PRELIM EXAM SOLUTIONS

THE GRAD STUDENTS + KEN

Contents

1. 2010F Complex Exam

2010F Complex Exam #1

Suppose that the function $f(z) = u(z) + iv(z)$ with u, v real-valued is analytic in domain D and $v(z) = u(z)^2$. Prove that f is constant on D.

Proof. Since f is analytic on D, then the Cauchy-Riemman equations hold. That is,

$$\begin{aligned} u_x(z) &= v_y(z) & (1) \\ u_y(z) &= -v_x(z) & (2) \end{aligned}$$

for all $z \in D$. Differentiating $v(z) = u(z)^2$ according to x and according to y yields the following

$$\begin{aligned} v_x(z) &= 2u(z)u_x(z) & (3) \\ v_y(z) &= 2u(z)u_x(z) & (4) \end{aligned}$$

for all $z \in D$.
By (2), $u_y(z) + v_x(z) = 0$.
By (3), $u_y(z) + 2u(z)u_x(z) = 0$.
By (1), $u_y(z) + 2u(z)v_y(z) = 0$.

By (4), $u_y(z) + 4u(z)^2 u_y(z) = 0$.
Then $u_y(z)(1 + 4u(z)^2) = 0$, which implies

$$u_y(z) = 0 \,\text{or}\, (1 + 4u(z)^2) = 0.$$

The equation $1 + 4u(z)^2 = 0$ has no real solution for $u(z)$. Thus $u_y(z) = 0$. Equation (4) implies $v_y(z) = 0$, and then f analytic on D implies

$$\begin{aligned} f'(z) &= v_y(z) - iu_y(z) \,\forall\, z \in D \\ f'(z) &= 0 \,\forall\, z \in D \end{aligned}$$

Therefore, f is constant on D. □

2010F Complex Exam #9

Find all entire functions for which $|f(z)| \leq |z|^2$ for $|z| \leq 1$ and $|f(z)| \leq |z|^3$ for $|z| \geq 1$.

Proof. Let f have the properties given in the statement. From the second inequality, we have that for $R \geq 1$,

$$\sup_{|x|=R} |f(z)| = R^3.$$

By the maximum principle $|f(z)| \leq R^3$ for all z such that $|z| \leq R$. Then, for instance, if $|z| = R_0 < R$ is fixed, the Cauchy inequality for derivatives implies

$$|f'''(z)| \leq \frac{3! R^3}{(R - R_0)^3}.$$

Since f''' is entire, the maximum principle tells us that $|f'''(z)| \leq \frac{3!R^3}{(R-R_0)^3}$ for $|z| < R_0$ as well. Since the inequality is true for all $R \geq 1$, we have by choosing an arbitrary R_0 and taking the limit as $R \to \infty$ that $|f'''(z)| \leq 6$ for all $z \in \mathbb{C}$. By Liouville's Theorem, we conclude that $f'''(z)$ is constant, so that

$$f(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0,$$

for some $a_3, a_2, a_1, a_0 \in \mathbb{C}$.

Next, consider the first inequality. The maximum principle for f and the Cauchy inequality for f' for $0 < r < 1$ centered at $z = 0$ yields

$$\begin{aligned} |a_0| = |f(0)| &\leq r^2, \\ |a_1| = |f'(0)| &\leq \frac{r^2}{r} = r. \end{aligned}$$

Since these inequalities are true for all $r > 0$, we have

$$a_0 = a_1 = 0.$$

Applying both inequalities when $z = e^{i\theta}$ for any $\theta \in \mathbb{R}$, we have

$$\begin{aligned} |a_3 e^{3i\theta} + a_2 e^{2i\theta}| &\leq 1, \text{ or} \\ |a_3 e^{i\theta} + a_2| &\leq 1 \end{aligned}$$

by factoring out $|e^{2i\theta}| = 1$. Then either one of a_2 and a_3 is zero, or there exists $\theta \in \mathbb{R}$ such that $a_3 e^{i\theta}$ and a_2 have the same argument. Then the inequality above becomes

$$|a_3| + |a_2| = |a_3 e^{i\theta} + a_2| \leq 1.$$

In summary, we have show that if f is entire and satisfies the two inequalities given, then

$$\begin{aligned} f(z) &= a_3 z^3 + a_2 z^2, \text{ with} \\ |a_3| &\leq 1, \\ |a_2| &\leq 1 - |a_3|. \end{aligned}$$

