Let $H^2(S^1)$ be the Hilbert subspace of $L^2(S^1)$ spanned by $\{z^n\}_{n \geq 0}$. Let

$$P : L^2(S^1) \rightarrow H^2(S^1)$$

denote the orthogonal projection. Now, for $f \in C(S^1)$, let

$$T_f : H^2(S^1) \rightarrow H^2(S^1)$$

be defined by $T_f q = Pf q$ for $q \in H^2(S^1)$. Interestingly, $T_f$ and $T_g$ do not commute. However, for $f, h \in C^\infty(S^1)$,

$$T_fT_h - T_hT_f \in L^1(\mathcal{H}^2(S^1)),$$

We define the smooth Toeplitz algebra

$$T^\infty = \left\{ T_f + L : f \in C^\infty(S^1), L \in L^1(\mathcal{H}^2(S^1)) \right\}.$$
Then we have the following short exact sequence
\[ 0 \rightarrow \mathcal{L}^1 \rightarrow \mathcal{T}^\infty \xrightarrow{\sigma} C^\infty (S^1) \rightarrow 0, \]
where
\[ \sigma (T_f + L) = f. \]
Note that the principal symbol \(\sigma\) is a homomorphism.

Now, suppose \(S, T \in \mathcal{T}^\infty\) are invertible, and \(\sigma (S) = f, \sigma (T) = g\). Consider
\[ STS^{-1}T^{-1}, \]
which has symbol
\[ \sigma (STS^{-1}T^{-1}) = 1, \]
which implies that \(STS^{-1}T^{-1} \in T_1 + \mathcal{L}^1 = 1 + \mathcal{L}^1\). So the determinant of this operator exists.

**Theorem 2.1. (Helton-Howe, 1973)**
\[ \det (STS^{-1}T^{-1}) = \exp \left( \frac{1}{2\pi i} \int_0^{2\pi} \ln (f) \frac{dg}{g} \right). \]

Question: How do you know that \(\det (STS^{-1}T^{-1})\) only depends on the symbols \(f\) and \(g\).

Let \(\tilde{S} = S + L, \tilde{T} = T + K\). Then...

Observation (Brown, 1975): \(\det (STS^{-1}T^{-1})\) is related to algebraic K-theory and only depends on the Steinberg symbol \(\{f, g\} \in K_2^{\text{alg}} (C^\infty (S^1))\) of \(f = \sigma (S)\) and \(g = \sigma (T)\).

### 3. Algebraic K-theory

Let \(R\) be a ring with unit (think \(R = C^\infty (S^1)\)). This is basically a homology theory for rings. Anyway, \(K_0 (R)\) is an abelian group constructed from finitely generated projective modules over \(R\). (eg sections of a vector bundle). “Projective” means that it is a direct summand of a free module. (For vector bundles, you can take a direct sum with another bundle to get a trivial bundle). We define \(K_1 (R)\), let \(GL (R)\) be the set of infinite matrices over \(R\) of the form
\[ \begin{pmatrix} * & 0 \\ 0 & I_\infty \end{pmatrix}, \]
where \(I_\infty\) is the infinite identity matrix, and * is an invertible finite-dimensional matrix over \(R\). For \(i \neq j, r \in R\), the elementary matrices \(e_{ij} (r)\) in \(GL (R)\) are the matrices whose \(a, b\) entry is
\[ (e_{ij} (r))_{ab} = \begin{cases} \delta_{ab} & \text{if } (a, b) \neq (i, j) \\ r & \text{if } (a, b) = (i, j) \end{cases} \]
Next, let \(E (R)\) be the normal subgroup generated by \(\{e_{ij} (r)\}\). We define
\[ K_1^{\text{alg}} (R) = GL (R) / E (R). \]

Properties of elementary matrices:
- \(e_{ij} (r) e_{ij} (s) = e_{ij} (r + s)\)
- \(e_{ij} (r) e_{kl} (s) = e_{kl} (s) e_{ij} (r)\) if \(i \neq k, j \neq l\)
- \(e_{ij} (r) e_{jk} (s) e_{ij} (r)^{-1} e_{jk} (s)^{-1} = e_{ik} (r), i, j, k\) distinct
These properties characterize elementary matrices.

Let the Steinberg group $St(R)$ be the group with generators

$$\{x_{ij}(r) : i, j \in \mathbb{N}, i \neq j, r \in R\}$$

with the relations

- $x_{ij}(r)x_{ij}(s) = x_{ij}(r + s)$
- $x_{ij}(r)x_{kl}(s) = x_{kl}(s)x_{ij}(r)$ if $i \neq k, j \neq l$
- $x_{ij}(r)x_{jk}(s)x_{ij}(r)^{-1}x_{jk}(s)^{-1} = x_{ik}(r)$, $i, j, k$ distinct

There is an obvious surjection

$$\phi : St(r) \to E(R)$$

defined by

$$\phi(x_{ij}(r)) = e_{ij}(r).$$

Then we define

$$K_{2}^{\text{alg}}(R) = \ker \phi$$

(Milnor, ~1972) This is an abelian group! There is a short exact sequence

$$0 \to K_{2}^{\text{alg}}(R) \to St(R) \xrightarrow{\phi} E(R) \to 0.$$ 

How can we write down elements of $K_{2}^{\text{alg}}(R)$? Suppose that $A, B \in E(R)$, and they commute. Choose $\tilde{A}, \tilde{B} \in St(R)$ with $\phi(\tilde{A}) = A, \phi(\tilde{B}) = B$. Define

$$A \ast B = \tilde{A}\tilde{B}\tilde{A}^{-1}\tilde{B}^{-1} \in K_{2}^{\text{alg}}(R).$$

Suppose that $R$ is a commutative ring. For $a, b$ invertible elements of $R$, let

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & I_{\infty} \end{pmatrix}, B = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b^{-1} & 0 \\ 0 & 0 & 0 & I_{\infty} \end{pmatrix}.$$ 

By the Whitehead Lemma, these are in $E(R)$. We define the Steinberg symbol $\{a, b\}$ as

$$\{a, b\} = A \ast B \in K_{2}^{\text{alg}}(R).$$

**Theorem 3.1.** (Brown, 1975) We have

$$\det (T_{f}T_{g}T_{f}^{-1}T_{g}^{-1})$$

depends only on the Steinberg symbol $\{f, g\}$. 

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