MOD k INDEX THEOREM

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1. The index problem

Let Q be a smooth compact spin^c manifold with nonempty boundary ∂Q. We assume that ∂Q consists of K isomorphic components. This goes back to work of Melrose and Freed in the late 1980's. Consider the Dirac operator on Q with APS boundary conditions. They showed that index $(D) \in \mathbb{Z}$ is not a topological invariant, but it is a topological invariant mod k. Question: can we compute index (D) topologically? Answer (Higson): Yes!

2. Index theory via operator algebras

Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a $\mathbb{Z}_2$-graded Hilbert space, and let $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \mathcal{H} \to \mathcal{H}$ be the grading operator. Define projections

$$\begin{aligned} Q^+ &= \frac{1}{2}(1+\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ Q^- &= \frac{1}{2}(1-\varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and suppose that $D : \mathcal{H} \to \mathcal{H}$ is self-adjoint of odd degree, ie

$$D = \begin{pmatrix} 0 & (D^+)^* \\ D^+ & 0 \end{pmatrix}$$

for some $D^+ : \mathcal{H}^+ \to \mathcal{H}^-$. Note that $\varepsilon D = -D\varepsilon$.

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded (continuous) linear maps $\mathcal{H} \to \mathcal{H}$. We say $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H})$ is a C^*-algebra if

- $\mathcal{C}$ is a subalgebra, not nec containing I
- $\mathcal{C}$ is closed under adjoint $*$
- $\mathcal{C}$ is closed in norm ($\|T\|_{op} = \sup\limits_{f \neq 0} \frac{\|Tf\|}{\|f\|}$.)

Suppose that $\mathcal{C}$ is a C^*-algebra such that

- $(D \pm i)^{-1} \in \mathcal{C}$
- $\varepsilon\mathcal{C} \subseteq \mathcal{C}$
- the projections Q^+, $Q^- \notin \mathcal{C}$ (prevents theorem from being trivial)

Example 2.1. *$D : L^2(S^1) \to L^2(S^1).\{z^n : n \in \mathbb{Z}\}$ is an orthonormal basis. The operator $D\left(z^n = e^{in\theta}\right) := nz^{n-1}$ is only densely defined and does not map L^2 to L^2. However, note that $(D+i)(z^n) = (n+i)z^n$, and $(D+i)^{-1}(z^n) = \frac{1}{(n+i)}z^n$, and this operator maps L^2 to L^2! Here $\mathcal{C}$ is the set of compact operators (norm closure of finite rank operators).*

Let $\widetilde{\mathcal{C}}$ be the C^*-algebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{C}$ and ε (and thus contains $I = \varepsilon^2$ and $Q^{\pm}$). Because $\varepsilon\mathcal{C} \subseteq \mathcal{C}$, $\mathcal{C}$ is an ideal of $\widetilde{\mathcal{C}}$, and $\widetilde{\mathcal{C}} / \mathcal{C} \cong C^*(\varepsilon) = C^*$-algebra generated by ε. So we get a short split-exact sequence

$$0 \to \mathcal{C} \to \widetilde{\mathcal{C}} \xrightarrow{\tau} C^*(\varepsilon) \to 0$$

with splitting $\rho : C^*(\varepsilon) \to \widetilde{\mathcal{C}}$ that is the inclusion. In K-theory, we get the short exact sequence

$$0 \to K_0(\mathcal{C}) \to K_0\left(\widetilde{\mathcal{C}}\right) \to K_0(C^*(\varepsilon)) \to 0.$$

We can actually do this in general. Let $\mathcal{A}$ be a C^*-algebra with unit. Let

$$\begin{aligned} M_\infty(\mathcal{A}) &= \lim_{n\to\infty} M_n(\mathcal{A}) \\ &= \text{infinite matrices over } \mathcal{A} \text{ with all but finitely many entries } 0 \\ (a) &\hookrightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \hookrightarrow \ldots \end{aligned}$$

Let $V(\mathcal{A})$ be the set of idempotents ($A^2 = A$) from $M_\infty(\mathcal{A})$ mod homotopy of idempotents (or similarity). $V(\mathcal{A})$ is an abelian monoid with sum $[P] + [Q] = \left[\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}\right]$. Let $K_0(\mathcal{A})$ be the Grothendieck completion of $V(\mathcal{A})$, the set of formal differences of elements of $V(\mathcal{A})$ with equivalence relation

$$\begin{aligned} [P] - [Q] &= [P'] - [Q'] \text{ when} \\ [P] + [Q'] + [R] &= [P'] + [Q] + [R] \end{aligned}$$

for some idempotent R.