Conversely, suppose that f has the form above. Then for $|z| \geq 1$,

$$\begin{aligned} |f(z)| &\leq |a_3 z^3 + a_2 z^2| \\ &\leq |a_3 z^3| + |a_2 z^2| \\ &\leq |a_3|\,|z|^3 + |a_2|\,|z|^2 \\ &\leq |a_3|\,|z|^3 + (1 - |a_3|)|z|^3 \\ &= |z|^3. \end{aligned}$$

Likewise, for $|z| \leq 1$,

$$\begin{aligned} |f(z)| &\leq |a_3 z^3 + a_2 z^2| \\ &\leq |a_3 z^3| + |a_2 z^2| \\ &\leq |a_3|\,|z|^3 + |a_2|\,|z|^2 \\ &\leq |a_3|\,|z|^2 + (1 - |a_3|)|z|^2 \\ &= |z|^2. \end{aligned}$$

Thus, we have shown that f satisfies the hypotheses if and only if f is a cubic polynomial of the form

$$\begin{aligned} f(z) &= a_3 z^3 + a_2 z^2, \text{ with} \\ |a_2| + |a_3| &\leq 1. \end{aligned}$$

□

2010F Complex Exam #10

Find the number of zeros of $f(z) = z^5 - 20z^4 + 5z^3 - z^2 + 50z - 17$ inside the annulus $1 \leq |z| \leq 5$.

Proof. Let f be given as above. Observe that when $|z| = 1$,

$$\begin{aligned} |f(z) - 50z| &= |z^5 - 20z^4 + 5z^3 - z^2 - 17| \\ &\leq |z^5| + |20z^4| + |5z^3| + |z^2| + 17 \\ &= 1 + 20 + 5 + 1 + 17 < 50 = |50z|. \end{aligned}$$

Thus, $|f(z) - 50z| < |50z|$ for z on the unit circle. By Rouche's Theorem, the number of zeros of f in the open unit disk is the same as that of $50z$, which is one zero.

Next, consider the case where $|z| = 5$. Then

$$\begin{aligned} |f(z) + 20z^4| &= |z^5 + 5z^3 - z^2 + 50z - 17| \\ &\leq |z^5| + |5z^3| + |z^2| + |50z| + 17 \\ &= 5^5 + 5^4 + 5^2 + 50 \cdot 5 + 17 < 20 \cdot 5^4 = |-20z^4|. \end{aligned}$$

Thus, $|f(z) + 20z^4| < |-20z^4|$ if $|z| = 5$. By Rouche's Theorem, the number of zeros of f on $\{z : |z| < 5\}$ is the same as that of $-20z^4$, which is four zeros.

Therefore, f has three zeros in the given annulus. (Throughout this proof, we have been counting multiplicities of zeros.) $\square$

2. 2012S Complex Exam

2012S Complex Exam #7

Suppose that g is a holomorphic function defined on $\{z \in \mathbb{C} : z \neq 0\}$ and that $|g'(z)| \leq \frac{1}{|z|^{3/2}}$ for $0 < |z| \leq 1$. Prove that $z = 0$ is a removable singularity.

