ARE DIRAC OPERATORS GOOD FOR NOTHING?

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1. INTRODUCTION

In this survey, we figure out what the %$#@&* a Dirac operator is good for.

2. THE ATIYAH-SINGER INDEX THEOREM

In this section I give a quick survey of index theory results. You can skip this section if you want. Given Banach spaces S and T, a bounded linear operator $L : S \to T$ is called *Fredholm* if its range is closed and its kernel and cokernel $T / L(S)$ are finite dimensional. The index of such an operator is defined to be

$$\operatorname{ind}(L) = \dim \ker(L) - \dim \operatorname{coker}(L),$$

and this index is constant on continuous families of such L. Suppose that D is an elliptic operator of order m on sections of a vector bundle $E^{\pm}$ over a smooth, compact manifold M. Let $H^s(\Gamma(M, E^{\pm}))$ denote the Sobolev s-norm completion of the space of sections $\Gamma(M, E)$, with respect to a chosen metric. Then D can be extended to be a bounded linear operator $\overline{D_s} : H^s(\Gamma(M, E^+)) \to H^{s-m}(\Gamma(M, E^-))$ that is Fredholm, and $\operatorname{ind}(D) := \operatorname{ind}(\overline{D_s})$ is well-defined and independent of s. In the 1960s, the researchers M. F. Atiyah and I. Singer proved that the index of an elliptic operator on sections of a vector bundle over a smooth manifold satisfies the following formula ([1],[2]):

$$
\begin{aligned}
\operatorname{ind}(D) &= \int_M \operatorname{ch}(\sigma(D)) \wedge \operatorname{Todd}(T_{\mathbb{C}}M) \\
&= \int_M \alpha(x) \ \operatorname{dvol}(x),
\end{aligned}
$$

where $\operatorname{ch}(\sigma(D))$ is a form representing the Chern character of the principal symbol $\sigma(D)$, and $\operatorname{Todd}(T_{\mathbb{C}}M)$ is a form representing the Todd class of the complexified tangent bundle $T_{\mathbb{C}}M$; these forms are characteristic forms derived from the theory of characteristic classes and depend on geometric and topological data. The local expression for the relevant term of the integrand, which is a multiple of the volume form $\operatorname{dvol}(x)$, can be written in terms of curvature and the principal symbol and is denoted $\alpha(x) \ \operatorname{dvol}(x)$.

If D is a first order elliptic differential operator, let $K^+(t, x, y)$, respectively $K^-(t, x, y)$, denote the kernel of the operator e^{-tD^*D}, respectively e^{-tDD^*}. Then for any $N \in \mathbb{N}$, there are smooth functions $c_j^{\pm}$ on M such that

$$\operatorname{tr} K^{\pm}(t, x, x) = \sum_{j=0}^{N} c_j^{\pm}(x) t^{j - \frac{\dim M}{2}} + \mathcal{O}\left(t^{N+1 - \frac{\dim M}{2}}\right).$$

If D were pseudodifferential, then we would have a similar asymptotic formula, but other terms would be present in the expansion, such as terms that include $\log t$. Continuing, we

1

have

$$\begin{aligned}
\operatorname{ind}(D) &= \dim \ker D - \dim \ker D^* \\
&= \operatorname{tr}\left(e^{-tD^*D}\right) - \operatorname{tr}\left(e^{-tDD^*}\right) \text{ for every } t > 0 \\
&= \int_M \operatorname{tr} K^+(t,x,x) \, \operatorname{dvol}(x) - \int_M \operatorname{tr} K^-(t,x,x) \, \operatorname{dvol}(x) \\
&= \int_M \left[c^+_{\frac{\dim M}{2}}(x) - c^-_{\frac{\dim M}{2}}(x) \right] \operatorname{dvol}(x).
\end{aligned}$$

In fact, we have that the integrand in the Atiyah-Singer Index Theorem satisfies

$$\alpha(x) = c^+_{\frac{\dim M}{2}}(x) - c^-_{\frac{\dim M}{2}}(x).$$

Note that this expression is identically zero if $\dim M$ is odd.

Typical examples of this theorem are some classic theorems in global analysis. First, let $D = d + d^*$ from the space of even forms to the space of odd forms on the manifold M of dimension $2n$, where d^* denotes the L^2-adjoint of the exterior derivative d. Then the elements of $\ker(d+d^*)$ are the even harmonic forms, and the elements of the cokernel can be identified with odd harmonic forms. Moreover,

$$\begin{aligned}
\operatorname{ind}(d+d^*) &= \dim H^{\text{even}}(M) - \dim H^{\text{odd}}(M) \\
&= \chi(M), \text{ and} \\
\int_M \operatorname{ch}(\sigma(d+d^*)) \wedge \operatorname{Todd}(T_{\mathbb{C}}M) &= \frac{1}{(2\pi)^n} \int_M \operatorname{Pf},
\end{aligned}$$

where Pf is the Pfaffian, which is, suitably interpreted, a characteristic form obtained using the square root of the determinant of the curvature matrix. In the case of 2-manifolds ($n = 1$), Pf is the Gauss curvature times the area form. Thus, in this case the Atiyah-Singer Index Theorem yields the generalized Gauss-Bonnet Theorem.

Another example is the operator $D = d + d^*$ on forms on a $2n$-dimensional manifold, this time mapping the self-dual to the anti-self-dual forms. This time the Atiyah-Singer Index Theorem yields the equation (called the Hirzebruch Signature Theorem)

$$\operatorname{Sign}(M) = \int_M L,$$

where $\operatorname{Sign}(M)$ is signature of the manifold, and L is the Hirzebruch L-polynomial applied to the Pontryagin forms.

Different examples of operators yield other classical theorems, such as the Hirzebruch-Riemann-Roch Theorem.

3. Bott Periodicity = Thom Isomorphism = Index Theory

3.1. **Vector bundles over the sphere.** Consider the following examples of complex line bundles over the sphere $S^2 = \mathbb{C}P^1$. Note that complex line bundles over the sphere can be classified (up to isomorphism) by their clutching functions $f : S^1 \to \mathbb{C}$. That is, since the upper and lower hemispheres are contractible and thus all vector bundles are trivial over these spaces, we may regard every line bundle as a copy of $\mathbb{C}$ over each hemisphere along with a gluing transformation (ie multiplication by a complex-valued function, the clutching function) over the equator. It is possible to show (and certainly believable) every such clutching function is homotopic to the function of the form $f(z) = z^j$, with $j \in \mathbb{Z}$. The

exponent of the clutching function gives the Chern number of the line bundle, which is the same as the Euler number of the realization of the line bundle, which is the same as the integral of the first Chern class (obtained from the curvature form). Thus, the (first) Chern number classifies the complex line bundle up to isomorphism.

(1) **Tangent Bundle to S^2.** Since S^2 (in fact, all oriented surfaces), can be given the structure of a complex manifold, and thus $TS^2 \to S^2$ is a complex vector bundle. Its Chern class is the standard Euler class, so the Chern number of this bundle is two. Note that the unit tangent bundle $SS^2 \to S^2$ is $SO(3) = \mathbb{R}P^3$.