Example 2.2. *The Groth. completion of a group is itself. The Groth. completion of* $\mathbb{N} \cup \{0\}$ *is* $\mathbb{Z}$. *The Groth. completion of* $\mathbb{N} \cup \{0\} \cup \{\infty\}$ *is* $\{0\}$.

3. Aside - Topological Quantum Computing

(from Greg – Zhenghung Wang CBMS conference in OK)

Overview of concept: Classically, a computer uses bits – sequence of zeros and ones — goes through logic gates — spits out difference sequence of zeros and ones.

Quantum computing: have a Hilbert Space, states, qubit: $\mathbb{C}^2$: basis is $\{|0\rangle, |1\rangle\}$. Other elements are $|011\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle \in \mathbb{C}^8$ etc. Think of these as logical zeros and ones. Could have a state that is $a|0\rangle + b|1\rangle$ with $a, b \in \mathbb{C}$.

Take $|0\rangle \in \mathbb{C}^2$. Start with a qubit arrangement, say $x = |0011\rangle$, then act on it by unitary operators, and the output is Ux. This is apparently more efficient for lots of kinds of computations. But these systems are prone to errors.

Want quantum computers that are not sensitive to perturbations. This is where topology comes in. Find a fractional quantum Hall liquid. Note the Hall effect – classical electromagnetic effect. When you run electric current through a conductor in the presence of a magnetic field. The Lorenz force will divert the electric field, and a voltage differential will occur on the conductor. Classically the change in the voltage difference is linear in the current and in the magnetic field. In the quantum realm (low temperature, high magnetic field), there are plateaus in the graph of emf difference versus electric field. What essentially happens is that the orbital energy levels make the electrons be trapped within levels – so you can't get voltage changes. (Quantum Hall effect). The fractional quantum Hall effect — refinement of the graph — more discrete levels – get symmetries and crystalline effects – extra blips.

New setup. Given fractional quantum Hall liquid – 2-d space. Fix quasiparticles (anyon – sort of acts like particle) in liquid. (these are states in a Hilbert space) Then you can grab these states and braid them. You look at the permutation / or braid. So each braid is a unitary operator — get representation of braid group. The Hilbert space is the Temperley-Lieb algebra (TL_n : braids from n points to n points). Note — nonlocal property of the system. — You can make logical gates out of this. This is a nonabelian process. Note inner product on braids — need trace. You close the braid "Plat closure" – gives a knot, an element of $TL_0 = \mathbb{C}$. The answer is the value of the Jones polynomial on a particular fourth root of unity.

4. Back to mod k index theory

Recap: $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is a graded Hilbert space, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the grading operator,

$$D : \mathcal{H}^+ \to \mathcal{H}^-$$

is an unbounded, self-adjoint operator; that is, it is densely defined and $\langle Df, g\rangle = \langle f, Dg\rangle$ for all $f, g \in dom\,(D)$. Also, $D\varepsilon = -\varepsilon D$, so $D = \begin{pmatrix} 0 & (D^+)^* \\ D^+ & 0 \end{pmatrix}$. We have $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ be a C^*-subalgebra, usually unital such that

- $(D \pm i)^{-1} \in \mathcal{C}$
- $\varepsilon\mathcal{C} \subset \mathcal{C}$
- $Q^\pm : \frac{1}{2}(1 \pm \varepsilon) \notin \mathcal{C}$

Let $\widetilde{\mathcal{C}}$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{C}$ and ε. Since $\varepsilon^2 = 1$, it is unital. It turns out the $\mathcal{C}$ is an ideal in $\widetilde{\mathcal{C}}$, because $\varepsilon\mathcal{C} \subset \mathcal{C}$. Then you have the (split) short exact sequence

$$0 \to \mathcal{C} \to \widetilde{\mathcal{C}} \xrightarrow{\pi} C^*(\varepsilon) \to 0,$$

where

$$C^*(\varepsilon) = \{\alpha + \beta\varepsilon : \alpha, \beta \in \mathbb{C}\}.$$