Proof. Since g is holomorphic on $\mathbb{C} \setminus \{0\}$, g has a Laurent expansion

$$g(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

valid on $\mathbb{C} \setminus \{0\}$ with $a_n \in \mathbb{C}$. Since g is holomorphic on our set, so is g'. We rewrite

$$g(z) = \ldots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 z + a_1 z + a_2 z^2 + \ldots$$

Thus,

$$\begin{aligned} |g'(z)| &= |\ldots + \frac{-2a_{-2}}{z^3} + \frac{-a_{-1}}{z^2} + a_1 + 2a_2 z + \ldots| \\ &\leq \frac{1}{|z|^{3/2}} \end{aligned}$$

on the annulus $0 < |z| \leq 1$. Thus, multiplying by $|z^2|$,

$$|\ldots + \frac{-2a_{-2}}{z} + \frac{-a_{-1}}{+} a_1 z^2 + a_2 z^3 + \ldots| \leq |z^{1/2}| \leq 1$$

on $0 < |z| \leq 1$. Since z^2 is holomorphic on the same annulus as $g(z)$ and $g'(z)$, then $z^2 g'(z)$ is holomorphic on the same annulus, and thus, its Laurent expansion (given above) converges on that annulus. Therefore, by the Riemann Removable Singularity Theorem, $z^2 g'(z)$ is holomorphic on the annulus, and therefore $a_k = 0$ for all $k < -1$.

By the Cauchy integral formula, since $g'(z)z^2$ is holomorphic on the disk (and beyond), for fixed r with $0 < r \leq 1$,

$$\begin{aligned} -a_{-1} &= \lim_{w \to 0} g'(w) w^2 \\ &= \lim_{w \to 0} \frac{1}{2\pi i} \int_{|z|=r} \frac{z^2 g'(z)}{z - w} \, dz \\ &= \frac{1}{2\pi i} \int_{|z|=r} z g'(z) \, dz \end{aligned}$$

We can switch the limit and integral above since the limit of the integrand exists at all points z such that $|z| = 1$, and the limit is absolutely integrable. Note that the denominator is bounded strictly away from zero in the circle of integration. Now, since $|g'(z)| \leq |z|^{-3/2}$, when $|z| = r$, the integrand is bounded in absolute value by $r^{-1/2}$. Therefore,

$$|a_{-1}| \leq \frac{1}{2\pi} (2\pi r) r^{-1/2} = r^{1/2}.$$

Since this inequality is true for all r such that $0 < r \leq 1$, we must have $a_{-1} = 0$. Therefore g is holomorphic at 0. $\square$

Proof. (Alternate proof) Since g is holomorphic on $\mathbb{C} \setminus \{0\}$, g has a Laurent expansion

$$g(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

valid on $\mathbb{C} \setminus \{0\}$ with $a_n \in \mathbb{C}$.

By the Laurent series formula for $g'(z) = \sum n a_n z^{n-1}$, for any $n \in \mathbb{Z}$,

$$n a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{g'(z)}{z^n} dz$$

Then for any r such that $0 < r \leq 1$,

$$|n|\,|a_n| \leq \frac{1}{2\pi} \int_{|z|=r} |z|^{-\frac{3}{2}-n} |dz| = r^{-\frac{1}{2}-n}.$$

Thus, since this is true as $r \to 0^+$ for $n \leq -1$ (i.e. , $-\frac{1}{2} - n \geq \frac{1}{2}$), $a_n = 0$. Thus $z = 0$ is a removable singularity for g. $\square$

Proof. (quickest proof) Since g is holomorphic on $\mathbb{C} \setminus \{0\}$, g has a Laurent expansion

$$g(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

valid on $\mathbb{C} \setminus \{0\}$ with $a_n \in \mathbb{C}$.

By the Cauchy inequality formula for $g'(z) = \sum n a_n z^{n-1}$, for any $n \in \mathbb{Z}$,

$$|n a_n| \leq \frac{M}{r^{n-1}},$$

where $M = \max_{|z|=r} |g'(z)| \leq r^{-3/2}$ for $0 < r \leq 1$.

Then for any r such that $0 < r \leq 1$,

$$|n|\,|a_n| \leq r^{-\frac{1}{2}-n}.$$

Thus, since this is true as $r \to 0^+$ for $n \leq -1$ (i.e. , $-\frac{1}{2} - n \geq \frac{1}{2}$), $a_n = 0$. Thus $z = 0$ is a removable singularity for g. $\square$

3. 2011S Real Exam

2011S Real Exam #4

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be smooth, and suppose that the graph of $F(x, y, z) = 0$ in $\mathbb{R}^3$ is a surface that is tangent to the plane $z = 2x - y + 3$ at $(1, -1, 6)$.