To figure out the clutching function, pretend first that we have the chosen the standard metric on S^2. Imagine that we have a tangent vector to the north pole, and we parallel translate it to the south pole via the geodesic whose initial velocity is that tangent vector. Next, take another geodesic through the north pole that makes an angle θ (counterclockwise) with the first geodesic at the north pole. Then if we parallel translate the original vector along this new geodesic, observe that the resulting vector at the south pole makes an angle 2θ with the vector that was parallel translated along the original geodesic. (It helps to visualize what happens when $\theta = \frac{\pi}{2}$. Thus the clutching function (which identifies the copy of $\mathbb{C}$ at the north pole with a copy of $\mathbb{C}$ at the south pole) satisfies $f\left(e^{i\theta}z\right) = e^{2i\theta}z$, so the function is $f(z) = z^2$. In general, the exponent of the clutching function gives the Chern number of the line bundle. Thus, the Chern number classifies the line bundle up to isomorphism.

(2) **The "Canonical Line Bundle" or "Tautological Line Bundle" or "Hyperplane Bundle" H over $\mathbb{C}P^1 = S^2$.**

The canonical line bundle of any complex projective space $\mathbb{C}P^n$ is the union of the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, and the projection to $\mathbb{C}P^n$ is given by projecting the elements of a complex line through the origin to the complex line itself, thought of as an element of $\mathbb{C}P^n$. The natural complex and smooth structure of this bundle is defined in a natural way, using the ways that the lines are embedded in $\mathbb{C}^{n+1}$. In other words, given a complex line $[z_0, ..., z_n] \in \mathbb{C}P^n$, the fiber is the set of points $(z_0, ..., z_n) \in \mathbb{C}^{n+1}$ that project to that line. The case $n = 1$ yields the canonical line bundle H over $\mathbb{C}P^1 = S^2$. We let the trivialization near the north pole to be $[z, 1] = [z_0, z_1]$, the points of the line bundle over a point $[z, 1]$ in the northern hemisphere be the points $(zw, w) \in \mathbb{C}^2$, where $w \in \mathbb{C}$. These points correspond to $[1, 1/z]$ near the south pole, where the elements of the line bundle are $(w, w/z) \in \mathbb{C}^2$ with $w \in \mathbb{C}$, so that the gluing transformation at $z \in S^1$ is multiplication by z. Since the clutching function is $f(z) = z$, the Chern number of this line bundle is 1 (or -1, if one uses $[z, 1]$ for the southern hemisphere). Traditionally, in the literature one uses $[z, 1]$ for the southern hemisphere, and thus the dual H^* has Chern number 1.

(3) **The "Clifford multiplication line bundle" over S^2.** Consider the following line bundle generated by Clifford multiplication on the spinor bundle over S^2. We will take the line bundle to be $\mathbb{S}^+$ over northern hemisphere and $\mathbb{S}^-$ over the southern hemisphere, with Clifford multiplication being the gluing transformation over the equator. In this case, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- = \mathbb{C}^2$, and Clifford multiplication by $v = (v_1, v_2, v_3) \in S^2$ is matrix multiplication by

$$v_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + v_3 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} iv_3 & -v_1 + iv_2 \\ v_1 + iv_2 & -iv_3 \end{pmatrix}.$$

On the equator $v_3 = 0$, $\mathbb{S}^+ = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbb{S}^- = \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the isomorphism at $(x, y, 0)$ is multiplication by $x + iy$, so the clutching function is $f(z) = z$, and thus this line bundle is topologically the same as H^*, and its Chern number is 1.

(4) **The complex exterior bundle** $\Lambda_{\mathbb{C}}(TM)$. (A similar calculation can be done on any complex manifold). Over each point, we identify the tangent space with $\mathbb{C}$, and $\Lambda_{\mathbb{C}}\mathbb{C}$ is the complex vector space with basis $\{1, dz\}$, and we consider the Clifford action of $\text{Cl}(TM)$ defined by $c(v_1, v_2) = (v_1 + iv_2)\, dz \wedge - (v_1 - iv_2)\, dz\lrcorner$, which maps $\Lambda_{\mathbb{C}}^{\text{even}}(TM) = \mathbb{C}$ to $\Lambda_{\mathbb{C}}^{\text{odd}}(TM) = \mathbb{C}dz$. This is the same as

$$c(v_1, v_2) = v_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

on $\text{span}\begin{pmatrix} 1 \\ dz \end{pmatrix}$, and this is clearly a Clifford action.

(5) **Any other line bundle over** S^2 is a tensor product of a number of copies of H with a number of copies of H^*. That is, since $H \otimes H^*$ is trivial, it is the tensor product of a number of copies of H, it is the trivial line bundle, or it is the tensor product of a number of copies of H^*. Note that the Chern number of a tensor product of two line bundles is simply the sum of the Chern numbers of the individual line bundles, because the clutching function of a tensor product is the product of the clutching functions.

(6) **Complex vector bundles over** S^2: More generally, suppose we are given any complex vector bundle of rank n over the sphere S^2. As in the line bundle case, since every such vector bundle is trivial over the hemisphere, the vector bundle is determined topologically by an isomorphism of two complex vector bundles of rank n over the equator S^1. Thus, the bundle is determined topologically by a homotopy class of maps from S^1 into $Gl(n, \mathbb{C})$, in other words the fundamental group of $Gl(n, \mathbb{C})$. Since $GL(n, \mathbb{C})$ is homotopic to $U(n)$,

$$\pi_1(Gl(n, \mathbb{C})) \cong \pi_1(U(n)) \cong \pi_1(U(1)) \cong \mathbb{Z}.$$

The second isomorphism is induced by the determinant homomorphism. The fact that $\pi_1 U(n) \cong \mathbb{Z}$ comes from the long exact sequence in homotopy for a fiber bundle. Clearly $\pi_1(U(1)) = \pi_1(S^1) \cong \mathbb{Z}$, and we have the following fiber bundle:

$$\begin{array}{ccc} U(n-1) & \to & U(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

Thus, the long exact sequence in homotopy is

$$...\pi_{j+1}(S^{2n-1}) \to \pi_j(U(n-1)) \to \pi_j(U(n)) \to \pi_j(S^{2n-1}) \to ...$$

Since $\pi_m(S^{2n-1}) = \{0\}$ for $m \le 2n - 2$, we have that $\pi_j(U(n-1)) \cong \pi_j(U(n))$ for $j < 2n - 2$, which implies in particular that $\pi_1(U(n)) = \mathbb{Z}$ for all $n \ge 1$. Back to the equator, we see that every homotopy class of maps from S^1 into $Gl(n, \mathbb{C})$ is

homotopic to

$$z \mapsto \begin{pmatrix} z^j & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, every complex vector bundle over S^2 is isomorphic to the direct sum of a line bundle and a trivial bundle. Thus, every complex vector bundle over S^2 is isomorphic to the direct sum of a tensor product of a number of copies of H with a number of copies of H^*, with a trivial bundle. Said differently, H (or the Clifford multiplication line bundle) generates the set of all vector bundles over S^2 using the operations $*$ (dual), $\otimes$, and $\oplus$.

