CLIFFORD ALGEBRAS AND DIRAC OPERATORS

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1. Algebras

Definition 1.1. *Let K be a field. A ring R is a K-*algebra *if there exists a map $\cdot : K \times R \longrightarrow R$ that makes R into a K-vector space and has the property that $k \cdot (rs) = (k \cdot r)s = r(k \cdot s)$ for all k in K and r and s in R.*

Example 1.2. *For any field K, the ring $K[x]$ of polynomials in one variable over K is a K-algebra.*

Example 1.3. *For any field K and natural number n, the ring $\mathrm{M}(n, K)$ of $n \times n$ matrices with entries in K is a K-algebra.*

How can we take a K-vector space and enlarge it to become a K-algebra?

Definition 1.4. *Let V and W be vector spaces over a field K. The* tensor product *of V and W is the vector space $V \otimes_K W$ spanned by the set of* simple tensors

$$\{v \otimes w : v \in V, w \in W\},$$

subject to the relations

- $(v + \tilde{v}) \otimes w = v \otimes w + \tilde{v} \otimes w$
- $v \otimes (w + \tilde{w}) = v \otimes w + v \otimes \tilde{w}$
- $k(v \otimes w) = (kv) \otimes w = v \otimes (kw)$

for all v and $\tilde{v}$ in V, all w and $\tilde{w}$ in W, and k in K.

When the field K is clear from context, we will usually just write $V \otimes W$.

If $\{e_i : 1 \leq i \leq m\}$ and $\{f_j : 1 \leq j \leq n\}$ are bases for V and W respectively, the set $\{e_1 \otimes f_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $V \otimes W$.

Definition 1.5. *Let V be a vector space over a field K. For each positive integer i, let $\bigotimes^i V$ denote the tensor product of i copies of V. Set $\bigotimes^0 V = K$. The* tensor algebra *is the K-algebra*

$$\mathcal{T}(V) = \bigoplus_{i=0}^{\infty} \left(\bigotimes^i V \right),$$

where we make the identification that for any finite collection $v_1, v_2, \ldots, v_n$ of elements of V and any element k of K, we identify $k \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n$ with $kv_1 \otimes v_2 \otimes \cdots \otimes v_n$.

This is a very large algebra – it is infinite dimensional. We would like a way to take a K-vector space and construct a finite-dimensional algebra from it.

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2. Clifford Algebras

Definition 2.1. *Let V be a vector space over a field K. A* symmetric bilinear form *is a function $\langle\ ,\ \rangle : V \times V \longrightarrow K$ such that*

- $\langle v, w\rangle = \langle w, v\rangle$
- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
- $\langle kv, w\rangle = k\langle v, w\rangle$

for all k in K and u, v, and w in V.

Note that when K is the field of complex numbers, a symmetric bilinear form is **not** an inner product, because complex inner products are conjugate linear in the second variable, while a complex symmetric bilinear form is linear in the second variable.

Definition 2.2. *Let V be a vector space over a field K and suppose that V is equipped with a symmetric bilinear form $\langle\ ,\ \rangle$. Let $\mathcal{I}(V)$ be the ideal in $\mathcal{T}(V)$ generated by the set $\{v \otimes v + \langle v, v\rangle \mathbf{1} : v \in V\}$; here $\mathbf{1}$ denotes the multiplicative identity in $K = \bigotimes^0 V \subseteq \mathcal{T}(V)$. The* Clifford algebra *of V is the quotient algebra*

$$\mathrm{C}\ell(V) = \mathcal{T}(V)/\mathcal{I}(V).$$

Note that $\mathrm{C}\ell(V)$ depends on the choice of symmetric bilinear form!

Warning: there is some inconsistency in the Clifford algebra literature. Authors who are interested Clifford algebras and their application in mathematical analysis and/or physics tend to define $\mathrm{C}\ell(V)$ the way we have here. Authors who are more interested in the algebraic side of Clifford algebras tend to define $\mathcal{I}(V)$ as the ideal in $\mathcal{T}(V)$ generated by the set $\{v \otimes v - \langle v, v\rangle \mathbf{1} : v \in V\}$.