(a) Prove that there exists an open disk D of some positive radius centered at $(-1, 6) \in \mathbb{R}^2$ and a function $g : D \to \mathbb{R}$ such that $F(g(u, v), u, v) = 0$ for all $u, v \in D$.

(b) Find all possible values of $\nabla g(-1, 6)$.

Proof. **(a)** Since the surface is tangent to $z = 2x - y + 3$ at $(1, -1, 6)$ (which has vector $(2, -1, -1)$ normal to the plane) and the gradient ∇F is perpendicular to the surface, we have

$$\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)_{(1,-1,6)} = \lambda (2, -1, -1)$$

for some nonzero scalar λ. In particular, $\frac{\partial F}{\partial x}(1,-1,6) \neq 0$. By the implicit function theorem, there exists an open disk D of some positive radius centered at $(-1,6) \in \mathbb{R}^2$ and a function $g : D \to \mathbb{R}$ such that $F(g(u,v),u,v) = 0$ for all $u, v \in D$.

(b) Since $F(g(u,v),u,v) = 0$, we differentiate with respect to u and v to get (from the chain rule):

$$\begin{aligned} \frac{\partial F}{\partial x}\frac{\partial g}{\partial u} + \frac{\partial F}{\partial y}(1) &= 0, \text{ i.e. } 2\lambda\frac{\partial g}{\partial u} + -\lambda = 0 \\ \frac{\partial F}{\partial x}\frac{\partial g}{\partial v} + \frac{\partial F}{\partial z}(1) &= 0, \text{ i.e. } 2\lambda\frac{\partial g}{\partial v} + -\lambda = 0. \end{aligned}$$

Thus, $\nabla g(-1,6) = \left(\frac{\partial g}{\partial u}, \frac{\partial g}{\partial v}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$. □

4. 2011F Real Exam

2011F Real Exam #3

Let $f : [a,b] \to \mathbb{R}$ be bounded.

(a) Prove that if f is Riemann integrable on $[a,b]$, then so is f.

(b) Prove that if f^2 is Riemann integrable on $[a,b]$, then so is f.

Proof. **(a) is True:** Suppose f is Riemann integrable and bounded on $[a,b]$. Let $\varepsilon > 0$. There exists $M > 0$ such that $|f(x)| < M$ for all $x \in [a,b]$. Observe that

$$\begin{aligned} \left|f^2(x) - f^2(y)\right| &= |f(x) + f(y)|\,|f(x) - f(y)| \\ &\leq (|f(x)| + |f(y)|)\,|f(x) - f(y)| \\ &\leq 2M\,|f(x) - f(y)|. \end{aligned}$$

Since f is Riemann integrable and $\varepsilon' = \frac{\varepsilon}{2M} > 0$, there exist a partition $P = (a = x_0 < x_1 < ... < x_m = b)$ of $[a,b]$ such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2M}$. Then

$$\begin{aligned} U\left(f^2,P\right) - L\left(f^2,P\right) &= \sum_{j=0}^{m-1}\left(\sup_{x_j \leq x \leq x_{j+1}} f^2(x) - \inf_{x_j \leq y \leq x_{j+1}} f^2(y)\right)(x_{j+1} - x_j) \\ &= \sum_{j=0}^{m-1}\left(\sup_{x_j \leq x,y \leq x_{j+1}} f^2(x) - f^2(y)\right)(x_{j+1} - x_j) \\ &\leq \sum_{j=0}^{m-1}\left(\sup_{x_j \leq x,y \leq x_{j+1}} 2M\,|f(x) - f(y)|\right)(x_{j+1} - x_j) \\ &= 2M\sum_{j=0}^{m-1}\left(\sup_{x_j \leq x,y \leq x_{j+1}} f(x) - f(y)\right)(x_{j+1} - x_j) \\ &= 2M\,(U(f,P) - L(f,P)) < \varepsilon. \end{aligned}$$

Thus, f^2 is Riemann integrable.