Notice that if we were trying to classify complex vector bundles over higher dimensional spheres (S^k with $k > 2$), we would need to be looking at higher homotopy groups $\pi_{k-1}(U(n))$, which is pretty darn hard. However, if $n \geq \frac{k}{2}$, then

$$\pi_{k-1}(U(n)) = \begin{cases} \{0\} & \text{if } k \text{ is odd} \\ \mathbb{Z} & \text{if } k \text{ is even} \end{cases}$$

(This is a result of Bott periodicity – as you increase n with k fixed, the homotopy group stabilizes, so these are actually the stable homotopy groups.) Thus the group of stable isomorphism classes of complex vector bundles over S^k (stable means that we mod out by the operation of direct-summing with trivial bundles) is

$\{\text{complex vector bundles over } S^k\}$ mod isomorphism, mod stability

$$\leftrightarrow \begin{cases} \{0\} & \text{if } k \text{ is odd} \\ \mathbb{Z} & \text{if } k \text{ is even} \end{cases}$$

3.2. Thom Isomorphism in de Rham Cohomology.

Proposition 3.1. *(Poincaré Lemma for compact supports) The compactly supported (de Rham) cohomology of $\mathbb{R}^n$ satisfies*

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

In fact, the generator of $H_c^n(\mathbb{R}^n)$ is the Thom form Φ, a bump function times the volume form whose total integral is 1. If we let $\pi : \mathbb{R}^n \to \{0\}$, the isomorphism is

$$\pi_* = \pi_! : H_c^n(\mathbb{R}^n) \to H^0(\{0\}) = \mathbb{R},$$

which is given by integration (over the fiber), and the inverse $i_! : \mathbb{R} \to H_c^n(\mathbb{R}^n)$ is given by multiplying the constant by Φ. Notice that this is the same as

$$i_!(u) = [\pi^* u \wedge \Phi],$$

where u is a constant (function on $\{0\}$). The notation comes from the inclusion map $i : \{0\} \to \mathbb{R}^n$ (the zero section).

Proposition 3.2. *(Thom Isomorphism for Cohomology) If M is a compact manifold and if $\pi : E \to M$ is an oriented real vector bundle of rank r, then*

$$H_c^k(E) \cong H^{k-r}(M).$$

In fact there exists a closed r-form Φ (Thom form) on E that restricts to a bump volume form on each fiber such that the inverse $i_!$ of the "integration over the fiber" map

$$\pi_* = \pi_! : H_c^k(E) \to H^{k-r}(M)$$

satisfies

$$i_!([u]) = [\pi^* u \wedge \Phi].$$

In particular, $[\Phi] = i_!([1])$. The cohomology class $[\Phi]$ is called the *Thom class* of E.

3.3. K-theory.
Let M be a compact manifold (actually compact Hausdorff is sufficient). We define the K-theory $K(M)$

$$
\begin{aligned}
K(M) \quad = \quad & \{\text{formal differences of vector bundles over } M\} \\
& \text{mod isomorphisms of vector bundles} \\
& \text{mod direct summing with trivial bundles.}
\end{aligned}
$$

This is an abelian group under the direct sum operation, and in fact you can also define a multiplication (we'll get to that later). An interesting observation: it is always possible to embed any vector bundle $E \to M$ in Euclidean space (as a vector bundle), so in fact there always exists a complementary vector bundle E' such that $E \oplus E'$ is trivial. Therefore, for any two vector bundles E and F over M,

$$[F - E] = [F \oplus E' - E \oplus E'],$$

so one may always take the second bundle to be trivial. So you can think of K-theory as stable isomorphism classes of vector bundles. Next, observe that

$$K(pt) = \mathbb{Z},$$

where the integer is the difference in ranks of the two bundles (ie dimensions of the vector spaces) — that is, the generator of $K(pt)$ is $[\mathbb{C} - \{0\}]$. Also, observe that every complex vector bundle over the circle S^1 is trivial, so

$$K(S^1) = \mathbb{Z}.$$

On the other hand, $K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$, where the standard generator of the second copy of $\mathbb{Z}$ is $[H - \mathbf{1}]$, where H is the canonical line bundle and $\mathbf{1}$ is the trivial bundle.

Note that homotopy equivalent spaces have the same K-theory.

Given a closed manifold M and a point $pt \in M$, we define the reduced K-theory $\widetilde{K}(M)$ as the kernel of the homomorphism

$$K(M) \to K(pt) = \mathbb{Z}$$

given by restricting bundles over M to $pt \in M$. In other words,

$$
\begin{aligned}
\widetilde{K}(M) \quad = \quad & \{\text{formal differences of vector bundles of the same rank over } M\} \\
& \text{mod isomorphisms of vector bundles} \\
& \text{mod direct summing with trivial bundles.}
\end{aligned}
$$

So $\widetilde{K}(pt) = \{0\}$, $\widetilde{K}(S^1) = \{0\}$, and $\widetilde{K}(S^2) = \mathbb{Z}$.

Given a compact pair (X, A) (meaning X is compact Hausdorff, A is a closed subspace), then we define the relative K-theory of the pair as

$$K(X, A) = \widetilde{K}(X/A),$$

where X/A means you have taken the quotient space by identifying A to a point. Thus, if $pt \in M$, then

$$K(X, pt) = \widetilde{K}(X).$$

Also,

$$K(X, \varnothing) = K(X).$$

If you think about it, there are other equivalent ways to understand relative K-theory:

$$
\begin{aligned}
K(X, A) \quad = \quad & \{\text{formal differences of vector bundles over } X \text{ that are isomorphic over } A\} \\
& \text{mod isomorphisms of vector bundles} \\
& \text{mod direct summing with trivial bundles.}
\end{aligned}
$$

Let the double $D(X, A)$ of X along A be defined as the disjoint union of two copies of X whose copies of A are identified. Then

$$
\begin{aligned}
K(X, A) \quad & = \quad \text{kernel of the homomorphism} \\
K(D(X, A)) \quad & \rightarrow \quad K(X),
\end{aligned}
$$

where the homomorphism is given by restricting the vector bundles to one copy of X in the double.

One typical way of exhibiting an element of $K(X, A)$ is as

$$[E, F, \phi],$$

where E and F are vector bundles over X, and $\phi : E|_A \rightarrow F|_A$ is an isomorphism.

