1. Introduction

Let $A$ be a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$, representing a physical quantity. Let $\Sigma$ be the spectrum of $A$. Let $E : \mathcal{B}(\Sigma) \rightarrow \mathcal{P}(\mathcal{H})$ be the spectral measure (Borel measurable subsets, $\mathcal{P}$ projections). The map $E(X)$ is a property, $X \in \mathcal{B}(\Sigma)$. Each normal state is defined by a density operator $\rho_0$ on $\mathcal{H}$, which is positive, trace-class, $\|\rho_0\|_1 = 1$.

Let $\omega_{\rho_0} : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ be the function defined by

$$\omega_{\rho_0}(B) = Tr(\rho_0 B).$$

The trace-class ideal $\mathcal{I}^1(\mathcal{H}) = \{\text{trace class operators}\}$. In general, the trace-class norm $\|A\|_1$ of a trace-class operator $A$ is equal to $Tr(|A|)$. The probability wrt $\rho_0$ of $E(X)$ is

$$\mathcal{P}^1_{\rho_0}[E(X)] = \omega_{\rho_0}(E(X)),$$

the probability of that the measurement of the observable $A$ will be in $X$.

Let $A_1, \ldots, A_n$ be observables with corresponding spectra $\Sigma_1, \ldots, \Sigma_n$ and spectral measures $E_1, \ldots, E_n$ and a sequence of properties $E_1(X_1), \ldots, E_n(X_n)$ corresponding to times $t_1 < \ldots < t_n$. The finite sequence $(E_1(X_1), \ldots, E_n(X_n))$ is called a history. Think of this as an apparatus that makes measurements. Another proposal is that we should think of a history as a product

$$\bar{\approx} = E_1(X_1) \odot \ldots \odot E_n(X_n),$$

where $\odot$ has the algebraic properties of the tensor product. We simplify $F_j = E_j(X_j)$. In the physics literature, the nth order probability of the history $\bar{\approx}$ is defined to be

$$\mathcal{P}^n_{\rho_0}(\bar{\approx}) = Tr(F_n F_{n-1} \ldots F_1 \rho_0 F_1 \ldots F_{n-1}).$$

For example, if

$$\bar{\approx}' = F_1 \odot F_2,$$

then

$$\mathcal{P}^2_{\rho_0}(\bar{\approx}') = Tr(F_2 F_1 \rho_0 F_1).$$
so then
\[
\rho_0 (X_1) = \frac{F_1 \rho_0 F_1}{\| F_1 \rho_0 F_1 \|} = \frac{F_1 \rho_0 F_1 }{Tr (F_1 \rho_0 F_1) } = \frac{F_1 \rho_0 F_1 }{Tr (\rho_0 F_1) } = \frac{F_1 \rho_0 F_1 }{P_1 (F_1) }
\]
is a density operator. Then
\[
\mathcal{P}^2_{\rho_0} \left( \sim' \right) = Tr (F_2 \rho_0 (X_1)) \mathcal{P}^1_{\rho_0} (F_1) = \mathcal{P}^1_{\rho_0 (X_1)} (F_2) \mathcal{P}^1_{\rho_0} (F_1) ,
\]
the first factor of which is like a conditional probability.

Let
\[
\sim' = F_1 \odot F_2
\]
Think of \( X_1 \times X_2 \) as a product, with \( X_1 = X^1_1 \cup X^2_1 \). Then
\[
E_1 (X_1) = E_1 (X^1_1) \lor E_1 (X^2_1) = F^1_1 \lor F^2_1 ,
\]
so that
\[
\sim' = (F^1_1 \odot F_2) \lor (F^2_1 \odot F_2) = \sim_1 \lor \sim_2
\]
Is it true that
\[
\mathcal{P}^2_{\rho_0} \left( \sim' \right) = \mathcal{P}^2_{\rho_0} \left( \sim_1 \lor \sim_2 \right) = \mathcal{P}^2_{\rho_0} \left( \sim_1 \right) + \mathcal{P}^2_{\rho_0} \left( \sim_2 \right) ?
\]
Well, not generally, because
\[
\mathcal{P}^2_{\rho_0} \left( \sim' \right) = Tr \left( F_2 \left( F^1_1 \lor F^2_1 \right) \rho_0 \left( F^1_1 \lor F^2_1 \right) \right) = Tr \left( F_2 F^1_1 \rho_0 F^1_1 \right) + Tr \left( F_2 F^2_1 \rho_0 F^2_1 \right) = \mathcal{P}^2_{\rho_0} \left( \sim_1 \right) + \mathcal{P}^2_{\rho_0} \left( \sim_2 \right) + Tr \left( F_2 F^1_1 \rho_0 F^1_1 \right) + Tr \left( F_2 F^2_1 \rho_0 F^2_1 \right) .
\]
The **consistency condition** for second order histories is that
\[
Tr \left( F_2 F^1_1 \rho_0 F^1_1 \right) + Tr \left( F_2 F^2_1 \rho_0 F^2_1 \right) = 0
\]
for every measurable decomposition of \( X_1 \) as \( X^1_1 \cup X^2_1 \).