(b) is False: Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then $f^2(x) = 1$, which is clearly Riemann integrable. But for any partition of P of the interval $[a,b]$, the upper sum is $U(f,P) = 1(b-a) = b-a$, and the lower sum is $L(f,P) =$

$-1(b-a) = a-b$, using the density of rationals (resp. irrationals) in $\mathbb{R}$. Thus f is not Riemann integrable. $\square$

2011F Real Exam #5

Let (x_k) be a sequence in $\mathbb{R}^n$ and let L be the set of all limits of subsequences of (x_k) that exist. Prove that L is a closed set.

Proof. Let x be a limit point of L. Then there exist $\square$

5. 2012S Real Exam

2012S Real Exam #7

For which $k \in \mathbb{R}$ is it true that $\int_{\mathbb{R}^n} |x|^k e^{-|x|}\, dx < \infty$?

Proof. We compute the integral in spherical coordinates. That is, let

$$\begin{aligned}
x_1 &= \rho\cos\varphi_1 \\
x_2 &= \rho\sin\varphi_1\cos\varphi_2 \\
x_3 &= \rho\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\
&\ldots \\
x_{n-1} &= \rho\sin\varphi_1\ldots\sin\varphi_{n-2}\cos\vartheta \\
x_n &= \rho\sin\varphi_1\ldots\sin\varphi_{n-2}\sin\vartheta
\end{aligned}$$

Then the integral is

$$\begin{aligned}
& \int_{\mathbb{R}^n} |x|^k e^{-|x|}\, dx \\
= & \int_0^\infty \int_{\vartheta=0}^{2\pi} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \rho^k e^{-\rho}\left(\rho^{n-1}\sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \ldots \sin^1\varphi_{n-2}\right) d\varphi_{n-2}\ldots d\varphi_1 d\theta d\rho \\
= & \int_0^\infty e^{-\rho}\rho^{n+k-1} d\rho \int_{\vartheta=0}^{2\pi} \int_{\varphi_1=0}^{\pi} \cdots \int_{\varphi_{n-2}=0}^{\pi} \left(\sin^{n-2}\varphi_1 \sin^{n-3}\varphi_2 \ldots \sin^1\varphi_{n-2}\right) d\varphi_{n-2}\ldots d\varphi_1 d\theta d\rho
\end{aligned}$$

This integral converges if and only if

$$\int_0^\infty e^{-\rho}\rho^{n+k-1}d\rho = \int_0^1 e^{-\rho}\rho^{n+k-1}d\rho + \int_1^\infty e^{-\rho}\rho^{n+k-1}d\rho$$

converges. The integral from 0 to 1 converges if and only if $\int_0^1 x^{n+k-1}dx$ converges, because for $x \in [0,1]$, $\frac{1}{e}x^s \le e^{-x}x^s \le x^s$ for all $s \in \mathbb{R}$. By the standard facts about integral convergence, the integral converges if and only if $n+k > 0$, i.e. $k > -n$. The integral from 1 converges no matter what k is. The reason is that for all s and for all x sufficiently large, $x^{-2} > e^{-x}x^s > 0$, so since $\int_1^\infty x^{-2}dx$ converges, by the comparison test, so does $\int_1^\infty e^{-\rho}\rho^{n+k-1}d\rho$. Thus the conclusion is that the integral converges if and only if $k > -n$. $\square$

Note: one can avoid the spherical coordinates by lumping the angular coordinates together to calculate the volume of S^{n-1}.

2012S Real Exam #8

Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $h'(0) = 0$ and $h''(0) = 1$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by $F(x) = h(|x|)$. Prove that F is differentiable on $\mathbb{R}^n$.