We denote multiplication in Clifford algebras by juxtaposition. Also, we use the map $k \mapsto k\mathbf{1}$, which turns out to be injective, to view K as a subalgebra of $\mathrm{C}\ell(V)$.

Example 2.3. *Let V be a one-dimensional vector space over $\mathbb{R}$ and fix any nonzero element e of V. Then $\{e\}$ is a vector space basis for V. Define a symmetric bilinear form on V by decreeing that $\langle e, e\rangle = 1$. Then $e^2 = -1$ in $\mathrm{C}\ell(V)$. Furthermore, each element of $\mathrm{C}\ell(V)$ can be uniquely written in the form $a + be$ for some real numbers a and b, and we have the following formulas for addition and multiplication:*

$$(a + be) + (c + de) = (a + c) + (b + d)e$$
$$(a + be)(c + de) = (ac - bd) + (ad + bc)e.$$

From these formulas it is not hard to see that $\mathrm{C}\ell(V) \cong \mathbb{C}$.

Example 2.4. *Let V be a one-dimensional vector space over $\mathbb{R}$ and fix any nonzero element e of V. Then $\{e\}$ is a vector space basis for V. Define a symmetric bilinear form on V by decreeing that $\langle e, e\rangle = -1$. Then $e^2 = 1$ in $\mathrm{C}\ell(V)$. Furthermore, each element of $\mathrm{C}\ell(V)$ can be uniquely written in the form $a + be$ for some real numbers a and b, and we have the following formulas for addition and multiplication:*

$$(a + be) + (c + de) = (a + c) + (b + d)e$$
$$(a + be)(c + de) = (ac + bd) + (ad + bc)e.$$

In this case, $\mathrm{C}\ell(V) \cong \mathbb{R}^2$ via the isomorphism $\phi(a + be) = \big(\frac{1}{2}(a + b), \frac{1}{2}(a - b)\big)$.

Example 2.5. *Let V be a one-dimensional vector space over $\mathbb{R}$ and fix any nonzero element e of V. Then $\{e\}$ is a vector space basis for V. Define a symmetric bilinear form on V by decreeing that $\langle e, e\rangle = 0$. Then $e^2 = 0$ in $\mathrm{C}\ell(V)$. Furthermore, each element of $\mathrm{C}\ell(V)$ can be uniquely written in the form $a + be$ for some real numbers a and b, and we have the following formulas for addition and multiplication:*

$$(a + be) + (c + de) = (a + c) + (b + d)e$$
$$(a + be)(c + de) = ac + (ad + bc)e.$$

In this case, $\mathrm{C}\ell(V) \cong \bigwedge \mathbb{R}$, the exterior algebra *of $\mathbb{R}$.*

More generally, if we define a symmetric bilinear form on V by setting $\langle e, e\rangle$ to be positive, then $\mathrm{C}\ell(V) \cong \mathbb{C}$; if $\langle e, e\rangle$ is negative, then $\mathrm{C}\ell(V) \cong \mathbb{R}^2$.

Given a symmetric bilinear form on a K-vector space V, define the associated *quadratic form* $q : V \longrightarrow K$ to be $q(v) = \langle v, v\rangle$ for each v in V. The definition of $\mathrm{C}\ell(V)$ only depends on q. We can recover the symmetric bilinear form $\langle\ ,\ \rangle$ from q by the *polarization identity*

$$\langle v, w\rangle = \frac{1}{2}\big(q(v + w) - q(v) - q(w)\big).$$

Theorem 2.6. *Let V be a n-dimensional $\mathbb{R}$-vector space equipped with a symmetric bilinear form $\langle\ ,\ \rangle$. There exist nonnegative integers r and s and a basis $\{e_i\}$ of V with the property that*

$$q(x_1e_1 + x_2e_2 + \cdots + x_ne_n) = x_1^2 + x_2^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2.$$

If $r + s = n$, we say that the quadratic form q is *nondegenerate*.

