

ALGEBRAIC CONSISTENT QUANTUM THEORY

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1. INTRODUCTION

Let A be a bounded, self-adjoint operator on a Hilbert space \mathcal{H} , representing a physical quantity. Let Σ be the spectrum of A . Let

$$E : \mathcal{B}(\Sigma) \rightarrow \mathcal{P}(\mathcal{H})$$

be the spectral measure (\mathcal{B} Borel measurable subsets, \mathcal{P} projections). The map $E(X)$ is a property, $X \in \mathcal{B}(\Sigma)$. Each normal state is defined by a density operator ρ_0 on \mathcal{H} , which is positive, trace-class, $\|\rho_0\|_1 = 1$.

Let $\omega_{\rho_0} : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$ be the function defined by

$$\omega_{\rho_0}(B) = \text{Tr}(\rho_0 B).$$

The trace-class ideal $\mathcal{I}^1(\mathcal{H}) = \{\text{trace class operators}\}$. In general, the trace-class norm $\|A\|_1$ of a trace-class operator A is equal to $\text{Tr}(|A|)$. The probability wrt ρ_0 of $E(X)$ is

$$\mathcal{P}_{\rho_0}^1[E(X)] = \omega_{\rho_0}(E(X)),$$

the probability of that the measurement of the observable A will be in X .

Let A_1, \dots, A_n be observables with corresponding spectra $\Sigma_1, \dots, \Sigma_n$ and spectral measures E_1, \dots, E_n and a sequence of properties $E_1(X_1), \dots, E_n(X_n)$ corresponding to times $t_1 < \dots < t_n$. The finite sequence $(E_1(X_1), \dots, E_n(X_n))$ is called a **history**. Think of this as an apparatus that makes measurements. Another proposal is that we should think of a history as a product

$$\approx = E_1(X_1) \odot \dots \odot E_n(X_n),$$

where \odot has the algebraic properties of the tensor product. We simplify $F_j = E_j(X_j)$. In the physics literature, the nth order probability of the history \approx is defined to be

$$\mathcal{P}_{\rho_0}^n(\approx) = \text{Tr}(F_n F_{n-1} \dots F_1 \rho_0 F_1 \dots F_{n-1}).$$

For example, if

$$\approx' = F_1 \odot F_2,$$

then

$$\mathcal{P}_{\rho_0}^2(\approx') = \text{Tr}(F_2 F_1 \rho_0 F_1),$$

so then

$$\begin{aligned}
\rho_0(X_1) &= \frac{F_1 \rho_0 F_1}{\|F_1 \rho_0 F_1\|} \\
&= \frac{F_1 \rho_0 F_1}{Tr(F_1 \rho_0 F_1)} \\
&= \frac{F_1 \rho_0 F_1}{Tr(\rho_0 F_1)} \\
&= \frac{F_1 \rho_0 F_1}{\mathcal{P}_{\rho_0}^1(F_1)}
\end{aligned}$$

is a density operator. Then

$$\begin{aligned}
\mathcal{P}_{\rho_0}^2(\tilde{\omega}') &= Tr(F_2 \rho_0(X_1)) \mathcal{P}_{\rho_0}^1(F_1) \\
&= \mathcal{P}_{\rho_0(X_1)}^1(F_2) \mathcal{P}_{\rho_0}^1(F_1),
\end{aligned}$$

the first factor of which is like a conditional probability.

Let

$$\tilde{\omega}' = F_1 \odot F_2$$

Think of $X_1 \times X_2$ as a product, with $X_1 = X_1^1 \cup X_1^2$. Then

$$\begin{aligned}
E_1(X_1) &= E_1(X_1^1) \vee E_1(X_1^2) \\
&= F_1^1 \vee F_1^2,
\end{aligned}$$

so that

$$\begin{aligned}
\tilde{\omega}' &= (F_1^1 \odot F_2) \vee (F_1^2 \odot F_2) \\
&= \tilde{\omega}_1 \vee \tilde{\omega}_2
\end{aligned}$$

