NONCOMMUTATIVE GEOMETRY AND TOPOLOGY

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1. Cyclic homology

Let A be a $\mathbb{C}$-algebra with unit, and for each nonnegative integer n, let $C_n^\lambda(A)$ be the A-module $A \otimes A \otimes \cdots \otimes A$ ($n+1$ factors) modulo the relation

$$a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} = (-1)^n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n.$$

The A-linear map $b : \otimes^{n+1} A \longrightarrow \otimes^n A$ determined by

$$\begin{aligned} b(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n \\ &\qquad + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

passes to a map $b : C_n^\lambda(A) \longrightarrow C_{n-1}^\lambda(A)$ with the property that $b^2 = 0$. The *cyclic homology* of A is the homology $H_*^\lambda(A)$ of the complex $(C_*^\lambda(A), b)$.

We sometimes write $a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n$ as a *noncommutative differential form* $a_0\, da_1\, da_2 \ldots da_n$. When $A = C^\infty(M)$ for some smooth compact manifold M, this determines isomorphisms

$$H_{2n}^\lambda(C^\infty(M)) \cong H_{dR}^{even}(M), \quad H_{2n+1}^\lambda(C^\infty(M)) \cong H_{dR}^{odd}(M)$$

for n sufficiently large.

Question: Where do interesting elements of $HC_*^\lambda(A)$ come from?

Answer: From the K-theory of A.

Let e be an idempotent in $\mathrm{M}(m, A)$. Then trace $\left(e\,(de)^{2n}\right)$ determines an element of $H_{2n}^\lambda(A)$ for each natural number n.
Let s be an element in $\mathrm{GL}(m, A)$. Then trace $\left((s^{-1}\, ds)^{2n+1}\right)$ determines an element of $H_{2n+1}^\lambda(A)$ for each natural number n.

2. Cyclic cohomology

Let A be a topological $\mathbb{C}$-algebra with unit, and for each natural number n, let $C^n(A)$ denote the A-module of continuous multilinear maps $\tau : A^{n+1} \longrightarrow \mathbb{C}$, and let $C_\lambda^n(A)$ be the A-submodule of elements τ in $C^n(A)$ that satisfy

$$\tau(a_n, a_0, a_1, \ldots, a_{n-1}) = (-1)^n \tau(a_0, a_1, \ldots, a_n)$$

Date: April 1, 2010.

for all $a_0, a_1, \dots a_n$ in A. The map $b : C^n(A) \longrightarrow C^{n+1}(A)$ by the formula

$$(b\tau)(a_0, a_1, \dots, a_{n+1}) = \tau(a_0a_1, a_2, \dots, a_{n+1}) \\ + \sum_{i=1}^{n}(-1)^i\tau(a_0, \dots, a_{i-1}, a_ia_{i+1}, a_{i+2}, \dots, a_{n+1}) \\ + (-1)^{n+1}\tau(a_{n+1}a_0, a_1, \dots, a_n)$$

has the property that $b^2 = 0$, and b restricts to a map from $C^n_\lambda(A)$ to $C^{n-1}_\lambda(A)$. The cohomology $H^*_\lambda(A)$ of the complex $(C^*_\lambda(A), b)$ is called the *cyclic cohomology* of A.

Question: Where do interesting elements of $HC^*_\lambda(A)$ come from?

Answer: From *Fredholm modules* over A.

3. Fredholm modules

Let $\mathcal{H}$ be a Hilbert space. A (bounded) *operator* on $\mathcal{H}$ is a continuous linear map from $\mathcal{H}$ to $\mathcal{H}$. The algebra of all operators on $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$. An element K in $\mathcal{B}(\mathcal{H})$ is *compact* if it is the norm limit of finite-rank operators on $\mathcal{H}$.