Definition 2.7. *For nonnegative integers r and s with $r + s = n$, let $\mathrm{C}\ell_{r,s}(\mathbb{R})$ denote the Clifford algebra on $\mathbb{R}^n$ equipped with the symmetric bilinear form*

$$q(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2.$$

Let $\{e_1, e_2, \ldots e_n\}$ denote the standard basis for $\mathbb{R}^n$. Then in $\mathrm{C}\ell_{r,s}(\mathbb{R})$,

$$\begin{aligned} e_ie_j + e_je_i &= (e_i + e_j)^2 - e_i^2 - e_j^2 \\ &= -q(e_1 + e_j) + q(e_i) + q(e_j) \\ &= -2\langle e_i, e_j\rangle \\ &= \begin{cases} -2\delta_{ij} & i \le r \\ 2\delta_{ij} & i > r. \end{cases} \end{aligned}$$

One consequence of the relation above is that $\mathrm{C}\ell_{r,s}(\mathbb{R})$, viewed as a $\mathbb{R}$-vector space, has basis

$$\{\mathbf{1}\} \cup \{e_i : 1 \le i \le n\} \cup \{e_ie_j : 1 \le i < j \le n\} \cup \{e_ie_je_k : 1 \le i < j < k \le n\} \cup \cdots \cup \{e_1e_2 \ldots e_n\},$$

whence $\mathrm{C}\ell_{r,s}(\mathbb{R})$ has vector space dimension 2^n.
We have already shown that $\mathrm{C}\ell_{1,0}(\mathbb{R}) \cong \mathbb{C}$ and that $\mathrm{C}\ell_{0,1}(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}$. We set $\mathrm{C}\ell_{0,0}(\mathbb{R}) = \mathbb{R}$. Here are some other isomorphisms:

$$\mathrm{C}\ell_{2,0}(\mathbb{R}) \cong \mathbb{H} \qquad \mathrm{C}\ell_{1,1}(\mathbb{R}) \cong \mathrm{M}(2, \mathbb{R}) \qquad \mathrm{C}\ell_{0,2}(\mathbb{R}) \cong \mathrm{M}(2, \mathbb{R}).$$

We are particularly interested in the Clifford algebras $\mathrm{C}\ell_{n,0}(\mathbb{R})$:

$$\begin{aligned}
&\mathrm{C}\ell_{0,0}(\mathbb{R}) \cong \mathbb{R}\\
&\mathrm{C}\ell_{1,0}(\mathbb{R}) \cong \mathbb{C}\\
&\mathrm{C}\ell_{2,0}(\mathbb{R}) \cong \mathbb{H}\\
&\mathrm{C}\ell_{3,0}(\mathbb{R}) \cong \mathbb{H} \oplus \mathbb{H}\\
&\mathrm{C}\ell_{4,0}(\mathbb{R}) \cong \mathrm{M}(2, \mathbb{H})\\
&\mathrm{C}\ell_{5,0}(\mathbb{R}) \cong \mathrm{M}(4, \mathbb{C})\\
&\mathrm{C}\ell_{6,0}(\mathbb{R}) \cong \mathrm{M}(8, \mathbb{R})\\
&\mathrm{C}\ell_{7,0}(\mathbb{R}) \cong \mathrm{M}(8, \mathbb{R}) \oplus \mathrm{M}(8, \mathbb{R})\\
&\mathrm{C}\ell_{8,0}(\mathbb{R}) \cong \mathrm{M}(16, \mathbb{R})
\end{aligned}$$

$$\mathrm{C}\ell_{n+8,0}(\mathbb{R}) \cong \mathrm{C}\ell_{n,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathrm{C}\ell_{8,0}(\mathbb{R}) \cong \mathrm{C}\ell_{n,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathrm{M}(16, \mathbb{R})$$

This last isomorphism shows that there is a sort of periodicity of order 8 for the Clifford algebras $\mathrm{C}\ell_{n,0}$.

Now let's look at complex Clifford algebras.