Let $\mathcal{A}$ be a C^*-algebra with unit. We can define $M_\infty(\mathcal{A}) = \varinjlim M_n(\mathcal{A})$, with $a \hookrightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $V(\mathcal{A}) = \{\text{idempotents in } M_\infty(\mathcal{A})\} / \sim$, where $\sim$ is similarity or homotopy through idempotents. The operation $+$ on $V(\mathcal{A})$ is defined by:

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right] = [q] + [p] = \left[\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}\right]$$

(the same by using a rotation matrix). The additive identity is $[0]$, and this gives the notion of an abelian monoid (associative abelian operation with identity but not nec inverses). Form the Grothendieck completion $K_0(\mathcal{A})$ of $V(\mathcal{A})$, which consists of formal differences of elements of $V(\mathcal{A})$ modulo the equivalence relation $[p] - [q] = [p'] - [q']$ when $[p] + [q'] + [r] = [p'] + [q] + [r]$ for some idempotent r. (Why not eliminate the r part? The Universal Mapping property of Grothendieck completion: Given any monoid homomorphism

$$\phi : V(\mathcal{A}) \to H,$$

where H is an abelian group, there exists a unique group homomorphism $K_0(\mathcal{A}) \to H$ such that $\phi = V(\mathcal{A}) \overset{[p]\mapsto[p]-[0]}{\longrightarrow} K_0(\mathcal{A}) \to H$. A good example is the Chern character for $\mathcal{A} = C(N) = \mathbb{C}$-valued continuous functions on N, which is defined for idempotents=vector bundles to $H^{\text{even}}(N,\mathbb{C})$, and thus it gives a Chern character on K_0.

Functor⊐ality of K_0: If $\psi : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra homomorphism between unital algebras, you get a map

$$\psi_* : K_0(\mathcal{A}) \to K_0(\mathcal{B}),$$

defined by $\psi_*[p] = [\psi(p)]$. A use of this: let $\mathcal{J}$ be a C^*-algebra **without** unit. We define

$$\mathcal{J}^+ = \{j + \lambda 1 : j \in \mathcal{J}, \lambda \in \mathbb{C}\},$$

so

$$\begin{aligned} \mathcal{J}^+/\mathcal{J} &\cong \mathbb{C} \\ q &: \mathcal{J}^+ \to \mathbb{C}, q(j+\lambda 1) = \lambda \end{aligned}$$

Then

$$q_* : K_0(\mathcal{J}^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}$$

We define

$$K_0(\mathcal{J}) := \ker q_* .$$

Recall that we had

$$0 \to \mathcal{C} \to \widetilde{\mathcal{C}} \overset{\pi}{\longrightarrow} C^*(\varepsilon) \to 0,$$

even if there is no unit, then we know that we have a map

$$K_0(\mathcal{C}) \to K_0\left(\widetilde{\mathcal{C}}\right) \overset{\pi}{\longrightarrow} K_0(C^*(\varepsilon)),$$

but we don't get the ends. But since we have a splitting, we do get

$$0 \to K_0(\mathcal{C}) \to K_0\left(\widetilde{\mathcal{C}}\right) \overset{\pi}{\longrightarrow} K_0(C^*(\varepsilon)) \to 0.$$

5. K-theory element associated to an unbounded operator

Choose a smooth function $f : \mathbb{R} \to [-1,1]$ such that f is odd, $\lim_{x\to\infty} f(x) = 1$. Let $g(x) = \sqrt{1 - f(x)^2}$. Apply f and g to D via the spectral theorem (can use since D commutes with its adjoint D):

$$f(D) = \int f(\lambda)\ P_\lambda\ d\mu(\lambda),$$

similarly for $g(D)$. Note that $f(D)$ has an odd grading, and $g(D)$ has an even grading and thus commutes with ε. Set

$$U = \varepsilon f(D) + g(D).$$

Proposition 5.1. *U is unitary.*

Proof. $U^* = f(D)\varepsilon + g(D)$, so

$$\begin{aligned} U^*U &= (f(D)\varepsilon + g(D))(\varepsilon f(D) + g(D)) \\ &= f(D)^2 + g(D)\varepsilon f(D) + f(D)\varepsilon g(D) + g(D)^2 \\ &= f(D)^2 + \varepsilon f(D) g(D) - \varepsilon f(D) g(D) + g(D)^2 \\ &= f(D)^2 + g(D)^2 = \mathbf{1}. \end{aligned}$$