Proof. Since $|x|$ is smooth in x at all points $p \in \mathbb{R}^n \setminus \{0\}$, $h(|x|)$ is definitely smooth at points $x \neq 0$ since it is a composition of smooth functions. Next, observe that for x near 0 in $\mathbb{R}^n$,

$$\begin{aligned} \frac{|F(x) - F(0)|}{|x|} &= \frac{|h(|x|) - h(0)|}{|x|} \\ &= \frac{|h'(c)||x|}{|x|} = |h'(c)|, \end{aligned}$$

by the mean value theorem, where c is a real number between $|x|$ and 0. Since h' is continuous and $h'(0) = 0$, the limit of the above quantity as $x \to 0$ is 0, by the squeeze theorem. Thus,

$$\lim_{x \to 0} \frac{|F(x) - F(0)|}{|x|} = 0,$$

(Note that the above is NOT related to $F'(0)$.)

So that F is differentiable at 0, with the derivative linear transformation being the zero map. Thus, F is differentiable on $\mathbb{R}^n$.

[Here, we are using the fact that for a function $F : U \to V$ of several variables, where $U \subseteq R^k$ and $V \subseteq R^n$, is differentiable at $a \in U$ if there exists a linear transformation $L : R^k \to R^n$ such that

$$\lim_{x \to a} \frac{\|F(x) - F(a) - L(x - a)\|}{\|x - a\|} = 0.$$

If this is indeed true, then $L = dF(a)$ is the derivative of F at a. We write $F'(a)$ for the matrix for the linear transformation, so that $L(x - a) = F'(a)(x - a)$. (and, yes, $F'(a)$ is the matrix of all the first partial derivatives).] ∎

6. 2012F Real Exam

2012F Real Exam #4

Consider the series $S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

(1) Find the interval of convergence of this series.
(2) Is the convergence on this interval uniform?
(3) Find $S(1/2)$.

Proof. **(a)** We use the ratio test:

$$\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)(n+2)} \frac{n(n+1)}{x^n} \right| = \lim_{n \to \infty} \left| \frac{nx}{(n+2)} \right| = |x|.$$

Thus, the series converges for $|x| < 1$. For $x = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2 + n} < \sum_{n=1}^{\infty} \frac{1}{n^2};$$

The latter sum converges since it is a p-series, so the first series converges by the comparison test. For $x = -1$, the series is alternating and the absolute value of the terms form a decreasing sequence with a limit of zero. By the alternating series test, the series converges at $x = -1$.

Therefore, the interval of convergence is $[-1, 1]$.

(b) Observe that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges as shown above. By the Weierstrass M-test, the series converges uniformly.

(c) Observe that

$$\begin{aligned} S'(x) &= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n(n+1)} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n+1)} \end{aligned}$$

Then

$$(x^2 S'(x))' = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

Thus,

$$x^2 S'(x) = -x - \log(1-x) + C.$$

After evaluating at $x = 0$ we obtain $C = 0$. Then we integrate by parts to get

$$S(x) = -\log(x) + \int \frac{\log(1-x)}{x^2} dx = -\log(x) + \frac{\log(1-x)}{x} + \int \frac{1}{x(1-x)} dx$$

by integrating by parts. Then

$$\begin{aligned} S(x) &= -\log(x) + \frac{\log(1-x)}{x} + \int \left(\frac{1}{1-x} + \frac{1}{x} \right) \\ &= \frac{\log(1-x)}{x} - \log(1-x) + C \\ &= \frac{(1-x)\log(1-x)}{x} + C \end{aligned}$$

Taking the limit as $x \to 0$, we may evaluate the constant C: $C = 1$. Thus

$$S(\frac{1}{2}) = -\log(2) + 1.$$

□

2012F Real Exam #5

Prove that every sequence of real numbers has a monotone subsequence.

Proof. Let (S_n) be a sequence of real numbers. We define a term S_i of the sequence to be a **dominant term** if $S_i \geq S_j$ for all $j > i$. Define $T = \{i \in \mathbb{N} : S_i \text{ is a dominant term }\}$. We have two cases.