Theorem 3.1. *The spectrum of a compact operator on a Hilbert space consists entirely of eigenvalues whose only accumulation point is zero. Furthermore, each nonzero eigenvalue has finite multiplicity.*

Let K be a compact operator on a Hilbert space, and let $\lambda_1, \lambda_2, \dots,$ be the eigenvalues of K^*K listed in nondecreasing order (counting multiplicities). We say K is in the *Schatten p-class* $L^p(\mathcal{H})$ if

$$\sum_{n=1}^{\infty} \lambda_n^{p/2} < \infty.$$

Elements of $L^1(\mathcal{H})$ are called *trace class operators.*

Definition 3.2. *Let A be a unital $\mathbb{C}$-algebra. A* Fredholm module *over A is a triple $(\mathcal{H}, \pi, F)$, where*

- *$\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space with grading operator ε;*
- *$\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$ is a representation of A on $\mathcal{H}$ that respects the grading;*
- *$F \in \mathcal{B}(\mathcal{H})$ anticommutes with the grading, $F^2 - I$, and $F\pi(a) - \pi(a)F$ is compact for every a in A.*

If there exists a positive number p such that $F\pi(a) - \pi(a)F$ is in $L^p(\mathcal{H})$ for all a in A, we say that $(\mathcal{H}, \pi, F)$ is p-summable*; if this condition only holds for some dense subalgebra $\mathcal{A}$ of A, we say that $(\mathcal{H}, \pi, F)$ is* essentially *p-summable.*

Prototypical example: $A = C(M)$ for some smooth compact manifold M, $\mathcal{H} = L^2(M, E)$ for some $\mathbb{Z}_2$-graded Hermitian vector bundle over M, A acts on $\mathcal{H}$ by pointwise multiplication, D is an operator of Dirac type acting on $\mathcal{H}$, and $F = D(1 + D^2)^{-1/2}$. This Fredholm module is essentially p-summable for $p > \dim M$.

The *character* of an essentially 1-summable Fredholm module $(\mathcal{H}, \pi, F)$ over a not necessarily commutative topological $\mathbb{C}$-algebra A is a linear function

$$\rho(a) = \frac{1}{2}\,\mathrm{trace}\,(\varepsilon F[F, \pi(a)])\,.$$

The linear map ρ determines an element of $H^1_\lambda(A)$. More generally, an n-summable Fredholm module over A determines an element of $H^n_\lambda(A)$.

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fred}(A) \times K_*(A) \arrow[r, "\mathrm{index}"] \arrow[d] & \mathbb{Z} \arrow[d] \\
H^*_\lambda(A) \times H^\lambda_*(A) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

4. An interesting application

Let Γ be a discrete group and let $\mathbb{C}\Gamma$ be its complex group algebra. We take the left regular representation of $\mathbb{C}\Gamma$ on the Hilbert space $\ell^2(\Gamma)$, and as we discussed in the last lecture, the closure of $\mathbb{C}\Gamma$ in $\mathcal{B}\big(\ell^2(\Gamma)\big)$ is the *reduced C^*-algebra $C^*_r(\Gamma)$* of Γ.

Bass Idempotent Conjecture: Let Γ be a torsion free discrete group. Then $\mathbb{C}\Gamma$ contains no nontrivial idempotents; i.e., the only idempotents in $\mathbb{C}\Gamma$ are 0 and 1.

Kadison's Conjecture: Let Γ be a torsion free discrete group. Then $C^*_r(\Gamma)$ contains no nontrivial idempotents.

Note the Kadison's Conjecture implies the Bass Idempotent Conjecture.

We are going to prove Kadison's Conjecture for $\Gamma = F_2$, the free group on two generators.

Definition: Let A be a C^*-algebra and suppose A admits a trace function $\tau : A \longrightarrow \mathbb{C}$ such that

- τ is *positive*; i.e., $\tau(a^*a) \geq 0$ for all a in A;
- τ is *faithful*; i.e., $\tau(a^*a) = 0$ if and only if $a = 0$.

Proposition: Let A be a C^*-algebra that admits a positive faithful trace τ such that $\tau(1) = 1$. Let $(\mathcal{H}, \pi, F)$ be a Fredholm module over A, and suppose that the subalgebra $\mathcal{A} = \{a \in A : F\pi(a) - \pi(a)F \in \mathcal{L}^1(\mathcal{H})\}$ is dense in A (Note that this means that $(\mathcal{H}, \pi, F)$ is also a Fredholm module over $\mathcal{A}$). Further suppose that the character ρ of $(\mathcal{H}, \pi, F)$ agrees with τ on $\mathcal{A}$. Then A contains no nontrivial idempotent.