Similarly, $UU^* = \mathbf{1}$. ☐

Recall

$$\begin{aligned} Q^{\pm} &= \frac{1}{2}(1 \pm \varepsilon), \\ \left(Q^{\pm}\right)^2 &= Q^{\pm} \end{aligned}$$

We define the idempotent

$$\begin{aligned} P &= UQ^{+}U^{*} = g(D)^2 \varepsilon - f(D)\, g(D) + Q^{-}, \text{ so} \\ P^2 &= P \end{aligned}$$

Recall we have

$$\begin{aligned} \pi &: \widetilde{\mathcal{C}} \longrightarrow C^{*}(\varepsilon), \\ \pi(c) &= 0 \text{ if } c \in \mathcal{C}, \\ \pi(\varepsilon) &= \varepsilon. \end{aligned}$$

The fact that $(D \pm i)^{-1} \in \mathcal{C}$ and the Stone-Weierstrauss theorem imply that $h(D)$ is in $\mathcal{C}$ for any smooth $h : \mathbb{R} \to [-1, 1]$. with $\lim\limits_{x \to \pm\infty} h(x) = 0$. Thus, since g satisfies this condition,

$$\pi(P) = \pi\left(Q^{-}\right) = Q^{-},$$

since $\varepsilon \mapsto \varepsilon$, $1 \mapsto 1$. We have

$$[P] - \left[Q^{-}\right] \in K_0\left(\widetilde{\mathcal{C}}\right),$$

and therefore

$$\pi_{*}\left([P] - \left[Q^{-}\right]\right) = 0.$$

Whence (from where)

$$[P] - \left[Q^{-}\right] \in \ker \pi_{*} = K_0(\mathcal{C}).$$

This element is defined to be the index class of D:

$$\mathrm{Index}_{\mathcal{C}}(D) := [P] - \left[Q^{-}\right] \in K_0(\mathcal{C}).$$

Why does this not depend on the choice of f ?

Proposition 5.2. $\mathrm{Index}_{\mathcal{C}}(D)$ *does not depend on the choice of f.*

Proof. Given f_0, f_1, do the straight line homotopy, and we use homotopy invariance of K_0. ☐

6. Digression: Partial Isometries

Polar form of complex numbers: Let $z = re^{i\theta}, r \geq 0, e^{i\theta}\overline{e^{i\theta}} = 1$. For operators on Hilbert space, $T = PV$, where $P = \sqrt{T^{*}T}$. If T is invertible, then $V = P^{-1}T$ is unitary. In general, you can define V as a *partial isometry.*

An operator $V \in \mathcal{B}(\mathcal{H})$ is called a **partial isometry (p.i.)** if $\|Vf\| = \|f\|$ for all $f \in (\ker V)^{\perp}$. Algebraically, V is a p.i. if and only if VV^{*} and $V^{*}V$ are projections (self-adjoint idempotents). Note that in infinite dimensions, isometries are not necessarily unitary. For example, let $\mathcal{H} = l^2(\mathbb{N} \cup \{0\})$. Then the delta functions $\{\delta_n\}$ form an orthonormal basis. Let $S : \mathcal{H} \to \mathcal{H}$ be defined by $S(\delta_n) = \delta_{n+1}$. This is injective but not surjective.

7. Motivating Example

Let M be a closed Riemannian manifold, and let $V = V^+ \oplus V^-$ be a smooth Hermitian vector bundle over M. Let $\mathcal{H}^\pm$ be the space of L^2-sections of $V^\pm$. Let D be a first order, symmetric elliptic operator. Let $\mathcal{K}$ be the ideal of compact operators, the norm closure of finite rank operators on $\mathcal{H}$. Then ellipticity implies

$$(D \pm i)^{-1} \in \mathcal{K}.$$

Also,

$$\varepsilon\mathcal{K} = \mathcal{K},$$

since $\mathcal{K}$ is an ideal. Also,

$$Q^\pm = \frac{1}{2}(1 \pm \varepsilon) \notin \mathcal{K}.$$

We can choose an orthonormal basis $\{\phi_n\}$ of eigensections of D.

