UNITARY EQUIVALENCE OF NORMAL MATRICES OVER COMPACT HAUSDORFF SPACES

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Let $M(n,\mathbb{C})$ be the set of $n\times n$ complex matrices. Let $A\in M(n,\mathbb{C})$, so that

$$A=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & \dots & \dots\\ a_{n1} & \dots & \dots & a_{nn}\end{pmatrix}.$$

The adjoint is

$$A^*=\begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \dots & \overline{a_{n1}}\\ \overline{a_{12}} & \overline{a_{22}} & \dots & \overline{a_{n2}}\\ \dots & \dots & \dots & \dots\\ \overline{a_{1n}} & \dots & \dots & \overline{a_{nn}}\end{pmatrix}.$$

The matrix A is called **normal** if $AA^*=A^*A$. Let the set of unitary matrices

$$U(n,\mathbb{C})=\{U\in M(n,\mathbb{C}):U^*U=UU^*=I\}.$$

Theorem 1. *(Spectral Theorem) Every normal matrix A in $M(n,\mathbb{C})$ is diagonalizable — i.e. there exists $U\in U(n,\mathbb{C})$ such that U^*AU is diagonal.*

Next, suppose X is a compact Hausdorff space. Let $C(X)$ be the set of complex-valued functions on X. Does the spectral theorem hold for $M(n,C(X))$? Answer: No!

Example 2. *Let $A\in M(2,C([-1,1]))$. Let*

$$A(x)=\begin{cases}\begin{pmatrix} x & x\\ x & x\end{pmatrix} & \text{if } x\geq 0\\ \begin{pmatrix} x & 0\\ 0 & 0\end{pmatrix} & \text{if } x<0\end{cases}$$

If U^*AU were diagonal for some $U\in U(n,\mathbb{C})$, then we would have

$$\begin{aligned} U(x) &= \begin{pmatrix} f(x) & g(x)\\ -f(x) & g(x)\end{pmatrix} \text{ or } \begin{pmatrix} g(x) & f(x)\\ g(x) & -f(x)\end{pmatrix} \text{ for } x\geq 0\\ U(x) &= \begin{pmatrix} 0 & h(x)\\ k(x) & 0\end{pmatrix} \text{ or } \begin{pmatrix} h(x) & 0\\ 0 & k(x)\end{pmatrix} \text{ for } x<0\end{aligned}$$

with $|f(x)|=|g(x)|=\frac{1}{\sqrt{2}}$, $|u(x)|=|k(x)|=1$. So this is impossible.

We will say that A is **multiplicity-free** if $A(x)$ has distinct eigenvalues for each $x\in X$. Equivalently, $p(\lambda,x)=\det(\lambda I-A(x))\in C(X)[\lambda]$, and $p(\lambda,x)$ has distinct roots for each $x\in X$.

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If $A \in M(n, C(X))$ is normal and multiplicity-free. Suppose U^*AU is diagonal. Let $d_j(x)$ =eigenvalue of $A(x)$ associated to the i^{th} column of $U(x)$. Then $d_i : X \to \mathbb{C}$ is continuous for $1 \leq i \leq n$. Then

$$p(\lambda, x) = \det(\lambda I - A(x)) = \prod_{i=1}^{n} (\lambda - d_i(x)).$$

Example 3. *Let* $A \in M(2, C(S^1))$. *Let* $A(z) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$, *so that* $p(\lambda, z) = \lambda^2 - z$. *This does not globally factor.*

Example 4. *Let* $A \in M(2, C(S^2))$. *Define*

$$A(x, y, z) = \frac{1}{2} \begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix}.$$

Then

$$p(\lambda, (x, y, z)) = \lambda^2 - \lambda = \lambda(\lambda - 1)$$

(Constant on the sphere.) But this matrix is NOT diagonalizable! Look at range $(A(x, y, z))$, *a* 1*-dimensional subspace of* $\mathbb{C}^2$. *This determines a* 1*-dimensional vector bundle which is not trivial. (But diagonalizability requires triviality of this.)*

Theorem 5. *(Grove-Pedersen) Suppose that* X *is* 2*-connected. [In other words,* X *is path-connected, every continuous function from* $S^1 \to X$ *can be deformed to a constant function, and every continous function from* $S^2 \to X$ *can be deformed to a constant function.] If* $A \in M(n, C(X))$ *is normal and multiplicity-free, then* A *can be diagonalized.*

We will recast this question. Let $A, B \in M(n, C(X))$ be unitarily equivalent. Then $B = U^*AU$, then

$$\begin{aligned} p_B(\lambda, x) &= \det(\lambda I - B(x)) = \det(\lambda I - U^*(x) A(x) U(x)) \\ &= \det(U^*(x)(\lambda I - A(x)) U(x)) = \det(\lambda I - A(x)). \end{aligned}$$

So similar matrices must have the same characteristic polynomial. Is this condition sufficient.

Theorem 6. *(Park) Let* $A, B \in M(n, C(S^1))$ *be normal and multiplicity-free. Then* A *and* B *are unitarily equivalent iff* A *and* B *have the same characteristic polynomial.*

Proof. Let $q : [0, 1] \to S^1$ by $q(t) = \exp(2\pi i t)$. Given $f \in C(S^1)$, we get $q^* f \in C([0, 1])$ by $q^* f(t) = f(q(t)) = f(\exp(2\pi i t))$. Then $A \in M(n, C(S^1))$, so $q^* A \in M(n, C([0, 1]))$. Since $[0, 1]$ is 2-connected, by Grove-Pedersen, $q^* A$ and $q^* B$ are diagonalizable. That is, there exist $U, V \in U(n, C([0, 1]))$. Then

$$U^* q^* A U = DIAG(\alpha_1, ..., \alpha_n) = V^* q^* B V.$$

Then $q^* B = (UV^*)^* q^* A (UV^*)$. Then "wiggle" U and/or V so that $UV^* = q^* W$ for some $W \in U(n, C(S^1))$. Then $q^* B = (q^* W)^* q^* A (q^* W) = q^* (W^* AW)$. Then $B = W^* AW$ because q^* is injective. $\square$