We will consider \( A_1, \ldots, A_n \) observables, \( \mathcal{A}_1, \ldots, \mathcal{A}_n \) maximal abelian von Neumann algebras. Each of these has Gel’fand spectrum \( S_1, \ldots, S_n \) and is endowed with a unique spectral integral
\[
P_i : \mathcal{B} (S_i) \to \mathcal{A}_i
\]
and spectral measure
\[
E_i : \mathcal{B} (S_i) \to \mathcal{P} (\mathcal{A}_i) .
\]
Then
\[ A^n_1 : = A_1 \otimes \cdots \otimes A_n \]
\[ S^n_1 : = S_1 \times \cdots \times S_n \]
\[ P^n_1 : = P_1 \otimes \cdots \otimes P_n \]
\[ E^n_1 : = E_1 \otimes \cdots \otimes E_n \]

Then
\[ E^n_1 (Y^n_1) = E_1 (Y_1) \circ \cdots \circ E_n (Y_n) = F_1 \circ \cdots \circ F_n = F^n_1 \]

Then
\[ \mathcal{P}^n_{\rho_0} [F^n_1] = Tr (F_n F_{n-1} \cdots F_1 \rho_0 F_1 \cdots F_{n-1}) = \omega^n_{\rho_0} (F_1) Tr (F_n F_{n-1} \cdots F_2 \rho_0 (Y_1) F_2 \cdots F_{n-1}) = \omega^n_{\rho_0} (F_1) \omega^n_{\rho_0 (Y_1)} (F_2) \cdots \omega^n_{\rho_0 (Y^n_{1-1})} (F_n) = \omega^n_{\rho_0} (Y^n_{1-1}) [E^n_1 (Y^n_1)] = (\omega^n_{\rho_0} (Y^n_{1-1}) \circ E^n_1) [Y^n_1] = \mu^n_{\rho_0} [Y^n_{1-1}] (Y^n_1). \]

**Notation and Summary of Results for Lecture 2**

1. Elementary history: \( \mathcal{E}^n_1 (Y^n_1) = F^n_1 \)

2. \( k \)th initial density operator: \( \rho_k = \rho_0 (Y_k^1) := \frac{E_k (Y_k) \rho_0 (Y^k_{1-1}) E_k (Y_k)}{Tr \{ E_k (Y_k) \rho_0 (Y^k_{1-1}) \}} \)

3. History state with respect to \( \rho_0 \) generated by \( Y^{n-1}_1 \):
   \[ \omega^n_{\rho_0} (Y^{n-1}_1) = \omega^k_{\rho_0} (Y^{k-1}_1) \otimes \omega_{\rho_k (Y_{k+1})} \otimes \cdots \otimes \omega_{\rho_k (Y^{n-1}_{k+1})} = \omega^k_{\rho_0} (Y^{k-1}_1) \otimes \omega^{n-k}_{\rho_k (Y^{n-1}_{k+1})} \]

4. History measure with respect to \( \rho_0 \) generated by \( Y^{n-1}_1 \):
   \[ \mu^n_{\rho_0} [Y^{n-1}_1] = \mu^k_{\rho_0} [Y^{k-1}_1] \otimes \mu_{\rho_k (Y_{k+1})} \otimes \cdots \otimes \mu_{\rho_k (Y^{n-1}_{k+1})} = \mu^k_{\rho_0} [Y^{k-1}_1] \otimes \mu^{n-k}_{\rho_k (Y^{n-1}_{k+1})} \]

