ALGEBRAIC K-THEORY AND DETERMINANTS OF TOEPLITZ OPERATORS

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1. An Example

Consider $M(n, \mathbb{C})$. Two quantities *trace* and *determinant* are invariants of matrices; can they be generalized to infinite dimensions?

Let $\mathcal{H}$ be a complex, separable, infinite dimensional Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded (continuous) linear maps (*operators*) from $\mathcal{H}$ to $\mathcal{H}$. Suppose we try to compute the trace of $A \in \mathcal{B}(\mathcal{H})$.

$$Tr(A) \text{"} = \text{"} \sum_{i=1}^{\infty} \langle Ae_i, e_i \rangle ,$$

where $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$. This doesn't always converge. Let

$$\mathcal{L}^1(\mathcal{H}) = \left\{ A \in \mathcal{B}(\mathcal{H}) : Tr\sqrt{A^*A} < \infty \right\}.$$

Then the trace is well-defined on $\mathcal{L}^1(\mathcal{H})$ — the ideal of trace-class operators on $\mathcal{H}$. The determinant is well-defined on operators in $\mathbf{1} + \mathcal{L}^1(\mathcal{H})$.

Question: Is there an "easy" way to compute $\det(A)$ for $A \in \mathbf{1} + \mathcal{L}^1(\mathcal{H})$? Why could you expect that the answer to be yes ever?

Answer: Scott Nollet

The most notable example is the Fredholm determinant, the index.

2. Toeplitz operators

Consider $\mathcal{H} = L^2(S^1)$. Let $\{z^n\}_{n\in\mathbb{Z}}$ be an o-n basis. For $f \in C(S^1)$, let

$$m_f : L^2\left(S^1\right) \to L^2\left(S^1\right)$$

be defined by $m_f(g) = fg$. Note that $m_f m_g = m_g m_f$ for all $f \in C(S^1)$.

Let $H^2(S^1)$ be the Hilbert subspace of $L^2(S^1)$ spanned by $\{z^n\}_{n\geq 0}$. Let

$$P : L^2\left(S^1\right) \to H^2\left(S^1\right)$$

denote the orthogonal projection. Now, for $f \in C(S^1)$, let

$$T_f : H^2\left(S^1\right) \to H^2\left(S^1\right)$$

be defined by $T_f q = Pfq$ for $q \in H^2(S^1)$. Interestingly, T_f and T_g do not commute. However, for $f, h \in C^\infty(S^1)$,

$$T_f T_h - T_h T_f \in \mathcal{L}^1\left(\mathcal{H}^2\left(S^1\right)\right).$$

We define the smooth Toeplitz algebra

$$\mathcal{T}^\infty = \left\{ T_f + L : f \in C^\infty\left(S^1\right), L \in \mathcal{L}^1\left(\mathcal{H}^2\left(S^1\right)\right)\right\}.$$

Then we have the following short exact sequence

$$0 \to \mathcal{L}^1 \to \mathcal{T}^\infty \xrightarrow{\sigma} C^\infty\left(S^1\right) \to 0,$$

where

$$\sigma\left(T_f + L\right) = f.$$

Note that the principal symbol σ is a homomorphism.

Now, suppose $S, T \in \mathcal{T}^\infty$ are invertible, and $\sigma(S) = f$, $\sigma(T) = g$. Consider

$$STS^{-1}T^{-1},$$

which has symbol

$$\sigma\left(STS^{-1}T^{-1}\right) = 1,$$

which implies that $STS^{-1}T^{-1} \in T_1 + \mathcal{L}^1 = \mathbf{1} + \mathcal{L}^1$. So the determinant of this operator exists.

Theorem 2.1. *(Helton-Howe,1973)*

$$\det\left(STS^{-1}T^{-1}\right) = \exp\left(\frac{1}{2\pi i}\int_0^{2\pi} \ln(f)\,\frac{dg}{g}\right).$$

Question: How do you know that $\det\left(STS^{-1}T^{-1}\right)$ only depends on the symbols f and g. Let $\widetilde{S} = S + L$, $\widetilde{T} = T + K$. Then...

Observation (Brown, 1975): $\det\left(STS^{-1}T^{-1}\right)$ is related to algebraic K-theory and only depends on the *Steinberg symbol* $\{f, g\} \in K_2^{\mathrm{alg}}\left(C^\infty\left(S^1\right)\right)$ of $f = \sigma(S)$ and $g = \sigma(T)$.