Note that if B^n is an n-dimensional ball with boundary S^{n-1}, we have

$$
\begin{aligned}
K\left(B^n, S^{n-1}\right) \quad & = \quad \widetilde{K}\left(B^n/S^{n-1}\right) \\
& = \quad \widetilde{K}(S^n).
\end{aligned}
$$

Finally, we define compactly supported K-theory as

$$
\begin{aligned}
K_{cpt}(M) \quad & = \quad \widetilde{K}\left(M^+\right) \\
& = \quad K\left(M^+, \infty\right),
\end{aligned}
$$

where M^+ is the one-point (∞) compactification of M. If M is compact, M^+ is the union of M and an isolated point, so $K_{cpt}(M) = K(M)$. Note that one may regard

$$
\begin{aligned}
K_{cpt}(M) \quad = \quad & \left\{ \begin{array}{c} \text{formal differences of vector bundles over } M \\ \text{that are isomorphic outside a compact set} \end{array} \right\} \\
& \text{mod isomorphisms of vector bundles} \\
& \text{mod direct summing with trivial bundles.}
\end{aligned}
$$

Thus,

$$K_{cpt}\left(\mathbb{R}^n\right) = K\left(B^n, S^{n-1}\right) = \widetilde{K}(S^n).$$

If $E \rightarrow M$ is a vector bundle over a compact manifold, then

$$K_{cpt}(E) = K(B(E), S(E)),$$

where $B(E)$ denotes the unit ball bundle of E and $S(E)$ denotes the unit sphere bundle. Thus, elements of $K_{cpt}(E)$ can be expressed as

$$[F_1, F_2, \beta],$$

where $F_j \to B(E)$ is a vector bundle for $j = 1, 2$ such that $\beta : F_1|_{S(E)} \to F_2|_{S(E)}$ is a bundle isomorphism.

Because of these descriptions, it is easy to see that

$$
\begin{aligned}
K_{cpt}(\mathbb{R}^n) &= \widetilde{K}(S^n) \\
&= K(B^n, S^{n-1}) \\
&= \left\{ \begin{array}{c} [E, F, \beta] : E \text{ and } F \text{ are vector bundles over } B^n \\ \text{that are isomorphic via } \beta \text{ on } S^{n-1} \end{array} \right\} \mod \left\{ \begin{array}{c} \text{stable} \\ \text{isomorphism} \end{array} \right\}.
\end{aligned}
$$

Observe that every vector bundle over S^n is a pair of complex vector bundles $E \to B^n$, $F \to B^n$ over two copies of B^n (with opposite orientations but whose orientation on S^n is the same) such that $E|_{S^{n-1}}$ and $F|_{S^{n-1}}$ are isomorphic. Thus,

$$
\begin{aligned}
K_{cpt}(\mathbb{R}^n) &= \widetilde{K}(S^n) = K(B^n, S^{n-1}) \\
&= \{\text{stable isomorphism classes of vector bundles over } S^n\}
\end{aligned}
$$

In particular,

$$
K_{cpt}(\mathbb{R}^2) = \widetilde{K}(S^2) = K(B^2, S^1) = \mathbb{Z}, \tag{3.1}
$$

with generator

$$
\begin{aligned}
[H - \mathbf{1}] &= [H, \mathbf{1}, \text{multiplication by } z \in S^1] \\
&= [\mathbb{S}^+, \mathbb{S}^-, c],
\end{aligned}
$$

as explained in Section 3.1. The comments at the end of that section imply that

$$
K_{cpt}(\mathbb{R}^k) = \widetilde{K}(S^k) = K(B^k, S^{k-1}) = \left\{ \begin{array}{ll} \{0\} & \text{if } k \text{ is odd} \\ \mathbb{Z} & \text{if } k \text{ is even} \end{array} \right.
$$

Lemma 3.3. *Let $\pi : E \to M$ be a smooth real vector bundle over a closed manifold M. Then every element of $K_{cpt}(E)$ can be represented by*

$$
[\pi^* F_1, \pi^* F_2, \sigma],
$$

where F_1 and F_2 are vector bundles over M and isomorphic via σ outside a compact set, and where $\sigma : \pi^* F_1 \to \pi^* F_2$ is homogeneous of degree zero on the fibers.

K-theory is a generalized cohomology theory, in that it satisfies all of the axioms of cohomology except the dimension axiom.

We define multiplication in relative K-theory as follows. Given $[V_0, V_1, \alpha], [W_0, W_1, \beta] \in K(X, A)$, we define

$$
[V_0, V_1, \alpha] \cdot [W_0, W_1, \beta] =
$$

$$
\left[(V_0 \otimes W_0) \oplus (V_1 \otimes W_1), (V_1 \otimes W_0) \oplus (V_0 \otimes W_1), \left(\begin{array}{cc} \alpha \otimes \mathbf{1} & -\mathbf{1} \otimes \beta^* \\ \mathbf{1} \otimes \beta & \alpha^* \otimes \mathbf{1} \end{array} \right) \right],
$$

where $*$ denotes an adjoint with respect to Hermitian metrics on the bundles. (We may alternatively replace adjoint by inverse.)

3.4. **Other K-groups.** We define the *smash product* $A \wedge B$ of two topological spaces A and B with basepoints a and b as

$$A \wedge B := A \times B \big/ (A \times \{b\} \cup \{a\} \times B).$$

The smash product is commutative and associative, since

$$
\begin{aligned}
(A \wedge B) \wedge C &= [A \times B \big/ (A \times \{b\} \cup \{a\} \times B)] \times C \big/ [(A \wedge B) \times \{c\} \cup \{(a,b)\} \times C] \\
&= [A \times B \times C \big/ (A \times \{b\} \times C \cup \{a\} \times B \times C)] \big/ [(A \wedge B) \times \{c\} \cup \{(a,b)\} \times C] \\
&= (A \times B) \times C \big/ (A \times \{b\} \times C \cup \{a\} \times B \times C \cup A \times B \times \{c\}).
\end{aligned}
$$

Note that if A and B are compact, the smash product of A and B is the one-point compactification of $(A \setminus \{pt\}) \times (B \setminus \{pt\})$. Also, observe that if we let ∞_A and ∞_B be the (added) basepoints in the one-point compactifications of A and B,

$$
\begin{aligned}
A^+ \wedge B^+ &= A^+ \times B^+ \big/ \left(A^+ \times \{\infty_B\} \cup \{\infty_A\} \times B^+\right) \\
&= (A \times B \cup \{\infty_A\} \times B \cup A \times \{\infty_B\} \cup \{\infty_A\} \times \{\infty_B\}) \\
&\quad \big/ (\{\infty_A\} \times B \cup A \times \{\infty_B\} \cup \{\infty_A\} \times \{\infty_B\}) \\
&= (A \times B)^+
\end{aligned}
$$

We define the *reduced suspension* ΣA of a topological space A with basepoint a as

$$\Sigma A := S^1 \wedge A,$$

where the basepoint on S^1 can be arbitrary. This is the same as the topological suspension $s(A)$ mod the images of the basepoint of the suspended space. These two suspensions are homotopy equivalent. In other words,

$$
\begin{aligned}
\Sigma A &= [0,1] \times A \big/ (\{0\} \times A \cup \{1\} \times A \cup [0,1] \times \{a\}) \\
&= s(A) \big/ ([0,1] \times \{a\})
\end{aligned}
$$

Note that

$$S^1 \wedge S^j = S^{j+1}$$

for each nonnegative integer j. If $A^+ = A \cup \{pt\}$ denotes the one-point compactification of a compact Hausdorff space A, observe that

$$
\begin{aligned}
\Sigma\left(A^+\right) &= s(A) \text{ with endpoints identified} \\
&= [0,1] \times A \big/ (\{0\} \times A \cup \{1\} \times A) \\
&\simeq s(A) \cdot S^1 \\
&\simeq (\Sigma A) \cdot S^1.
\end{aligned}
\tag{3.2}
$$

The last line means that this space is homotopic to the standard suspension $s(A)$ glued to a circle.