**Theorem 3.1.** The \( n \)th order probability of \( \mathcal{E}^n_1 (Y^n_1) \) with respect to \( \rho_0 \) is given by
\[ p^n_{\rho_0} [F^n_1] = \omega^n_{\rho_0 (Y^{n-1}_1)} [F^n_1] = \mu^n_{\rho_0} [Y^{n-1}_1] (Y^n_1). \]

5. Measurable decomposition: \( Y_k = Y^1_k \sqcup Y^2_k \) for any \( k, 1 \leq k < n \).

6. For \( i = 1, 2 \):
   \[ Y^{m}_{1} ([ik]) := Y^{k-1}_{1} \times Y^{i}_{k} \times Y^{m}_{k+1}, \quad F^{m}_{1} (ik) = \mathcal{E}^m_{1} (Y^{m}_{1} (ik)) = F^{k-1}_{1} \times F^{i}_{1} \times F^{m}_{k+1}, \]
   for \( k < m \leq n \).

   \[ \rho^i_k := \frac{E_k (Y^i_k) \rho_0 (Y^{k-1}_1) E_k (Y^i_k)}{Tr \{ E_k (Y^i_k) \rho_0 (Y^{k-1}_1) \}} = \frac{F^i_k \rho_0 (Y^{k-1}_1) F^i_k}{Tr \{ F^i_k \rho_0 (Y^{k-1}_1) \}}. \] (7)
Definition. An elementary history $E^n_1(Y^n)$ is consistent with respect to a density operator $\rho_0$ if the probability is additive. That is, for each binary measurable decomposition $Y_k = Y^n_k \sqcup Y^n_{k+1}$ for any $k$, $1 \leq k \leq n$,

$$p^n_{\rho_0}[F^n_1] = \omega^n_{\rho_0}(Y^n_{k-1})[F^n_1] = \omega^n_{\rho_0}(Y^n_{k-1}Y^n_k) + \omega^n_{\rho_0}(Y^n_{k-1}Y^n_{k+1}) = p^n_{\rho_0}(F^n_1) + p^n_{\rho_0}(F^n_2).$$

$$\eta^i_k := \frac{\omega_{\rho_0(Y^n_{k-1})}(F^n_1)}{\omega_{\rho_0(Y^n_{k-1})}(F^n_2)} = \frac{\mu_{\rho_0(Y^n_{k-1})}(Y^n_k)}{\mu_{\rho_0(Y^n_{k-1})}(Y^n_{k+1})}; 0 \leq \eta^i_k \leq 1; \text{ and } \eta^1_k + \eta^2_k = 1. \hspace{1cm} (8)$$

Proposition 4.8(i). $\rho_k = \eta^1_k \rho_1 + \eta^2_k \rho_2$.

Corollary 6.4. $\mu^n_{\rho_0}(Y^n_{n-1}k) \ll \mu^n_{\rho_0}(Y^n_{k-1})$ everywhere in $\mathfrak{B}(S^n_1)$.

It follows that $\mu^{n-k}_{\rho_k}[Y^n_{k+1}] \ll \mu^{n-k}_{\rho_k}(Y_{n+1})$, with Radon-Nikodym derivative denoted by $\delta^{n-k}_{\rho_k}[Y^n_{k+1}]$.

Theorem [4.1, 4.9, 6.5]. For an elementary history $E^n_1(Y^n)$ and a density operator $\rho_0$ the following four conditions are equivalent:

(i) $E^n_1(Y^n)$ is consistent with respect to $\rho_0$;

(ii) (CH1) $Tr\{G^n_k[G^n_kF^n_k\rho_0(Y^n_{k-1})F^n_k + F^n_k\rho_0(Y^n_{k-1})F^n_{k+1}]\} = 0$, where $G^n_k = F^n_{n-1} \cdots F^n_{k+1}$;

