1. Some context.
   - Banach-Tarski (1924): measure theory
   - von Neumann (1929): first definition

2. Def. and permanence properties.
   \[ \Gamma = \text{discrete, countable grp.} \]

**Def.**
- A mean for \( \Gamma \) on \( L^\infty(\Gamma) \) is a linear functional
  \[ \delta : L^\infty(\Gamma) \to \mathbb{C} \]
  that is positive and unital.
  (Means are automatically bounded; c.f. states on C*-alg's.)
- A mean \( \delta \) is left invariant if \( \delta(\sigma f) = \delta(f) \)
  \( \forall \sigma \in \Gamma \forall f \in L^\infty(\Gamma) \). (Here \( \sigma f = f(\sigma^{-1}) \).)
\[ \Gamma \text{ is amenable if } \exists \text{ a left invariant mean for } \Gamma \text{ on } \ell^\infty(\Gamma). \]

**Ex.** \( \mathbb{Z} \) is amenable.

For this, recall that \( \ell^1(\mathbb{Z}) \to (\ell^\infty(\mathbb{Z}))^* \) by \( g \mapsto \hat{g} \), where, writing \( f = \sum_{s=-\infty}^{\infty} a_s \delta_s \),

\[ \hat{g}(f) = f(g) = \sum_{s=-\infty}^{\infty} a_s f(s). \]

Let \( P(\mathbb{Z}) = \left\{ \sum_{s=-\infty}^{\infty} a_s \delta_s : a_s \geq 0 \forall s \text{ and } \sum_{-\infty}^{\infty} a_s = 1 \right\} \).

This is the set of all \( g \in \ell^1(\mathbb{Z}) \) s.t. \( \hat{g} \) is a mean on \( \ell^\infty(\mathbb{Z}) \). Want to find an "increasingly invariant" seq. in \( P(\mathbb{Z}) \) and use the fact that the unit ball in \( (\ell^\infty(\mathbb{Z}))^* \) is compact in the weak-* topology (Alaoglu).

A cluster pt. of the seq. will be our left-inv. mean.
Just take
\[ \widehat{g}_n = \frac{1}{2n+1} \sum_{s=-n}^{n} g_s. \]

Then \[ |\widehat{g}_n(sf) - \widehat{g}_n(f)| \leq \frac{1}{2n+1} 2s \| f \|_{\infty}. \]

Thus any cluster pt. of \( (\widehat{f}_n) \) does the job. \( \square \)

**Proposition.**

(i) \( |\Gamma| < \infty \Rightarrow \Gamma \) is amenable;

(ii) \( \Gamma \) is amenable \( \Rightarrow \) every quotient and subgroup of \( \Gamma \) is amenable;

(iii) \( N \triangleleft \Gamma \) and both \( N \) and \( \Gamma/N \) are amenable \( \Rightarrow \)
\( \Rightarrow \) \( \Gamma \) is amenable;

(iv) \( \Gamma, \Lambda \) are amenable \( \Rightarrow \) \( \Gamma \times \Lambda \) is amenable;

(v) an increasing union of amenable groups is amenable;

(vi) \( \Gamma \) is locally amenable \( \Rightarrow \) \( \Gamma \) is amenable
\( \uparrow \)
\( \text{(every f.g. subgroup . . . . )} \)

(To be proved.)
Corollary. \( \Gamma = \text{abelian} \Rightarrow \Gamma = \text{amenable} \)

(N.B. We're claiming this for \( \Gamma = \text{discrete + countable} \),
but it's true in general.)

Def. The class of elementary amenable groups is
the smallest class containing finite \& abelian
groups, the finite groups and the abelian groups that is
closed under taking subgrps., quotients, extensions,
and direct limits.

Theorem (Grigorchuk). There are amenable grps. that
are not elementary amenable.