Is it true that

$$\mathcal{P}_{\rho_0}^2(\tilde{\omega}') = \mathcal{P}_{\rho_0}^2(\tilde{\omega}_1 \vee \tilde{\omega}_2) = \mathcal{P}_{\rho_0}^2(\tilde{\omega}_1) + \mathcal{P}_{\rho_0}^2(\tilde{\omega}_2)?$$

Well, not generally, because

$$\begin{aligned}
\mathcal{P}_{\rho_0}^2(\tilde{\omega}') &= Tr(F_2 (F_1^1 \vee F_1^2) \rho_0 (F_1^1 \vee F_1^2)) \\
&= Tr(F_2 F_1^1 \rho_0 F_1^1) + Tr(F_2 F_1^2 \rho_0 F_1^2) \\
&= \mathcal{P}_{\rho_0}^2(\tilde{\omega}_1) + \mathcal{P}_{\rho_0}^2(\tilde{\omega}_2) \\
&\quad + Tr(F_2 F_1^1 \rho_0 F_1^2) + Tr(F_2 F_1^2 \rho_0 F_1^1).
\end{aligned}$$

The **consistency condition** for second order histories is that

$$Tr(F_2 F_1^1 \rho_0 F_1^2) + Tr(F_2 F_1^2 \rho_0 F_1^1) = 0$$

for every measurable decomposition of X_1 as $X_1^1 \cup X_1^2$.

We will consider A_1, \dots, A_n observables, $\mathcal{A}_1, \dots, \mathcal{A}_n$ maximal abelian von Neumann algebras. Each of these has Gel'fand spectrum S_1, \dots, S_n and is endowed with a unique spectral integral

$$P_i : \mathbf{B}(S_i) \rightarrow \mathcal{A}_i$$

and spectral measure

$$E_i : \mathcal{B}(S_i) \rightarrow \mathcal{P}(\mathcal{A}_i).$$

Then

$$\begin{aligned}\mathbb{A}_1^n & : = \mathcal{A}_1 \bar{\otimes} \dots \bar{\otimes} \mathcal{A}_n \\ \mathbb{S}_1^n & : = S_1 \times \dots \times S_n \\ \mathbb{P}_1^n & : = P_1 \bar{\otimes} \dots \bar{\otimes} P_n \\ \mathbb{E}_1^n & : = E_1 \bar{\otimes} \dots \bar{\otimes} E_n\end{aligned}$$

Then

$$\begin{aligned}E_1^n(\mathbb{Y}_1^n) & = E_1(Y_1) \odot \dots \odot E_n(Y_n) \\ & = F_1 \odot \dots \odot F_n \\ & = \mathbb{F}_1^n\end{aligned}$$

Then

$$\begin{aligned}\mathcal{P}_{\rho_0}^n[\mathbb{F}_1^n] & = Tr(F_n F_{n-1} \dots F_1 \rho_0 F_1 \dots F_{n-1}) \\ & = \omega_{\rho_0}(F_1) Tr(F_n F_{n-1} \dots F_2 \rho_0(Y_1) F_2 \dots F_{n-1}) \\ & = \omega_{\rho_0}(F_1) \omega_{\rho_0(Y_1)}(F_2) \dots \omega_{\rho_0(\mathbb{Y}_1^{n-1})}(F_n) \\ & = \boldsymbol{\omega}_{\rho_0}^n(\mathbb{Y}_1^{n-1})[\mathbb{E}_1^n(\mathbb{Y}_1^n)] = (\boldsymbol{\omega}_{\rho_0}^n(\mathbb{Y}_1^{n-1}) \circ \mathbb{E}_1^n)[\mathbb{Y}_1^n] \\ & = \boldsymbol{\mu}_{\rho_0}^n[\mathbb{Y}_1^{n-1}](\mathbb{Y}_1^n).\end{aligned}$$

Notation and Summary of Results for Lecture 2

(1) Elementary history: $\mathbf{E}_1^n(\mathbf{Y}_1^n) = \mathbf{F}_1^n$

(2) k th initial density operator: $\rho_k = \rho_0(\mathbf{Y}_1^k) := \frac{E_k(Y_k) \rho_0(\mathbf{Y}_1^{k-1}) E_k(Y_k)}{Tr\{E_k(Y_k) \rho_0(\mathbf{Y}_1^{k-1})\}}$