$$D\phi_n = \lambda_n \phi_n$$

Assume these are ordered so that

$$... \leq \lambda_{-2} \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq ...$$

Define f to be a cutoff function that is $+1$ for $x > x_0 > 0$ and -1 for $x < -x_0 < 0$ with x_0 less than any positive eigenvalue and $-x_0$ greater than any negative eigenvalue. Then

$$f(D) = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}$$

with V a partial isometry, and

$$\ker W = \ker D^+, \ \ker W^* = \ker \left(D^+\right)^*.$$

Moreover,

$$g(D) = \begin{pmatrix} P_0 & 0 \\ 0 & P_1 \end{pmatrix},$$

where P_0 is the projection onto $\ker(D^+)$, P_1 is the projection onto $\ker(D^+)^*$. Then

$$P = UQ^+U^* = \begin{pmatrix} P_0 & 0 \\ 0 & 1 - P_1 \end{pmatrix},$$

so

$$\begin{aligned}
[P] - [Q^-] &= \left[\begin{pmatrix} P_0 & 0 \\ 0 & 1-P_1 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & 0 \\ 0 & 1-P_1 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} P_0 & 0 \\ 0 & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & P_1 \end{pmatrix}\right] = [P_0] - [P_1].
\end{aligned}$$

Next, $K_0(\mathcal{K}) \cong \mathbb{Z}$, and

$$[\rho] - [\rho'] \leftrightarrow Tr(\rho) - Tr(\rho'),$$

so

$$[P_0] - [P_1] \leftrightarrow Tr(P_0) - Tr(P_1) = \dim\ker\left(D^+\right) - \dim\ker\left(D^+\right)^*.$$

8. The analytical index

Definition 8.1. *A* Z_k***-structure on a Hilbert space*** H *is a separable infinite dimensional Hilbert space* $\mathcal{L}$ *together with* k *isometries* $e_i : \mathcal{L} \to \mathcal{H}$ *whose ranges are pairwise orthogonal.*

Define partial isometries

$$e_{ij} := e_i e_j^*, \ i \neq j$$

and projections

$$p_i = e_i e_i^*, \ i = 1, 2, ..., k$$

Also define

$$p_0 = 1 - p_1 - p_2 - ... - p_k$$

The subalgebra $\mathcal{D}_k(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ is defined as

$$\mathcal{D}_k(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : [A, e_{ij}] \in \mathcal{K}(\mathcal{H}) \text{ for all } i \neq j \text{ and } Ap_0 \in \mathcal{K}(\mathcal{H})\}.$$

Proposition 8.2. $\mathcal{K}(\mathcal{H}) \lhd \mathcal{D}_k(\mathcal{H})$ *(is an ideal of)*

Define

$$\alpha : \mathcal{B}(\mathcal{L}) \to \mathcal{D}_k(\mathcal{H})$$

by

$$\alpha(S) = \sum_{i=1}^{k} e_i S e_i^*$$

Check: since $e_l^* e_i = 0$ unless $l = i$ and is the identity if $l = i$

$$\begin{aligned}
[\alpha(S), e_{ij}] &= \alpha(S) e_{ij} - e_{ij} \alpha(S) \\
&= \sum_l e_l S e_l^* e_i e_j^* - e_i e_j^* e_l S e_l^* \\
&= e_i S e_i^* e_i e_j^* - e_i e_j^* e_j S e_j^* \\
&= e_i \left(S e_i^* e_i - e_j^* e_j S\right) e_j^* \\
&= e_i (S - S) e_j^* = 0
\end{aligned}$$

Also

$$\begin{aligned}
\alpha(S) p_0 &= \sum_l e_l S e_l^* \left(1 - \sum_j e_j e_j^*\right) \\
&= \sum_l e_l S e_l^* - \sum_l e_l S e_l^* e_l e_l^* \\
&= \sum_l e_l S e_l^* - \sum_l e_l S e_l^* = 0.
\end{aligned}$$

Proposition 8.3. *The quotient map*

$$\widetilde{\alpha} : \mathcal{B}(\mathcal{L}) \diagup \mathcal{K}(\mathcal{L}) \to \mathcal{D}_k(\mathcal{H}) \diagup \mathcal{K}(\mathcal{H})$$

is an isomorphism.