Proof: The inclusion map $\mathcal{A} \longrightarrow A$ determines an isomorphism in K-theory, so we may assume that any idempotent e in A is actually in $\mathcal{A}$, and also that it is self adjoint; in other words, e is a projection. By our commutative diagram, $\rho(e) = \tau(e)$ is an integer. Next, because $e = e^2 = e^*e$, we know that $0 \leq \tau(e)$. On the other hand, $1-e$ is also a projection, so $0 \leq \tau(1-e) = \tau(1)-\tau(e) = 1-\tau(e)$, or $\tau(e) \leq 1$. Therefore $\tau(e)$ equals 0 or 1. If $\tau(e) = 0$, then $e = 0$ by the faithfulness of the trace. If $\tau(e) = 1$, then $\tau(1-e) = 0$, whence $e = 1$ by the faithfulness of the trace.

There is a positive faithful trace $\tilde{\tau}$ on $\mathbb{C}F_2$

$$\tilde{\tau}\Big(\sum_{\gamma \in F_2} c_\gamma \gamma\Big) = c_1$$

that extends to a positive faithful trace τ on $C_r^*(F_2)$.

Let T be a tree upon which F_2 freely and transitively. Let T_0 and T_1 denote the vertices and edges of T respectively. Fix a vertex v_0 in T and define a map $\phi : T_0 \backslash \{v_0\} \longrightarrow T_1$ by requiring that for each vertex $v \neq v_0$, the edge $\phi(v)$ is the one that has v as one of its endpoints and is part of the unique arc between v and v_0. The map ϕ is "almost equivariant"; that is, for each γ in F_2, we have $\phi(\gamma v) = \gamma\phi(v)$ for all but finitely many vertices v. Set $\mathcal{H}^+ = \ell^2(T_0)$ and $\mathcal{H}^- = \ell^2(T_1) \oplus \mathbb{C}$. The action of F_2 on T_0 determines a representation π^+ of $C_r^*(\Gamma)$ on $\mathcal{H}^+ = \ell^2(T_0)$. In addition, the action of F_2 on T_1 determines a representation $\tilde{\pi}$ of $C_r^*(\Gamma)$ on $\ell^2(T_1)$ which in turn gives us a representation π^- of $C_r^*(\Gamma)$ on $\mathcal{H}^-$ via the formula $\pi^-(a)(\xi, \alpha) = (\tilde{\pi}(a)\xi, 0)$.

Define $P : \mathcal{H}^+ \longrightarrow \mathcal{H}^-$ by the formula

$$P\delta_v = \begin{cases} (0,1) & v = v_0 \\ (\delta_{\phi(v)}, 0) & v \neq v_0. \end{cases}$$

Proposition: Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, let $\pi = \pi^+ \oplus \pi^-$, and let

$$F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}.$$

Then $(\mathcal{H}, \pi, F)$ is an essentially 1-summable Fredholm module.

Proof: For every γ in F_2, the commutator $P\pi(\gamma) - \pi(\gamma)P$ has finite rank, because ϕ is almost equivariant. Therefore $\pi(\mathbb{C}F_2)$ is contained in the subalgebra

$$\mathcal{A} = \{a \in C_r^*(\Gamma) : \pi(a)F - F\pi(a) \in \mathcal{L}^1(\mathcal{H})\}$$

of $C_r^*(F_2)$, and because $\pi(\mathbb{C}F_2)$ is dense in $C_r^*(F_2)$, the algebra $\mathcal{A}$ is dense in $C_r^*(F_2)$ as well.

When we compute the character of $(\mathcal{H}, \pi, F)$, we find that

$$\rho(a) = \frac{1}{2}\,\mathrm{trace}\,(\varepsilon F[F, \pi(a)]) = \mathrm{trace}(\pi(a) - P^{-1}\pi(a)P) = \tau(a)$$

for every a in $\mathcal{A}$. Therefore, we have

Theorem: The group C^*-algebra $C_r^*(F_2)$ contains no nontrivial idempotent.