For $i \in \mathbb{N}$, we define the *i-fold reduced suspension* $\Sigma^i A$ of a topological space A with basepoint a by

$$
\begin{aligned}
\Sigma^1 A &:\ = \Sigma A \\
\Sigma^i A &:\ = \Sigma\left(\Sigma^{i-1} A\right) \text{ for } i > 1.
\end{aligned}
$$

This implies that

$$\Sigma^j A = S^j \wedge A$$

for each $j \in \mathbb{N}$.

For each $i \in \mathbb{N}$, we now define $K^{-i}(X)$, $K^{-i}(X,Y)$, and $\widetilde{K}^{-i}(X)$, where X is a compact Hausdorff space and Y is a closed subspace, and X^+ is the one-point compactification.

$$
\begin{aligned}
K^{-i}(X) &: = \widetilde{K}\left(\Sigma^i\left(X^+\right)\right) \\
K^{-i}(X,Y) &: = \widetilde{K}\left(\Sigma^i\left(X/Y\right)\right) \\
\widetilde{K}^{-i}(X) &: = \widetilde{K}\left(\Sigma^i\left(X\right)\right)
\end{aligned}
$$

These definitions imply that

$$
\begin{aligned}
\widetilde{K}^{-i}(X) &= \widetilde{K}\left(S^i \wedge X\right) \\
K^{-i}(X,Y) &= \widetilde{K}^{-i}\left(X/Y\right) \\
K^{-i}(X,\varnothing) &= \widetilde{K}\left(\Sigma^i\left(X^+\right)\right) = K^{-i}(X) \\
\widetilde{K}^{-i}\left(X^+\right) &= K^{-i}(X)
\end{aligned}
$$

Note also that if X is compact, then

$$
\begin{aligned}
K^{-1}(X) &= \widetilde{K}^{-1}\left(X^+\right) = \widetilde{K}\left(\Sigma\left(X^+\right)\right) \\
&= \widetilde{K}\left((\Sigma X) \cdot S^1\right) \\
&= \widetilde{K}(\Sigma X) \\
&= \widetilde{K}^{-1}(X) = K^{-1}(X, pt),
\end{aligned}
$$

where the second to last line is true because all complex vector bundles of a given rank over the circle are isomorphic. Equivalently, identifying two points on a compact space does not change the reduced K-theory. Also, note that

$$
\begin{aligned}
K^{-i}(X) &= \widetilde{K}\left(\Sigma^i\left(X^+\right)\right) \\
&= \widetilde{K}\left(S^i \wedge\left(X^+\right)\right) \\
&= \widetilde{K}\left(\mathbb{R}^{i+} \wedge\left(X^+\right)\right) \\
&= \widetilde{K}\left(\left(\mathbb{R}^i \times X\right)^+\right).
\end{aligned}
$$

For example,

$$
K^{-1}(pt) = \widetilde{K}\left(\Sigma\left(pt \cup pt_2\right)\right) = \widetilde{K}\left(S^1\right) = \{0\}.
$$

For another example,

$$
K^{-1}\left(S^1\right) = \widetilde{K}\left(S^2\right) = \mathbb{Z},
$$

and more generally

$$
\begin{aligned}
K^{-k}(pt) &= \widetilde{K}\left(S^k \wedge\left(pt \cup pt_2\right)\right) = \widetilde{K}\left(S^k\right) \\
&= K\left(B^k, S^{k-1}\right) = \begin{cases} \{0\} & \text{if } k \text{ is odd} \\ \mathbb{Z} & \text{if } k \text{ is even} \end{cases}
\end{aligned}
$$

Furthermore, if X and Y are spaces with basepoints, then

$$
\widetilde{K}^{-i}(X \times Y) \cong \widetilde{K}^{-i}(X \wedge Y) \oplus \widetilde{K}^{-i}(X) \oplus \widetilde{K}^{-i}(Y)
$$

Given complex vector bundles $E \to A$ and $F \to B$ over base-pointed spaces A and B, the exterior tensor product $E \otimes F$ lives naturally over $A \times B$. There is a natural map

from $\widetilde{K}(A \times B)$ to $\widetilde{K}(A) \oplus \widetilde{K}(B)$, given by restricting to one factor at a base point of the other. The kernel of this map is the exactly $\widetilde{K}(A \wedge B)$ by construction. To obtain a map from $\widetilde{K}(A \times B)$ to $\widetilde{K}(A \wedge B)$, first take an equivalence class of formal differences of vector bundles $[V_1 - V_2]$ of the same rank over $A \times B$, and then restrict this difference to $\{a\} \times B$. Then extend the result trivially to obtain $[V_1^a - V_2^a]$ over $A \times B$. Similarly, obtain $[V_1^b \times V_2^b]$ for the basepoint $b \in B$. Finally, let $[V_1^{ab} - V_2^{ab}]$ denote the restriction of $[V_1 - V_2]$ to $\{a\} \times \{b\}$ and then extended trivially to $A \times B$. We define the map $\widetilde{K}(A \times B) \to \widetilde{K}(A \wedge B)$ by

$$[V_1 - V_2] \mapsto [V_1 - V_2]^{\wedge} := \left[V_1 - V_1^a - V_1^b + V_1^{ab} - V_2 + V_2^a + V_2^b - V_2^{ab} \right].$$

The right hand side clearly is 0 over $\{a\} \times B$ and $A \times \{b\}$, so it is defined on $A \wedge B$. Observe that it is really not necessary to add $V_1^{ab} - V_2^{ab}$, since adding and subtracting a trivial bundle does not change the class. Another way to say this is that the exact sequence below splits if the spaces are pointed.

$$\widetilde{K}(A) \oplus \widetilde{K}(B) = \widetilde{K}(A \vee B) \to \widetilde{K}(A \times B) \to \widetilde{K}(A \wedge B) \to 0 \qquad (3.3)$$

We thus obtain a pairing

$$\widetilde{K}(A) \otimes \widetilde{K}(B) \quad \to \quad \widetilde{K}(A \wedge B), \text{ defined using}$$
$$[E] \otimes [F] \quad \mapsto \quad [E \boxtimes F]^{\wedge} = \left[E \boxtimes F - E^a \boxtimes F - E \boxtimes F^b + E^a \boxtimes F^b \right].$$