3. Algebraic K-theory

Let R be a ring with unit (think $R = C^\infty\left(S^1\right)$). This is basically a homology theory for rings. Anyway, $K_0(R)$ is an abelian group constructed from finitely generated projective modules over R. (eg sections of a vector bundle). "Projective" means that it is a direct summand of a free module. (For vector bundles, you can take a direct sum with another bundle to get a trivial bundle). We define $K_1(R)$, let $GL(R)$ be the set of infinite matrices over R of the form

$$\begin{pmatrix} * & 0 \\ 0 & I_\infty \end{pmatrix},$$

where I_∞ is the infinite identity matrix, and $*$ is an invertible finite-dimensional matrix over R. For $i \neq j$, $r \in R$, the elementary matrices $e_{ij}(r)$ in $GL(R)$ are the matrices whose a, b entry is

$$\left(e_{ij}(r)\right)_{ab} = \begin{cases} \delta_{ab} & \text{if } (a,b) \neq (i,j) \\ r & \text{if } (a,b) = (i,j) \end{cases}$$

Next, let $E(R)$ be the normal subgroup generated by $\{e_{ij}(r)\}$. We define

$$K_1^{\mathrm{alg}}(R) = GL(R) \diagup E(R).$$

Properties of elementary matrices:

- $e_{ij}(r)\, e_{ij}(s) = e_{ij}(r+s)$
- $e_{ij}(r)\, e_{kl}(s) = e_{kl}(s)\, e_{ij}(r)$ if $i \neq k$, $j \neq l$
- $e_{ij}(r)\, e_{jk}(s)\, e_{ij}(r)^{-1}\, e_{jk}(s)^{-1} = e_{ik}(r)$, i, j, k distinct

These properties characterize elementary matrices.

Let the Steinberg group $St\left(R\right)$ be the group with generators

$$\left\{x_{ij}\left(r\right): i,j\in\mathbb{N}, i\neq j, r\in R\right\}$$

with the relations

- $x_{ij}\left(r\right)x_{ij}\left(s\right)=x_{ij}\left(r+s\right)$
- $x_{ij}\left(r\right)x_{kl}\left(s\right)=x_{kl}\left(s\right)x_{ij}\left(r\right)$ if $i\neq k$, $j\neq l$
- $x_{ij}\left(r\right)x_{jk}\left(s\right)x_{ij}\left(r\right)^{-1}x_{jk}\left(s\right)^{-1}=x_{ik}\left(r\right)$, i,j,k distinct

There is an obvious surjection

$$\phi: St\left(r\right)\to E\left(R\right)$$

defined by

$$\phi\left(x_{ij}\left(r\right)\right)=e_{ij}\left(r\right).$$

Then we define

$$K_2^{\mathrm{alg}}\left(R\right)=\ker\phi$$

(Milnor, ˜1972) This is an abelian group! There is a short exact sequence

$$0\to K_2^{\mathrm{alg}}\left(R\right)\to St\left(R\right)\xrightarrow{\phi}E\left(R\right)\to 0.$$

How can we write down elements of $K_2^{\mathrm{alg}}\left(R\right)$? Suppose that $A,B\in E\left(R\right)$, and they commute. Choose $\widetilde{A},\widetilde{B}\in St\left(R\right)$ with $\phi\left(\widetilde{A}\right)=A$, $\phi\left(\widetilde{B}\right)=B$. Define

$$A*B=\widetilde{A}\widetilde{B}\widetilde{A}^{-1}\widetilde{B}^{-1}\in K_2^{\mathrm{alg}}\left(R\right).$$

Suppose that R is a commutative ring. For a,b invertible elements of R, let

$$A=\begin{pmatrix} a & 0 & 0\\ 0 & a^{-1} & 0\\ 0 & 0 & I_\infty\end{pmatrix}, B=\begin{pmatrix} b & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & b^{-1} & 0\\ 0 & 0 & 00 & I_\infty\end{pmatrix}.$$

By the Whitehead Lemma, these are in $E\left(R\right)$. We define the *Steinberg symbol* $\{a,b\}$ as

$$\{a,b\}=A*B\in K_2^{\mathrm{alg}}\left(R\right).$$

Theorem 3.1. *(Brown, 1975) We have*

$$\det\left(T_fT_gT_f^{-1}T_g^{-1}\right)$$

depends only on the Steinberg symbol $\{f,g\}$.

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