Definition 2.8. *Let V be an $\mathbb{R}$-vector space. The* complexification *of V is the set*

$$V \otimes_{\mathbb{R}} \mathbb{C} = \{v + wi : v, w \in V\}.$$

The set $V \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-vector space via the following operations: for all $v + wi$ and $\tilde{v} + \tilde{w}i$ in $V \otimes_{\mathbb{R}} \mathbb{C}$ and $x + yi$ in $\mathbb{C}$,

$$\begin{aligned}
(v + wi) + (\tilde{v} + \tilde{w}i) &= (v + \tilde{v}) + (w + \tilde{w})i\\
(x + yi)(v + wi) &= (xv - yw) + (xw + yv)i.
\end{aligned}$$

If A is a $\mathbb{R}$-algebra, then $A \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-algebra as well:

$$(a + bi)(\tilde{a} + \tilde{b}i) = (a\tilde{a} - b\tilde{b}) + (a\tilde{b} + \tilde{a}b)i$$

Let V be an $\mathbb{R}$-vector space equipped with a symmetric bilinear form $\langle\ ,\ \rangle$. Extend $\langle\ ,\ \rangle$ to $V \otimes_{\mathbb{R}} \mathbb{C}$ via bilinearity:

$$\langle v + wi, \tilde{v} + \tilde{w}i\rangle = \langle v, \tilde{v}\rangle + \langle v, \tilde{w}\rangle i + \langle w, \tilde{v}\rangle i - \langle w, \tilde{w}\rangle.$$

Then it is not hard to show that $\mathrm{C}\ell(V \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathrm{C}\ell(V) \otimes_{\mathbb{R}} \mathbb{C}$ as $\mathbb{C}$-algebras.
Suppose that the quadratic form q on V associated with our original symmetric bilinear form is nondegenerate. Then, as we discussed earlier, there exist a nonnegative integer r and a basis $\{e_i\}$ of V with the property that

$$q(x_1e_1 + x_2e_2 + \cdots + x_ne_n) = x_1^2 + x_2^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2$$

for all real numbers $x_1, x_2, \ldots x_n$. By our construction, the set $\{e_i\}$ is also a basis for $V \otimes_{\mathbb{R}} \mathbb{C}$. Let $q_{\mathbb{C}} : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C}$ be the quadratic form associated to the symmetric bilinear form on $V \otimes_{\mathbb{R}} \mathbb{C}$ we defined above. Then

$$q_{\mathbb{C}}(z_1e_1 + z_2e_2 + \cdots + z_ne_n) = z_1^2 + z_2^2 + \cdots + z_r^2 - z_{r+1}^2 - \cdots - z_n^2$$

for all complex numbers $z_1, z_2, \ldots z_n$. But $e_1, e_2, \ldots, e_r, ie_{r+1}, \ldots ie_n$ is also a basis for $V \otimes_{\mathbb{R}} \mathbb{C}$, and this basis makes all the signs above positive! This implies the following:

$$C\ell_{n,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong C\ell_{n-1,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong C\ell_{n-2,2}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cdots \cong C\ell_{0,n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

In light of this, we denote all of these algebras by $C\ell_n(\mathbb{C})$. We have

$$C\ell_0(\mathbb{C}) \cong \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}, \qquad C\ell_1(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2;$$

one can define an isomorphism $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ by setting

$$\phi(1,0) = \frac{1}{2}(1 \otimes 1 + i \otimes i), \qquad \phi(0,1) = \frac{1}{2}(1 \otimes 1 - i \otimes i)$$

and extending linearly. Also,

$$C\ell_2(\mathbb{C}) \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{M}(2, \mathbb{C}).$$

In addition

$$C\ell_{n+2}(\mathbb{C}) \cong C\ell_n(\mathbb{C}) \otimes_{\mathbb{C}} C\ell_2(\mathbb{C}) \cong C\ell_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{M}(2, \mathbb{C})$$

for all $n \geq 0$. Thus we have

$$\begin{aligned}
C\ell_0(\mathbb{C}) &\cong \mathbb{C} \\
C\ell_1(\mathbb{C}) &\cong \mathbb{C} \oplus \mathbb{C} \\
C\ell_2(\mathbb{C}) &\cong \mathrm{M}(2, \mathbb{C}) \\
C\ell_3(\mathbb{C}) &\cong \mathrm{M}(2, \mathbb{C}) \oplus \mathrm{M}(2, \mathbb{C}) \\
C\ell_4(\mathbb{C}) &\cong \mathrm{M}(4, \mathbb{C}) \\
C\ell_5(\mathbb{C}) &\cong \mathrm{M}(4, \mathbb{C}) \oplus \mathrm{M}(4, \mathbb{C}) \\
C\ell_6(\mathbb{C}) &\cong \mathrm{M}(8, \mathbb{C}) \\
C\ell_7(\mathbb{C}) &\cong \mathrm{M}(8, \mathbb{C}) \oplus \mathrm{M}(8, \mathbb{C}) \\
C\ell_8(\mathbb{C}) &\cong \mathrm{M}(16, \mathbb{C}).
\end{aligned}$$