This map is actually a ring homomorphism. We now apply this pairing to $A = S^i \wedge X$ and $B = S^j \wedge Y$. Since

$$\left(S^i \wedge X \right) \wedge \left(S^j \wedge Y \right) = S^{i+j} \wedge (X \wedge Y),$$

we obtain a pairing of the reduced K-theory groups:

$$\widetilde{K}^{-i}(X) \otimes \widetilde{K}^{-j}(Y) \to \widetilde{K}^{-i-j}(X \wedge Y)$$

Then you replace X and Y with their one-point compactifications, and you get corresponding pairings of nonreduced K-theory:

$$K^{-i}(X) \otimes K^{-j}(Y) \to K^{-i-j}(X \times Y),$$

using the fact that $X^+ \wedge Y^+ = (X \times Y)^+$. (Note that I see no need for base points in the pairing above.) Recalling that

$$K^{-i}(X) = \widetilde{K}\left(\left(\mathbb{R}^i \times X \right)^+ \right),$$

we can make sense of the pairing as follows. Given an equivalence class $[E]$ of formal differences of vector bundles of the same rank over $\left(\mathbb{R}^i \times X \right)^+$, you can just think of $[E]$ as a difference of vector bundles over $\mathbb{R}^i \times X$ that are isomorphic outside a compact set. Similarly, let $[F]$ be an equivalence class of formal differences of vector bundles over $\mathbb{R}^j \times Y$ that are isomorphic outside a compact set. Then we form the class $[E \boxtimes F - E^\infty \boxtimes F - E \boxtimes F^\infty + E^\infty \boxtimes F^\infty]$ over $\mathbb{R}^{i+j} \times X \times Y$, where for example E^∞ denotes the isomorphism class of E at ∞. Note that this class has the property that the bundles are isomorphic outside a compact set in $\mathbb{R}^{i+j} \times X \times Y$, so that the class lives in $K^{-i-j}(X \times Y)$. Also, since $E^\infty \boxtimes F^\infty$ is zero as a class, it could be deleted.

We apply this when X and Y are points to get:

Proposition 3.4. *The pairing* $K^{-i}(pt) \otimes K^{-j}(pt) \to K^{-i-j}(pt)$ *makes* $K^{-*}(pt)$ *a graded ring. More generally, if* (X, pt) *is a pointed space, the pairing*

$$K^{-i}(pt) \otimes K^{-j}(X) \to K^{-i-j}(X)$$

makes $K^{-*}(X)$ *a graded module over* $K^{-*}(pt)$.

Tracing through the constructions above, we can understand the pairing as follows. First, note that

$$
\begin{aligned}
K^{-i}(pt) &= \widetilde{K}\left(\left(\mathbb{R}^i\right)^+\right) = \widetilde{K}\left(S^i\right), \text{ and} \\
K^{-j}(X) &= \widetilde{K}\left(\left(\mathbb{R}^j \times X\right)^+\right).
\end{aligned}
$$

Next, given an equivalence class $[E]$ of formal differences of vector bundles of the same rank over $S^i = \mathbb{R}^i \cup \infty$ and an equivalence class $[F]$ of formal differences of vector bundles of the same rank over $(\mathbb{R}^j \times X)^+ = \mathbb{R}^j \times X \cup \infty'$, the pairing produces the equivalence class $\left[E \boxtimes F - E^\infty \boxtimes F - E \boxtimes F^{\infty'} + E^\infty \boxtimes F^{\infty'}\right]$ over $\mathbb{R}^{i+j} \times X \cup \infty$, where ∞ and ∞' are identified.

3.4.1. *Another approach to understanding* K^{-1}. Suppose that X is a compact Hausdorff space. Then consider the set

$$\widehat{K^{-1}}(X) = \bigcup_{n \in \mathbb{N}} Gl(n, C(X)) \Big/ \bigcup_{n \in \mathbb{N}} Gl(n, C(X))_0,$$

where the subscript 0 denotes the connected component of the identity. We will compare $\widehat{K^{-1}}(X)$ to

$$K^{-1}(X) = \widetilde{K}\left(S^1 \wedge X^+\right) = \widetilde{K}\left(\left(\mathbb{R} \times X\right)^+\right).$$

Observe that $\widehat{K^{-1}}(X)$ is the set of homotopy classes of isomorphisms of trivial complex vector bundles over X. Note that a bundle is trivial if and only if it extends to the cone over X. An isomorphism of trivial complex vector bundles over X can be thought of a bundle over the standard suspension of X, where the gluing map is the isomorphism; every bundle over the standard suspension can be constructed this way. Thus

$$\widehat{K^{-1}}(X) = \widetilde{K}\left(s(X)\right).$$

Since $s(X)$ is homotopy equivalent to the reduced suspension $\Sigma X = S^1 \wedge X$, we have

$$
\begin{aligned}
\widehat{K^{-1}}(X) &= \widetilde{K}\left(S^1 \wedge X\right) \\
&= \widetilde{K}\left(\left(S^1 \wedge X\right) \cdot S^1\right),
\end{aligned}
$$

where $\cdot$ means glued at a point, which is true since the reduced K-theory of the circle is trivial. But since $(S^1 \wedge X) \cdot S^1$ is homotopy equivalent to $S^1 \wedge (X^+)$, as seen in (3.2), we get

$$\widehat{K^{-1}}(X) = \widetilde{K}\left(S^1 \wedge X^+\right) = K^{-1}(X).$$

3.5. **Bott Periodicity.** Using (3.1) and Section 3.4, we have:

Proposition 3.5.

$$K^{-2}\left(pt\right) = \widetilde{K}\left(S^2\right) = K_{cpt}\left(\mathbb{R}^2\right) = K\left(B^2, S^1\right) = \mathbb{Z} = K^0\left(pt\right),$$

with generator

$$\begin{aligned}
\left[H - \mathbf{1}\right] &= \left[H, \mathbf{1}, \text{multiplication by } z \in S^1\right] \\
&= \left[\mathbb{S}^+, \mathbb{S}^-, c\right].
\end{aligned}$$

We now use this fact to prove the following Lemma:

Lemma 3.6. *(Product Theorem) If X is a compact Hausdorff space, then*

$$\widetilde{K}\left(X \times S^2\right) \cong \widetilde{K}\left(X\right) \otimes \widetilde{K}\left(S^2\right) \cong \widetilde{K}\left(X\right).$$

