1. Characteristic Classes from the viewpoint of Operator Theory

2. Introduction

Overarching Question: How can you tell if two vector bundles over a manifold are isomorphic?

Let X be a compact Hausdorff space. There is a category equivalence between vector bundles over X and idempotents $\{E : E^2 = E\}$ in $M(C(X)) = \varinjlim M(n, C(X))$, the set of infinite matrices with entries in $C(X)$ (complex-valued continuous functions), with all but finitely many entries zero. The inclusion is

$$\begin{array}{ccc} A & \hookrightarrow & \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \\ M(n, C(X)) & \hookrightarrow & M(n+1, C(X)) . \end{array}$$

The equivalence is

$$\begin{array}{rcl} V & = & \{Range(E_x)\}_{x \in X} \longleftrightarrow E \ . \\ V & \subset & X \times \mathbb{C}^n, E \subset M(n, C(X)) . \end{array}$$

Note that every vector bundle is a subbundle of a trivial bundle. The equivalence relation (isomorphism between categories) is similarity (or homotopy). The addition is

$$[E] + [F] = \left[\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \right] .$$

Note that $\Gamma(E)$ a (projective) module over $C(X)$, and in fact projective modules are in one-to-one correspondence with vector bundles.

3. Chern-Weil Theory

How can you tell if idempotents over X are similar?

3.1. Invariant Polynomials.

A complex-valued polynomial $P : M(n, \mathbb{C}) \to \mathbb{C}$ is **invariant** if $P(SAS^{-1}) = P(A)$ for all $A \in M(n, \mathbb{C})$, $S \in GL(n, \mathbb{C})$. Examples include: If

$$\det(I + xA) = 1 + c_1(A)x + ... + c_n(A)x^n$$

Each c_j is an invariant polynomial. For example, $c_1(A) = Tr(A)$, $c_n(A) = \det(A)$.

Theorem 1. *The ring of invariant polynomials is generated by* $\{c_j(A)\}$.

Let $E \in M(n, C^{\infty}(X))$ be an idempotent. Define

$$D : (\Omega^*(X))^n = \Omega^*(X) \oplus \ldots \oplus \Omega^*(X) \to (\Omega^*(X))^{n+1}$$

by

$$D(\omega_1, \omega_2, \ldots, \omega_n) = (d\omega_1, \ldots, d\omega_n).$$

Define

$$\nabla_E = EDE : (\Omega^*(X))^n \to (\Omega^*(X))^{n+1}.$$

Important point: ∇_E is not C^{∞}-linear, but ∇_E^2 is. In fact,

$$\nabla_E^2 = E(dE)^2 = E(dE) \wedge (dE).$$

(Note $DE = D \circ E$, $dE = d$ applied to entries of E).

Theorem 2. ***(Chern-Weil)*** *If P is an invariant polynomial, then $P(\nabla_E^2)$ is a closed form whose de Rham class only depends on the similarity class of E.*

Example 3. *We have $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Let*

$$E = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix} \in M\left(2, C^{\infty}\left(S^2\right)\right)$$

Then

$$dE = \frac{1}{2}\begin{pmatrix} dx & dy+idz \\ dy-idz & -dx \end{pmatrix}$$

$$\begin{aligned} (dE)^2 &= \frac{1}{2}\begin{pmatrix} dx & dy+idz \\ dy-idz & -dx \end{pmatrix} \wedge \frac{1}{2}\begin{pmatrix} dx & dy+idz \\ dy-idz & -dx \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -idy \wedge dz & dx \wedge dy + idx \wedge dz \\ -dx \wedge dy + idx \wedge dz & idy \wedge dz \end{pmatrix} \end{aligned}$$

So

$$E(dE)^2 = \frac{1}{4}\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix},$$

where each α_{ij} are two-forms. Note that there are no nontrivial 4-forms on the sphere, so we have

$$\det\left(1 + \nabla_E^2\right) = 1 + c_1\left(\nabla_E^2\right) + 0\ldots$$

So

$$\begin{aligned} c_1\left(\nabla_E^2\right) &= \frac{1}{4}(\alpha_{11} + \alpha_{22}) \\ &= -\frac{i}{2}(zdxdy - ydxdz + xdydz). \end{aligned}$$

We integrate this over the sphere to get

$$\begin{aligned} & -\frac{i}{2}\int_{S^2}(zdxdy - ydxdz + xdydz) \\ = & -\frac{i}{2}\int_{B^3} 3dzdxdy \ \textit{(Stokes)} \\ = & -\frac{i}{2}3\frac{4}{3}\pi = -2\pi i \neq 0. \end{aligned}$$

So this bundle is nontrivial. Note that if it is a trivial idempotent, so the Chern numbers would be zero.

4. Differential K-theory

Differential K-theory is a refinement of K-theory that takes into account differentiable structures. First we consider ordinary K-theory and what is new in the J. Simons and D. Sullivan theory.