In general,

$$C\ell_n(\mathbb{C}) \cong \begin{cases} \mathrm{M}(2^{n/2}, \mathbb{C}) & n \text{ even} \\ \mathrm{M}(2^{(n-1)/2}, \mathbb{C}) \oplus \mathrm{M}(2^{(n-1)/2}, \mathbb{C}) & n \text{ odd} \end{cases}$$

3. Representations of Clifford Algebras

Definition 3.1. *Let W be a K-vector space. We let* $\mathrm{Hom}(W, W)$ *denote the set of K-vector space homomorphisms (a.k.a. K-linear maps) from W to W. The set* $\mathrm{Hom}(W, W)$ *is a K-algebra under pointwise addition and composition.*

Definition 3.2. *A* representation *of a K-algebra A on a K-vector space W is a K-algebra homomorphism $\rho : A \longrightarrow$* $\mathrm{Hom}(W, W)$.

While one can consider cases where W is infinite dimensional, we will always assume that W is finite dimensional.

A representation of a K-algebra A on a K-vector space W makes W into an A-module: $a \cdot w := \rho(a)(w)$. When A is a Clifford algebra, we call this module action *Clifford multiplication.*

Definition 3.3. *A representation of a K-algebra A on a K-vector space W is* reducible *if there exists subspaces W_1 and W_2 of W with the properties that $W \cong W_1 \oplus W_2$ and that $\rho(a)$ maps W_1 to W_1 and W_2 to W_2 for all a in A. A representation that is not reducible is called* irreducible.

In the case described in the preceding definition, we can decompose ρ as a direct sum of representation $\rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are representations of A on W_1 and W_2 respectively.

Theorem 3.4. *Every (finite dimensional) representation of a K-algebra can be expressed as a direct sum of irreducible representations.*

In light of the preceding theorem, we see that to understand representations of an algebra, we need only focus on the irreducible representations. We also want to consider some representations as being "the same".

Definition 3.5. *Let ρ and $\widetilde{\rho}$ be representations of a K-algebra A on K-vector spaces W and $\widetilde{W}$ respectively. We say that ρ and $\widetilde{\rho}$ are* equivalent *if there exists a vector space isomorphism $F : W \longrightarrow \widetilde{W}$ such that $\widetilde{\rho}(a)(\tilde{w}) = (F\rho(a)F^{-1})(\tilde{w})$ for all $\tilde{w}$ in $\widetilde{W}$ and a in A.*

The next theorem shows that there are not very many irreducible representations of complex Clifford algebras. Recall from linear algebra that $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \cong \mathrm{M}(n, \mathbb{C})$ for each natural number n.

Theorem 3.6. *For each natural number n, the only irreducible representation (up to equivalence) of $\mathrm{M}(n, \mathbb{C})$ is the obvious representation of $\mathrm{M}(n, \mathbb{C})$ on $\mathbb{C}^n$. The algebra $\mathrm{M}(n, \mathbb{C}) \oplus \mathrm{M}(n, \mathbb{C})$ has two equivalence classes of irreducible representations: $\rho_i : \mathrm{M}(n, \mathbb{C}) \oplus \mathrm{M}(n, \mathbb{C}) \longrightarrow \mathrm{M}(n, \mathbb{C})$, $i = 1, 2$, defined by $\rho_1(A_1, A_2) = A_1$ and $\rho_1(A_1, A_2) = A_2$.*

Let's write down these irreducible representations. First consider the case $n = 2m$. Then our irreducible representation of $\mathrm{C}\ell_n(\mathbb{C})$ is on the vector space $\mathbb{C}^{2^m}$; In other words, we have an algebra homomorphism ϕ_n from $\mathrm{C}\ell_n(\mathbb{C})$ to $\mathrm{M}(2^m, \mathbb{C})$.