Proof. There is a 1-1 correspondence between complex vector bundles over $X \times S^2$ and pairs $[E, f]$ where E is a complex vector bundle over X and the "clutching function" $f : E \times S^1 \to E$ is an S^1-parameter family of bundle automorphisms. The idea is that every vector bundle over $X \times S^2$ is a pair of vector bundles over $X \times D^2$ that is "clutched" along the boundary. First any such clutching function is homotopic ($\simeq$) to a Laurent polynomial of the form $\sum_{|j| \leq n} a_j\left(x\right) z^j$ for some n. Next, one proves that $[E, z^{-m}q] \simeq [E, q] \otimes (H^*)^{\otimes m}$, so it suffices to consider polynomials of the form $\sum_{0 \leq j \leq n} a_j\left(x\right) z^j$. Next, one can show that if q is one such polynomial, then $[E, q] \oplus [nE, \mathbf{1}] \simeq [(n + 1) E, a\left(x\right) z + b\left(x\right)]$, where $a\left(x\right)$ and $b\left(x\right)$ are bundle maps. (Basically, we use the trick that $[H \otimes H, \mathbf{1}] \simeq [H, H]$.) *By* doing another homotopy, we can reduce to the case $\left[\widetilde{E}, z + B\left(x\right)\right]$, where $B\left(x\right)$ is an invertible bundle map. Then, one can perform another homotopy of this bundle to $\left[\widetilde{E}_+, \mathbf{1}\right] \oplus \left[\widetilde{E}_-, z\right] = \widetilde{E}_+ \oplus \widetilde{E}_- \otimes H$, where $\widetilde{E}_+$, respectively $\widetilde{E}_-$, is the direct sum of eigenspaces of $\widetilde{E}$ corresponding to eigenvalues of $B\left(x\right)$ outside the unit circle S^1, respectively inside S^1. This shows that every virtual complex vector bundle over $X \times S^2$ is stably homotopic to an element of $\widetilde{K}\left(X\right) \otimes \widetilde{K}\left(S^2\right)$. Since the map $\widetilde{K}\left(X\right) \otimes \widetilde{K}\left(S^2\right) \to \widetilde{K}\left(X \times S^2\right)$ given by exterior tensor product is injective and surjective, the result follows. $\qquad\square$

Theorem 3.7. *(Bott periodicity, complex case) The following sequence of maps*

$$\widetilde{K}\left(X\right) \overset{[H-\mathbf{1}]\boxtimes \cdot}{\to} \widetilde{K}\left(S^2 \times X\right) \overset{\wedge}{\to} \widetilde{K}\left(S^2 \wedge X\right) \cong \widetilde{K}\left(s\left(s\left(X\right)\right)\right)$$

is a ring isomorphism, for all compact Hausdorff spaces X.

Proof. The first map comes from the proposition and lemma above, and the second map comes from the isomorphism in (3.3):

$$\widetilde{K}\left(S^2 \times X\right) \cong \widetilde{K}\left(S^2 \wedge X\right) \oplus \widetilde{K}\left(S^2\right) \oplus \widetilde{K}\left(X\right).$$

The product theorem shows that $[H - \mathbf{1}] \boxtimes \cdot$ is an isomorphism, and the projection $\wedge$ is 1-1, by construction. Recall that $[E \boxtimes F]^{\wedge} = \left[E \boxtimes F - E^a \boxtimes F - E \boxtimes F^b + E^a \boxtimes F^b\right]$ is the reduced external product, which is a 1-1 map from $\widetilde{K}\left(A\right) \otimes \widetilde{K}\left(B\right)$ onto $\widetilde{K}\left(A \wedge B\right)$. $\qquad\square$

Corollary 3.8. *(restatement of Bott periodicity, complex case)Module multiplication by* $[H - 1] = [\mathbb{S}^+, \mathbb{S}^-, c]$ *is an isomorphism*

$$K^{-i}(X) \to K^{-i-2}(X)$$

for all compact Hausdorff spaces X *and all* $i \geq 0$. *If* X *is a pointed space, then a similar isomorphism holds in reduced K-theory.*

Proof. $K^{-i}(X) = \widetilde{K}(S^i \wedge X^+) = \widetilde{K}(S^i \wedge X^+)$, and $K^{-i-2}(X) = \widetilde{K}(S^{i+2} \wedge X^+) = \widetilde{K}(S^2 \wedge S^i \wedge X^+)$. $\square$

Corollary 3.9. *We have the following results: For all* $k \geq 0$,

$$K^{-k}(pt) = K_{cpt}(\mathbb{R}^k) = \widetilde{K}(S^k) = K(B^k, S^{k-1}) = \begin{cases} \{0\} & \textit{if } k \textit{ is odd} \\ \mathbb{Z} & \textit{if } k \textit{ is even} \end{cases},$$

and in even dimensions, the generator is $[\mathbb{S}^+, \mathbb{S}^-, c]$.

Proof. In even dimensions, the map $[\mathbb{S}^+, \mathbb{S}^-, c]_{2-\dim} \cdot$ is an isomorphism from $K_{cpt}(\mathbb{R}^j)$ to $K_{cpt}(\mathbb{R}^{j+2})$. My the construction of even-dimensional complex spinors,

$$\left[\mathbb{S}^+, \mathbb{S}^-, c\right]_{2-\dim} \cdot \left[\mathbb{S}^+, \mathbb{S}^-, c\right]_{j-\dim} = \left[\mathbb{S}^+, \mathbb{S}^-, c\right]_{(j+2)-\dim}.$$

The result follows. $\square$

There are some nice consequences of Bott periodicity to exact sequences in K-theory. First, observe the following.

Lemma 3.10. *Let* (A, X) *be a compact pair of pointed spaces, meaning that* A *is a closed subspace of the compact Hausdorff space* X. *Then the set of maps*

$$A \hookrightarrow X \to X/A$$

induces the exact sequence in reduced K-theory:

$$\widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A),$$

which in turn yields a long exact sequence

$$\ldots \to \widetilde{K}(s(X/A)) \to \widetilde{K}(s(X)) \to \widetilde{K}(s(A)) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A).$$

Proof. If $B \subseteq Y$ and $B \subseteq Z$, we define the notation $Y \cup_A Z$ to mean the union of disjoint copies of Y and Z that are identified along B. Observe that if $c(Y)$ denotes the cone on Y, then $X \cup_A c(A)$ is homotopy equivalent to X/A. The first exact sequence comes from the sequence of inclusions $A \hookrightarrow X \hookrightarrow X \cup_A c(A)$. To get the long exact sequence, consider the sequence of inclusions:

$$A \hookrightarrow X \hookrightarrow X \cup_A c(A) \hookrightarrow X \cup_A c(A) \cup_X c(X) \hookrightarrow c(X) \cup_X c(X)$$

Note that as before $X \cup_A c(A)$ is homotopy equivalent to X/A, $X \cup_A c(A) \cup_X c(X)$ is homotopy equivalent to $s(A)$, and $c(X) \cup_X c(X) = s(X)$. Using the fact that $s(X/A) = s(X)/s(A)$, the result follows. $\square$

Corollary 3.11. *If* (A, X) *is a compact pair, then the K-theory groups form the following exact sequence:*

$$
\begin{array}{ccccc}
K^0(X/A) & \to & K^0(X) & \to & K^0(A) \\
\uparrow & & & & \downarrow \\
K^{-1}(A) & \leftarrow & K^{-1}(X) & \leftarrow & K^{-1}(X/A)
\end{array}
$$

If (A, X) is a compact pair of pointed spaces, then we have a similar exact sequence in reduced K-theory.