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K}(\mathcal{H}) & \to & \mathcal{D}_k(\mathcal{H}) & \to & \mathcal{D}_k(\mathcal{H})\diagup\mathcal{K}(\mathcal{H}) & \to & 0 \\ & & \alpha\uparrow & & \alpha\uparrow & & \widetilde{\alpha}\uparrow & & \\ 0 & \to & \mathcal{K}(\mathcal{L}) & \to & \mathcal{B}(\mathcal{L}) & \to & \mathcal{B}(\mathcal{L})\diagup\mathcal{K}(\mathcal{L}) & \to & 0 \end{array}$$

We apply the K-theory exact sequence to get (assuming that $\mathcal{L}$ is infinite dimensional)

$$\begin{array}{ccccccc} K_1(\mathcal{D}_k(\mathcal{H})\diagup\mathcal{K}(\mathcal{H})) & \longrightarrow & K_0(\mathcal{K}(\mathcal{H}))=\mathbb{Z} & \longrightarrow & K_0(\mathcal{D}_k(\mathcal{H})) & \longrightarrow & K_0(\mathcal{D}_k(\mathcal{H})\diagup\mathcal{K}(\mathcal{H}))=0 \\ \widetilde{\alpha}_*\uparrow & & \alpha_*\uparrow & & \alpha_*\uparrow & & \alpha_*\uparrow \\ K_1(\mathcal{B}(\mathcal{L})\diagup\mathcal{K}(\mathcal{L})) & \longrightarrow & K_0(\mathcal{K}(\mathcal{L}))=\mathbb{Z} & \longrightarrow & K_0(\mathcal{B}(\mathcal{L})) & \longrightarrow & K_0(\mathcal{B}(\mathcal{L})\diagup\mathcal{K}(\mathcal{L}))=0 \end{array}$$

As a consequence, the image of α_* is $k\mathcal{K}(\mathcal{H})\cong k\mathbb{Z}$. This implies that $K_0(\mathcal{D}_k(\mathcal{H}))\cong\mathbb{Z}\diagup k\mathbb{Z}=\mathbb{Z}_k$. This is where the analytic and topological indices live.

9. The analytic index

Let M be an oriented, even-dimensional, complete Riemannian manifold without boundary that is not necessarily compact. Let S be a smooth (Hermitian or orthogonal) vector bundle equipped with a smooth action of the Clifford bundle $Cl(TM)$ and compatible connections. We'll call S a *Dirac bundle*. For any vector bundle $E\to M$, the bundle $S\otimes E$ becomes a Dirac bundle in a natural way. For any Dirac bundle S, there is an associated Dirac operator, which is locally

$$D_S=\sum_i f_i\cdot\nabla_{f_i}:C^\infty(S)\to C^\infty(S)$$

and is formally self-adjoint. This can be closed up to a self-adjoint operator, but the closure depends on the metric.

Let Q be a compact, oriented, even-dimensional manifold whose boundary consists of k diffeomorphic pieces $(\partial Q)_1,(\partial Q)_2,...,(\partial Q)_k$. Recall that Q admits a $\mathbb{Z}_k$-structure if there exists an oriented Riemannian manifold P and orientation-preserving diffeomorphisms

$$\gamma_i:V_i\to[0,1]\times P, 1\le i\le k,$$

where V_i is a collared neighborhood of $(\partial Q)_i$. The next step is to attach cylindrical ends to the V_i. This becomes M, a manifold with cylindrical ends. A Riemannian metric on Q is a choice of Riemannian metric on Q and the ends so that the γ_i's are isometries. A $\mathbb{Z}_k$ bundle over Q is a smooth vector bundle E over Q, a smooth vector bundle F over P and liftings of the γ_i's to "isomorphisms" $E|_{V_i}\cong\pi^*F$, where $\pi:[0,1]\times P\to P$ is given by $\pi(t,p)=p$. A spin^c structure on an oriented even-dimensional $\mathbb{Z}_k$ vector bundle $\mathcal{V}$ over Q is a Hermitian $\mathbb{Z}_k$ bundle S equipped with a Clifford action on $\mathcal{V}$. On the V_i's, the Clifford action is pulled back from $[0,1]\times P$. A spin^c structure on Q is a spin^c structure on TQ.