When $m = 1$, we define $\phi_2 : \mathrm{C}\ell_2(\mathbb{C}) \longrightarrow \mathrm{M}(2, \mathbb{C})$ by decreeing that

$$\phi_2(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \phi_2(e_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $\{e_1, e_2\}$ is the standard basis for $\mathbb{C}^2$. From here, we proceed inductively. Suppose we know $\phi_n : \mathrm{C}\ell_n(\mathbb{C}) \longrightarrow \mathrm{M}(2^m, \mathbb{C})$ and let $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}\}$ be the standard basis for $\mathbb{C}^{n+2}$. Set

$$\phi_{n+2}(e_k) = \begin{pmatrix} 0 & \phi_n(e_k) \\ \phi_n(e_k) & 0 \end{pmatrix}$$

for $1 \leq k \leq n$. Then define

$$\phi_{n+2}(e_{n+1}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

and

$$\phi_{n+2}(e_{n+2}) = \begin{pmatrix} 0 & 0 & iI & 0 \\ 0 & 0 & 0 & -iI \\ iI & 0 & 0 & 0 \\ 0 & -iI & 0 & 0 \end{pmatrix}.$$

By counting dimensions, we see that each ϕ_n is an algebra isomorphism when n is even.

Now let's look at the case where $n = 2m + 1$. An irreducible representation of $C\ell_n(\mathbb{C})$ is an algebra homomorphism into $\mathrm{M}(2^m, \mathbb{C})$. For $m = 0$, we have $\phi_1(e_1) = -i$. When $m > 0$, take the standard basis $\{e_1, e_2, \ldots, e_n\}$, define $\phi_n(e_k) = \phi_{n-1}(e_k)$ for $1 \leq k \leq n$, and then set

$$\phi_n(e_n) = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}.$$

4. Dirac operators

We are going to use Clifford algebras and their representations to define some very important partial differential operators. We begin with PDOs acting on $C_c^\infty(\mathbb{R}^n)$, the collection of smooth complex-valued compactly supported functions on $\mathbb{R}^\ltimes$. For each $1 \leq k < n$, let $\frac{\partial}{\partial x_k}$ denote partial differentiation in the direction of x_k. For each natural number n, we define a matrix operator

$$D = \sum_{k=1}^n E_k \frac{\partial}{\partial x_k},$$

where $E_k = \phi_n(e_k)$ in the notation of the previous section.
Let's look at some examples. When $n = 1$,

$$D = -i \frac{d}{dx}.$$

For $n = 2$,

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x_2} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & 0 \end{pmatrix}.$$

When $n = 3$,

$$\begin{aligned} D &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x_2} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \frac{\partial}{\partial x_3} \\ &= \begin{pmatrix} -i\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & i\frac{\partial}{\partial x_3} \end{pmatrix}. \end{aligned}$$

When $n = 4$, we get

$$D = \begin{pmatrix} 0 & 0 & -\frac{\partial}{\partial x_3} + i\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} - i\frac{\partial}{\partial x_4} \\ \frac{\partial}{\partial x_3} + i\frac{\partial}{\partial x_4} & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & 0 & 0 \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} - i\frac{\partial}{\partial x_4} & 0 & 0 \end{pmatrix}$$

Note that when we square each of these matrices, we get a matrix with the Laplacian on the diagonal. This is not a coincidence:

$$\begin{aligned} D^2 &= \left(\sum_{j=1}^n E_j \frac{\partial}{\partial x_j} \right) \left(\sum_{k=1}^n E_j \frac{\partial}{\partial x_k} \right) \\ &= \left(\sum_{j=k} E_k \frac{\partial^2}{\partial x_k^2} \right) \left(\sum_{j<k} (E_j E_k - E_j E_k) \frac{\partial^2}{\partial x_k \partial x_j} \right) \\ &= -\sum_{k=1}^n I_n \frac{\partial^2}{\partial x_k^2}. \end{aligned}$$