(3) History state with respect to ρ_0 generated by \mathbf{Y}_1^{n-1} :

$$\begin{aligned}\boldsymbol{\omega}_{\rho_0}^n(\mathbf{Y}_1^{n-1}) & = \boldsymbol{\omega}_{\rho_0}^k(\mathbf{Y}_1^{k-1}) \bar{\otimes} \omega_{\rho_k} \bar{\otimes} \omega_{\rho_k(Y_{k+1})} \bar{\otimes} \dots \bar{\otimes} \omega_{\rho_k(\mathbf{Y}_{k+1}^{n-1})} \\ & = \boldsymbol{\omega}_{\rho_0}^k(\mathbf{Y}_1^{k-1}) \bar{\otimes} \boldsymbol{\omega}_{\rho_k}^{n-k}(\mathbf{Y}_{k+1}^{n-1})\end{aligned}$$

(4) History measure with respect to ρ_0 generated by \mathbf{Y}_1^{n-1} :

$$\begin{aligned}\boldsymbol{\mu}_{\rho_0}^n[\mathbf{Y}_1^{n-1}] & = \boldsymbol{\mu}_{\rho_0}^k[\mathbf{Y}_1^{k-1}] \bar{\otimes} \mu_{\rho_k} \bar{\otimes} \mu_{\rho_k(Y_{k+1})} \bar{\otimes} \dots \bar{\otimes} \mu_{\rho_k(\mathbf{Y}_{k+1}^{n-1})} \\ & = \boldsymbol{\mu}_{\rho_0}^k[\mathbf{Y}_1^{k-1}] \bar{\otimes} \boldsymbol{\mu}_{\rho_k}^{n-k}[\mathbf{Y}_{k+1}^{n-1}]\end{aligned}$$

Theorem 3.1. *The n th order probability of $\mathbf{E}_1^n(\mathbf{Y}_1^n)$ with respect to ρ_0 is given by*

$$p_{\rho_0}^n[\mathbf{F}_1^n] = \boldsymbol{\omega}_{\rho_0}^n(\mathbf{Y}_1^{n-1})[\mathbf{F}_1^n] = \boldsymbol{\mu}_{\rho_0}^n[\mathbf{Y}_1^{n-1}](\mathbf{Y}_1^n).$$

(5) Measurable decomposition: $Y_k = Y_k^1 \sqcup Y_k^2$ for any $k, 1 \leq k < n$.

(6) For $i = 1, 2$: $\mathbf{Y}_1^m([ik]) := \mathbf{Y}_1^{k-1} \times Y_k^i \times \mathbf{Y}_{k+1}^m$, $\mathbf{F}_1^m(ik) = \mathbf{E}_1^m(\mathbf{Y}_1^m(ik)) = \mathbf{F}_1^{k-1} \times F_k^i \times \mathbf{F}_{k+1}^m$, for $k < m \leq n$.

$$\rho_k^i := \frac{E_k(Y_k^i) \rho_0(\mathbf{Y}_1^{k-1}) E_k(Y_k^i)}{Tr\{E_k(Y_k^i) \rho_0(\mathbf{Y}_1^{k-1})\}} = \frac{F_k^i \rho_0(\mathbf{Y}_1^{k-1}) F_k^i}{Tr\{F_k^i \rho_0(\mathbf{Y}_1^{k-1})\}}. \quad (7)$$

Definition. An elementary history $\mathbf{E}_1^n(\mathbf{Y}_1^n) = \mathbf{F}_1^n$ is **consistent with respect to a density operator** ρ_0 if the probability is additive. That is, for each binary measurable decomposition $Y_k = Y_k^1 \sqcup Y_k^2$ for any $k, 1 \leq k \leq n$,