Proof. For every pointed compact Hausdorff space Y, $\widetilde{K}(s(Y)) = \widetilde{K}(S^1 \wedge Y) = \widetilde{K}^{-1}(Y)$. The reduced K-theory sequence follows from Bott periodicity and the lemma above. Replacing each space with its one-point compactification yields the K-theory exact sequence. $\square$

3.6. **Thom Isomorphism in K-theory.** In the following, if $V \to M$ is an even dimensional vector bundle over M that is spin^c, let $\mathbb{S}(V) = \mathbb{S}^+(V) \oplus \mathbb{S}^-(V)$ be the corresponding complex spinor bundles over M, and let c denote Clifford multiplication by vectors in V.

If $V \to M$ is any even dimensional vector bundle over M (not necessarily spin^c), let $\mathbb{C}l(V) = \mathbb{C}l^+(V) \oplus \mathbb{C}l^-(V)$ be the corresponding bundle of complex Clifford algebras, and again let c denote the Clifford multiplication by vectors in V.

Note that if $\pi : E \to X$ is an oriented real vector bundle and if $[\pi^*V, \pi^*W, \sigma]$ is a class in $K_{cpt}(E) = K(B(E), S(E))$ and if $[F] \in K(X)$, then the multiplication in $K_{cpt}(E)$ gives

$$[\pi^*V, \pi^*W, \sigma] \cdot \pi^*[F] = [\pi^*V \otimes F, \pi^*W \otimes F, c \otimes \mathbf{1}].$$

Theorem 3.12. *(Thom Isomorphism in K-theory) Suppose that $\pi : E \to X$ is an oriented real vector bundle of even dimension over a closed manifold X.*

(1) *If E is spin^c, then there is an (additive) isomorphism*

$$i_!^s : K(X) \to K_{cpt}(E)$$

given by

$$i_!^s(u) = s(E) \cdot \pi^*u,$$

where

$$\begin{aligned} s(E) &= \left[\pi^*\mathbb{S}^+(E), \pi^*\mathbb{S}^-(E), c\right] \\ &\in K_{cpt}(E) = K(B(E), S(E)). \end{aligned}$$

(2) *There is always an (additive) isomorphism*

$$i_!^\delta : K(X) \otimes \mathbb{Q} \to K_{cpt}(E) \otimes \mathbb{Q}$$

given by

$$i_!^\delta(u) = \delta(E) \cdot \pi^*u,$$

where

$$\begin{aligned} \delta(E) &= \left[\pi^*\mathbb{C}l^+(E), \pi^*\mathbb{C}l^-(E), c\right] \\ &\in K_{cpt}(E) = K(B(E), S(E)). \end{aligned}$$

Proof. By Corollary 3.9, the spin version of this proposition follows immediately from Bott periodicity if X is instead a contractible closed subset, because Clifford multiplication generates the compactly supported K-theory of $\mathbb{R}^{2m}$. Now since every closed manifold can be written as the union of a finite number of closed contractible sets, to prove the spin^c part it suffices to show that the result can be extended to a union of N contractible closed sets. We prove this using induction; we have already shown the case $N = 1$. Now, suppose the result is true for the base being a union of $N - 1$ or less contractible closed sets. Letting A denote the union of the first $N - 1$ sets and B denote the Nth set, we use the Mayer-Vietoris

sequence, and then note that multiplication by the restriction of $s(E)$ gives a morphism of exact sequences.

$$\begin{array}{ccccccccc}
... \to & K_{cpt}^{-i}(A \cup B) & \to & K_{cpt}^{-i}(A) \oplus K_{cpt}^{-i}(B) & \to & K_{cpt}^{-i}(A \cap B) & \to & K_{cpt}^{-i+1}(A \cup B) & \to ... \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
... \to & K_{cpt}^{-i}(E|_{A \cup B}) & \to & K_{cpt}^{-i}(E|_A) \oplus K_{cpt}^{-i}(E|_B) & \to & K_{cpt}^{-i}(E|_{A \cap B}) & \to & K_{cpt}^{-i+1}(E|_{A \cup B}) & \to ...
\end{array}$$

Using the Five-lemma and the induction hypothesis, we get the isomorphism on $A \cup B$, since multiplication by the restriction of $s(E)$ on the other pieces is an isomorphism by the induction hypothesis.

Next, if E is of real rank $2m$ over X but not spin^c, we do the same thing over a closed contractible set $C \subseteq X$ with the homomorphism that is multiplication by

$$\begin{aligned}
\delta(E|_C) &= \left[\mathbb{S}^+ \otimes \mathbb{S}^*, \mathbb{S}^- \otimes \mathbb{S}^*, c \otimes \mathbf{1}\right] \\
&= \left[\mathbb{S}^+, \mathbb{S}^-, c\right] \cdot [\mathbb{S}^*] \\
&= 2^m \left[\mathbb{S}^+, \mathbb{S}^-, c\right] \\
&= 2^m s(E|_C).
\end{aligned}$$

Thus, using the same argument, we obtain an isomorphism up to torsion. $\qquad\square$

3.7. **Dirac operators and Index Theory.** Note that if $L : \Gamma(M, E) \to \Gamma(M, F)$ is a zeroth order differential operator with principal symbol $\sigma(L)$, then if $\pi : T^*M \to M$ is the tangent bundle, then

$$[\pi^*E, \pi^*F, \sigma(L)]$$

represents a class in $K_{cpt}(T^*M)$. Furthermore, any two such operators represent the same class if and only if the principal symbols are stably homotopic.

Suppose M is a compact spin^c manifold, and let $P : \Gamma(M, E) \to \Gamma(M, F)$ be any elliptic pseudodifferential operator. Then $P_0 = (1 + P^*P)^{-1/2} P$ has the same index and same symbol class $\sigma \in K_{cpt}(TM)$, but it is zeroth order. By the Thom Isomorphism,

$$\sigma = s(TM) \cdot \pi^*u,$$

where $u \in K(M)$. Writing $u = [V_1 - V_2]$, we see

$$\begin{aligned}
\sigma &= \left[\pi^*\mathbb{S}^+(E) \otimes V_1, \pi^*\mathbb{S}^-(E) \otimes V_1, c \otimes \mathbf{1}\right] \\
&\quad - \left[\pi^*\mathbb{S}^+(E) \otimes V_2, \pi^*\mathbb{S}^-(E) \otimes V_2, c \otimes \mathbf{1}\right].
\end{aligned}$$

Thus, if D_V^+ denotes the spin^c Dirac operator twisted by the vector bundle V, we have that $P \oplus D_{V_1}^+$ is stably homotopic to $D_{V_2}^+$. Thus,

$$\begin{aligned}
\text{index}(P) &= \text{index}\left(D_{V_1}^+\right) - \text{index}\left(D_{V_2}^+\right) \\
&= \left\{(ch(V_2) - ch(V_2)) \cdot \widehat{A}(M)\right\}[M].
\end{aligned}$$

So the Atiyah-Singer Index Theorem reduces to a calculation of the index of Dirac operators.

If M is not spin^c, a similar calculation can be done with $\delta(TM)$ and the signature operator.

4. Dirac operators in Global Analysis

One important result concerning the Dirac operator is the Bochner-Lichnerowicz formula for the square of the Dirac operator.

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