Now form a complete Riemannian manifold M by attaching cylinders $[1,\infty)\times P$ to each of $(\partial Q)_i$, $1\le i\le k$. If E is a Hermitian $\mathbb{Z}_k$-bundle over Q, extend in the obvious ways. Also, take the Dirac bundle S over Q and extend to M. Let D_E denote the Dirac operator on $L^2(S\otimes E)$.

The Hilbert space $L^2(S\otimes E)$ admits the $\mathbb{Z}_k$-structure with

$$\mathcal{L}=L^2\left(S\otimes E|_{[1,\infty)\times P}\right).$$

The isometries $e_1,...,e_k$ come from the inclusion of $[1,\infty)\times P$ into M (each end).

Proposition 9.1. $(D_E\pm 1)^{-1}\in\mathcal{D}_k(L^2(S\otimes E))=\{Y\in\mathcal{B}(L^2(S\otimes E)):[Y,e_{ij}]\in\mathcal{K},Yp_0\in\mathcal{K}\}$.

(the proof is related to unit propagation speed.)

From our earlier work, this D_E determines a class

$$\mathrm{Index}_k (D_E) \in K_0 \left(\mathcal{D}_k \left(L^2 (S \otimes E)\right)\right) \cong \mathbb{Z}_k$$

This is called the **analytical index**.

Theorem 9.2. *(N. Higson, D. Freed, R. Melrose)*

$$\mathrm{Index}_k (D_E)$$

coincides with the index problem with APS boundary conditions on Q (mod k).

10. Topological Index

Let $H = \left\{x \in \mathbb{R}^{2d} : x_1 < 0\right\}$. Construct a $\mathbb{Z}_k$-manifold X by attaching k disjoint open disks of radius 1 to H. Let

$$\widehat{X} = X \big/ _{\text{disks identified}}$$

A proof similar to the one that showed $K_0 (\mathcal{C}) \cong \mathbb{Z}_k$ shows that the inclusion of H in $\widehat{X}$ induces an isomorphism

$$K^0 \left(\widehat{X}\right) \cong K^0 (H) \big/ k K^0 (H) \cong \mathbb{Z}_k \ .$$

Note that $K^0 (H) \cong \mathbb{Z}$ comes from Bott periodicity.

Alternate description of $K^0 \left(\widehat{X}\right)$: Look at triples (U, S, G), where U is an open $\mathbb{Z}_k$ submanifold of X, S is a $\mathbb{Z}_2$-graded $\mathbb{Z}_k$ bundle over U, and G is a self-adjoint bundle endomorphism of S such that G^2 is positive off a compact subset of U. Let $\widehat{U}$ be the locally compact space obtained by identifying the k boundary pieces of U. This produces a vector bundle $\widehat{S}$ over $\widehat{U}$ and endomorphism $\widehat{G}$ which is invertible off of a compact subset of $\widehat{U}$.

The triple $\left(\widehat{U}, \widehat{S}, \widehat{G}\right)$ determines an element of $K^0 \left(\widehat{U}\right)$ and the inclusion $\widehat{U} \hookrightarrow \widehat{X}$ determines a homomorphism $K^0 \left(\widehat{U}\right) \to K^0 \left(\widehat{X}\right)$.

Imbed Q into $\widehat{X}$ (for some d) as a $\mathbb{Z}_k$ submanifold (ends inside ends of $\widehat{X}$). The normal bundle νQ is a $\mathbb{Z}_k$-bundle over Q, and the spin^c structures on Q and on $\widehat{X}$ determines a spin^c structure on the normal bundle. Let N be the total space of νQ; let S^* be the complex conjugate of the Dirac bundle S over νQ. Then $\pi^* (S^*)$ is the pullback to N. Let J be a self-adjoint endomorphism of $\pi^* (S^*)$ determined by the formula

$$J (v) = \varepsilon v,$$

where v acts by Clifford multiplication and ε is the grading operator. Let $\mathcal{N}$ be a tubular neighborhood of Q in N, imbedded into X. Then,

$$(\mathcal{N}, \pi^* S^*, J^*) \in K^0 \left(\widehat{X}\right) .$$

Definition: Let E be a $\mathbb{Z}_k$-bundle over Q, and set $\tau_k (E) := (\mathcal{N}, \pi^* (S^* \otimes E), J^*) \in K^0 \left(\widehat{X}\right)$.

Theorem 10.1. $\tau_k (E) = \mathrm{Index}_k (D_E)$.

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