$$\begin{aligned} p_{\rho_0}^n[\mathbf{F}_1^n] &= \omega_{\rho_0}^n(\mathbf{Y}_1^{n-1})[\mathbf{F}_1^n] \\ &= \omega_{\rho_0}^n(\mathbf{Y}_1^{n-1}1k)(\mathbf{F}_1^n1k) + \omega_{\rho_0}^n(\mathbf{Y}_1^{n-1}2k)(\mathbf{F}_1^n2k) \\ &= p_{\rho_0}^n(\mathbf{F}_1^n1k) + p_{\rho_0}^n(\mathbf{F}_1^n2k). \end{aligned}$$

$$\eta_k^i := \frac{\omega_{\rho_0(\mathbf{Y}_1^{k-1})}(F_k^i)}{\omega_{\rho_0(\mathbf{Y}_1^{k-1})}(F_k)} = \frac{\mu_{\rho_0(\mathbf{Y}_1^{k-1})}(Y_k^i)}{\mu_{\rho_0(\mathbf{Y}_1^{k-1})}(Y_k)}; \quad 0 \leq \eta_k^i \leq 1; \quad \text{and} \quad \eta_k^1 + \eta_k^2 = 1. \quad (8)$$

Proposition 4.8(i). $\rho_k = \eta_k^1 \rho_k^1 + \eta_k^2 \rho_k^2$.

Corollary 6.4. $\mu_{\rho_0}^n(\mathbf{Y}_1^{n-1}ik) \ll \mu_{\rho_0}^n(\mathbf{Y}_1^{n-1})$ everywhere in $\mathfrak{B}(\mathcal{S}_1^n)$.

It follows that $\mu_{\rho_k^i}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] \ll \mu_{\rho_k}^{n-k}(\mathbf{Y}_{k+1}^{n-1})$, with Radon-Nikodym derivative denoted by $\delta_{\rho_k^i}^{n-k}[\mathbf{Y}_{k+1}^{n-1}]$.

Theorem [4.1, 4.9, 6.5]. For an elementary history $\mathbf{E}_1^n(\mathbf{Y}_1^n)$ and a density operator ρ_0 the following four conditions are equivalent:

- (i) $\mathbf{E}_1^n(\mathbf{Y}_1^n)$ is consistent with respect to ρ_0 ;
- (ii) (CH1) $\text{Tr}\{G_k^* G_k [F_k^1 \rho_0(\mathbf{Y}_1^{k-1}) F_k^2 + F_k^2 \rho_0(\mathbf{Y}_1^{k-1}) F_k^1]\} = 0$, where $G_k = F_n \cdots F_{k+1}$;
- (iii) (CH4s) $\omega_{\eta_k^1 \rho_k^1 + \eta_k^2 \rho_k^2}^{n-k}(\mathbf{Y}_{k+1}^{n-1})[\mathbf{F}_{k+1}^n]$
 $= \left\{ \eta_k^1 \omega_{\rho_k^1}^{n-k}(\mathbf{Y}_{k+1}^{n-1}) + \eta_k^2 \omega_{\rho_k^2}^{n-k}(\mathbf{Y}_{k+1}^{n-1}) \right\} [\mathbf{F}_{k+1}^n];$

and,

- (iv) (CH5) the Radon-Nikodym derivatives $\delta_{\rho_k^i}^{n-k}[\mathbf{Y}_{k+1}^{n-1}]$ satisfy the relations:

$$\delta_{\rho_k^1}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] + \delta_{\rho_k^2}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] = 1$$

almost everywhere on \mathbf{Y}_{k+1}^n , and

$$\varphi_{Y_k^1} \otimes \delta_{\rho_k^2}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] + \varphi_{Y_k^2} \otimes \delta_{\rho_k^1}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] = 0$$

almost everywhere on \mathbf{Y}_k^n .

Note that the condition for a second order history is that

$$\text{Tr}\{F_2 (F_1^1 \rho_0 F_1^2 + F_1^2 \rho_0 F_1^1)\} = 0,$$

and (CH1) is a generalization of this.