(iii) (CH4s) $\omega^{n-k}_{\rho_k}[Y^n_{k+1}][F^n_{k+1}] = \{\eta^i_k \omega^{n-k}_{\rho_k}(Y^n_{k+1}) + \eta^2_k \omega^{n-k}_{\rho_k}(Y^n_{k+1})\}[F^n_{k+1}]$;

and,

(iv) (CH5) the Radon-Nikodym derivatives $\delta^{n-k}_{\rho_k}[Y^n_{k+1}]$ satisfy the relations:

$$\delta^{n-k}_{\rho_k}[Y^n_{k+1}] + \delta^{n-k}_{\rho_k}[Y^n_{k+1}] = 1$$

almost everywhere on $Y^n_{k+1}$, and

$$\phi_{Y^n_{k+1}} \otimes \delta^{n-k}_{\rho_k}[Y^n_{k+1}] + \phi_{Y^n_{k+1}} \otimes \delta^{n-k}_{\rho_k}[Y^n_{k+1}] = 0$$

almost everywhere on $Y^n_{k+1}$.

Note that the condition for a second order history is that

$$Tr\{F^n_2[F^n_1\rho_0F^n_2 + F^n_2\rho_0F^n_1]\} = 0,$$

and (CH1) is a generalization of this.

Proof:

$$\mathcal{P}^{n}_{\rho_0}[F^n_1] = \omega^{n-k}_{\rho_0}(Y^n_{k-1})[F^n_{k+1}] \omega^{n-k}_{\rho_{k+1}}(Y^n_{k+1})[F^n_{k+1}].$$
and
\[
\omega^{n-k+1} (Y_{n-2}^k) [F_k^n] = Tr \left\{ G_k F_k \rho_0 (Y_{k-1}^1) \left[ F_k^1 \vee F_k^2 \right] G_k^* \right\} = Tr \left\{ G_k F_k \rho_0 (Y_{k-1}^1) \left[ F_k^1 \right] G_k^* \right\} + Tr \left\{ G_k F_k \rho_0 (Y_{k-1}^1) \left[ F_k^2 \right] G_k^* \right\}
\]
+ cross terms

So the cross terms have to be zero. This implies (CH1).

For \( i = 1, 2 \),
\[
\eta_i \omega^{n-k+1} (Y_{n-1}^1 [i]) [F_k^n] = \frac{\omega \rho_0 (Y_{k-1}^1) (F_k^i)}{\omega \rho_0 (Y_{k-1}^1) (F_k)} \omega \rho_0 (Y_{k+1}^1) \omega_{\rho_{k+1}^i} \omega (Y_{n-1}^1 [i]) [F_k^n]
\]
\[
= \omega \rho_0 (Y_{k-1}^1) (F_k^i) \omega \rho_{\rho_{k+1}^i} (Y_{k+1}^1) [F_k^n [i]]
\]
\[
= \omega \rho_{\rho_{k+1}^i} (Y_{k-1}^1 [i]) [F_k^n [i]]
\]

Then
\[
\eta_1 \omega_{\rho_{k+1}^1} (Y_{n-1}^1 [1]) [F_k^n] + \eta_2 \omega_{\rho_{k+1}^2} (Y_{n-1}^1 [2]) [F_k^n] = \omega \rho_{\rho_{k+1}^1} (Y_{n-1}^1 [1]) [F_k^n [1]] + \omega \rho_{\rho_{k+1}^2} (Y_{n-1}^1 [2]) [F_k^n [2]]
\]
\[
= \mathcal{P}_{\rho_{k+1}} (F_k^n [1]) + \mathcal{P}_{\rho_{k+1}} (F_k^n [2]).
\]

Multiplying through by \( \omega (Y_{k-2}^1) (F_k^{i-1}) \) we obtain the sum of probabilities
\[
\mathcal{P}_{\rho_0} (F_1^n) = \left\{ \eta_1 \omega (Y_{n-1}^1) [F_1^n] + \eta_2 \omega (Y_{n-1}^1) [F_1^n] \right\}
\]

Then (CH4s) follows.

**Corollary 6.9.** If \( E_1^n (Y_1^n) \) is consistent with respect to \( \rho_0 \) then
\[
\omega^{n-k} (Y_{k+1}^n) = \eta_1 \omega_{\rho_{k+1}^1} (Y_{k+1}^n) + \eta_2 \omega_{\rho_{k+1}^2} (Y_{k+1}^n)
\]
everywhere on \( \mathcal{A}_{k+1}^n \).