Let X be a compact smooth manifold, and let $\mathrm{Vect}(X)$ be the set of isomorphism classes of smooth complex vector bundles over X. Note that each continuous vector bundle has a smooth structure. Let $K^0(X)$ be the Grothendieck completion of the abelian monoid $\mathrm{Vect}(X)$. Every element of $K^0(X)$ can be written as $[V] - [\theta^n]$, where θ^n is the trivial n-dimensional vector bundle.

Next, let ∇ be a connection on a vector bundle V over X. We define

$$ch(\nabla) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2\pi i}\right)^k Tr(R\wedge ... \wedge R) \in \Omega^{even}(X),$$

where R is the curvature of ∇. The class $[ch(\nabla)] \in H^{even}_{dR}(X) = H^{even}_{dR}(X;\mathbb{C})$ only depends on the isomorphism class of V. One can show that the Chern character extends to a homomorphism

$$ch : K^0(X) \to H^{even}_{dR}(X),$$

and

$$ch \otimes \mathbf{1} : K^0(X)\otimes\mathbb{C} \to H^{even}_{dR}(X).$$

is an isomorphism.

Consider a smooth path $\gamma_t = \nabla_t$ of connections in V. Let $A_t = \dot{\nabla}_t \in \Omega^1(X, \mathrm{End}(V))$, and define

$$cs(\gamma) = \int_0^1 \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\frac{1}{2\pi i}\right)^k Tr(A_t\wedge R_t \wedge ... \wedge R_t)\, dt \in \Omega^{odd}_{\mathbb{C}}(X).$$

Then

$$d\ cs(V) = ch(\nabla_1) - ch(\nabla_0).$$

Important Fact: If ∇_0, ∇_1 are two connections, then $(1-t)\nabla_0 + t\nabla_1$ is a connection for $0 \le t \le 1$.

Theorem 4. *If γ and α are smooth paths from ∇_0 to ∇_1, then*

$$cs(\alpha) = cs(\gamma) + \text{ exact form.}$$

Define

$$CS(\nabla_0, \nabla_1) = cs(\gamma) \operatorname{mod}(\text{exact forms}).$$

We say that ∇_0 are ∇_1 are **equivalent** if $CS(\nabla_0, \nabla_1) = 0$. This is an equivalence relation, which can be shown using these properties: $CS(\nabla_0, \nabla_0) = 0$, and $CS(\nabla_0, \nabla_1) = -CS(\nabla_1, \nabla_0)$, and $CS(\nabla_0, \nabla_1) + CS(\nabla_1, \nabla_2) = CS(\nabla_0, \nabla_2)$. A pair $(V, [\nabla])$ is called a **structured vector bundle**. Isomorphism: if $\phi : V \to W$ is a bundle isomorphism, then we say that $(W, [\nabla]) \sim (V, [\phi^* \nabla])$. Addition: Direct Sum (Note $CS\left(\nabla_V \oplus \nabla_W, \widetilde{\nabla_V} \oplus \widetilde{\nabla_W}\right) = CS\left(\nabla_V, \widetilde{\nabla_V}\right) + CS\left(\nabla_W, \widetilde{\nabla_W}\right)$.)

Let $\widehat{K^0}(X) =$ Grothendieck completion of this abelian monoid. Every element of $\widehat{K^0}(X)$ can be written in the form

$$(V, [\nabla_V]) - (\theta^n, [d]).$$

There is a map

$$\delta : \widehat{K^0}(X) \to K^0(X)$$

given by

$$\delta\left((V, [\nabla_V]) - (\theta^n, [d])\right) = [V] - [\theta^n].$$

Is there a nontrivial kernel of this map δ? Let $GL(\mathbb{C}) = \varinjlim GL(n, \mathbb{C})$. Let $\theta = A^{-1}dA \in \Omega^1(GL(\mathbb{C}), M(\mathbb{C}))$. Define

$$\Theta = \sum_{k=1}^{\infty} b_k Tr(\theta \wedge \ldots \wedge \theta), \ (2k-1 \text{ form}),$$

where

$$b_k = \frac{1}{(k-1)!}\left(\frac{1}{2\pi i}\right)^k \int_0^1 \left(t^2 - t\right)^{k-1} dt.$$

This generates the cohomology of $GL(\mathbb{C})$. Define the abelian group

$$\Lambda_{GL(\mathbb{C})}(X) = \left\{ g^*\theta + \Omega^{odd}_{exact}(X) : g : X \to GL(\mathbb{C}) \text{ smooth} \right\}.$$

Proposition 5. *If ∇ and $\widetilde{\nabla}$ are flat connections on a trivial bundle, then $CS\left(\nabla, \widetilde{\nabla}\right) \in \Lambda_{GL(\mathbb{C})}(X) \operatorname{mod}(\text{exact forms})$.*

Proof. We can write

$$\begin{aligned}\widetilde{\nabla} &= \nabla + g^{-1}dg = \nabla + g^{*}(\theta) \\ &\ldots \\ CS\left(\nabla, \widetilde{\nabla}\right) &= g^{*}(\Theta).\end{aligned}$$

□

Theorem 6. *There exists a short exact sequence*

$$0 \to \Omega^{odd}(X) \diagup \Lambda_{GL}(X) \to \widehat{K^{0}}(X) \to K^{0}(X) \to 0.$$