Proof:

$$\mathcal{P}_{\rho_0}^n[\mathbf{F}_1^n] = \omega_{\rho_0}^{k-1}(\mathbf{Y}_1^{k-2})[\mathbf{F}_1^{n-1}] \omega_{\rho_{k-1}}^{n-k+1}(\mathbf{Y}_k^{n-2})[\mathbf{F}_k^n],$$

and

$$\begin{aligned}\omega_{\rho_{k-1}}^{n-k+1}(\mathbf{Y}_k^{n-2})[\mathbf{F}_k^n] &= \text{Tr} \{ G_k F_k \rho_0(\mathbf{Y}_1^{k-1}) [F_k^1 \vee F_k^2] G_k^* \} \\ &= \text{Tr} \{ G_k F_k^1 \rho_0(\mathbf{Y}_1^{k-1}) [F_k^1] G_k^* \} + \text{Tr} \{ G_k F_k^2 \rho_0(\mathbf{Y}_1^{k-1}) [F_k^2] G_k^* \} \\ &\quad + \text{cross terms}\end{aligned}$$

So the cross terms have to be zero. This implies (CH1).

For $i = 1, 2$,

$$\begin{aligned}\eta_k^i \omega_{\rho_k^1}^{n-k+1}(\mathbf{Y}_k^{n-1} [i])[\mathbf{F}_k^1] &= \frac{\omega_{\rho_0}(\mathbf{Y}_1^{k-1})(F_k^i)}{\omega_{\rho_0}(\mathbf{Y}_1^{k-1})(F_k)} \omega_{\rho_0(\mathbf{Y}_1^{k-1})}(F_k) \omega_{\rho_k^i}^{n-k}(\mathbf{Y}_{k+1}^{n-1})[\mathbf{F}_{k+1}^n] \\ &= \omega_{\rho_0}(\mathbf{Y}_1^{k-1})(F_k^i) \omega_{\rho_k^i}^{n-k}(\mathbf{Y}_{k+1}^{n-1})[\mathbf{F}_k^n [i]] \\ &= \omega_{\rho_{k-1}}^{n-k+1}(\mathbf{Y}_k^{n-1} [i])[\mathbf{F}_k^n [i]]\end{aligned}$$

Then

$$\begin{aligned}\eta_k^1 \omega_{\rho_k^1}^{n-k+1}(\mathbf{Y}_k^{n-1} [1])[\mathbf{F}_k^1] + \eta_k^2 \omega_{\rho_k^1}^{n-k+1}(\mathbf{Y}_k^{n-1} [2])[\mathbf{F}_k^1] &= \omega_{\rho_{k-1}}^{n-k+1}(\mathbf{Y}_k^{n-1} [1])[\mathbf{F}_k^n [1]] + \omega_{\rho_{k-1}}^{n-k+1}(\mathbf{Y}_k^{n-1} [2])[\mathbf{F}_k^n [2]] \\ &= \mathcal{P}_{\rho_{k-1}}^{n-k+1}(\mathbf{F}_k^n [1]) + \mathcal{P}_{\rho_{k-1}}^{n-k+1}(\mathbf{F}_k^n [2]).\end{aligned}$$

Multiplying through by $\omega_{\rho_0}^{k-1}(\mathbf{Y}_1^{k-2})(\mathbf{F}_1^{k-1})$ we obtain the sum of probabilities

$$\mathcal{P}_{\rho_0}^k[\mathbf{F}_1^n] = \left\{ \eta_k^1 \omega_{\rho_k^1}^{n-k}(\mathbf{Y}_{k+1}^{n-1})[\mathbf{F}_1^n] + \eta_k^2 \omega_{\rho_k^2}^{n-k}(\mathbf{Y}_{k+1}^{n-1})[\mathbf{F}_1^n] \right\}.$$

Then (CH4s) follows.

Corollary 6.9. *If $\mathbf{E}_1^n(\mathbf{Y}_1^n)$ is consistent with respect to ρ_0 then*

$$\omega_{\eta_k^1 \rho_k^1 + \eta_k^2 \rho_k^2}^{n-k}(\mathbf{Y}_{k+1}^{n-1}) = \eta_k^1 \omega_{\rho_k^1}^{n-k}(\mathbf{Y}_{k+1}^{n-1}) + \eta_k^2 \omega_{\rho_k^2}^{n-k}(\mathbf{Y}_{k+1}^{n-1})$$

everywhere on \mathcal{A}_{k+1}^n .

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