Case 1: T is finite. Let i_0 be greater than any element of T. Therefore S_{i_0} is not dominant. Then there exists $i_1 > i_0$ such that $S_{i_1} \geq S_{i_0}$. Similarly, for $k > 1$, there exists $i_k > i_{k-1}$ such that $S_{i_k} \geq S_{i_{k-1}}$ since $S_{i_{k-1}}$ is not dominant. Therefore, $(S_{i_k})_{k \geq 0}$ is a monotone increasing sequence.

Case 2: $T = \{t_1, t_2, ...\}$ is infinite, with

$$t_1 < t_2 < ...$$

Then the sequence $(S_{t_p})_{p \geq 0}$ is by construction decreasing. □

2012F Real Exam #6
Let f be continuous on $[0,1]$. For $x \in [0,1]$, let

$$g_n(x) = \int_0^x f(y)(x-y)^n \, dy.$$

(1) Find the pointwise limit of g_n as $n \to \infty$.
(2) Is the convergence uniform?

Proof. With the given information, by the extreme value theorem, f assumes a maximum and minimum on $[0,1]$, so $|f(y)| \leq M$ for some constant M, for all $y \in [0,1]$. Now, for some fixed n,

$$\begin{aligned} |g_n(x)| &= \left| \int_0^x f(y)(x-y)^n \, dy \right| \leq \int_0^x |f(y)|(x-y)^n \, dy \leq M \int_0^x (x-y)^n \, dy \\ &= M \left[-\frac{(x-y)^{n+1}}{n+1} \right]_0^x \\ &= \frac{M}{n+1} x^{n+1}. \end{aligned}$$

Hence

$$\sup_{x \in [0,1]} |g_n(x)| \leq \frac{M}{n+1}.$$

Since $\frac{M}{n+1} \to 0$ and $\frac{M}{n+1}$ is a uniform bound for $|g_n(x)|$, the sequence of functions converges uniformly. □

7. 2013S Real Exam

2013S Real Exam #2
Evaluate the integral $\int \int_S \left(y\widehat{i} + y\widehat{j} + z\widehat{k} \right) \cdot \widehat{n} \, dS$, where S is the surface $\{(x,y,z) : x^2 + y^2 = z, \ z \leq 4\}$ and $\widehat{n}$ is the unit normal to S that points away from the z-axis.

Proof. **(Method 1)**
The surface S is a portion of the paraboloid $z = x^2 + y^2$ that is below the plane P defined by $z = 4$. If we include P in the integral, the surface $S \cup P$ is a closed surface, and we can extend $\widehat{n}$ to be the outward normal (so $\widehat{n} = \widehat{k}$ on P), and we have the setting of the divergence theorem. Let Ω be the interior of the paraboloid below $z = 4$. We have

$$\begin{aligned} \int \int_{S \cup P} \left(y\widehat{i} + y\widehat{j} + z\widehat{k} \right) \cdot \widehat{n} \, dS &= \int \int \int_\Omega div \left(y\widehat{i} + y\widehat{j} + z\widehat{k} \right) \, dV \\ &= \int \int \int_\Omega \left(\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right) \, dV \\ &= 2 \cdot \int \int \int_\Omega 1 \, dV = 2\text{volume}(\Omega). \end{aligned}$$

Next, using cylindrical coordinates with $r^2 = x^2 + y^2$ and $x = r\cos\theta$, $y = r\sin\theta$, we can compute:

$$\begin{aligned}
2\text{volume}(\Omega) &= 2\int_0^{2\pi}\int_0^2 \left(4 - r^2\right) r\ dr\ d\theta \\
&= 2\int_0^{2\pi}\int_0^2 \left(4r - r^3\right)\ dr\ d\theta \\
&= 2\int_0^{2\pi} \left(2(2)^2 - \frac{(2)^4}{4}\right)\ d\theta \\
&= 16\pi.
\end{aligned}$$

Next, we integrate the portion over P (disk of radius 2 at $z = 4$).

$$\begin{aligned}
\int\int_P \left(y\widehat{\imath} + y\widehat{\jmath} + z\widehat{k}\right)\cdot\widehat{n}\ dS &= \int_0^{2\pi}\int_0^2 \left(y\widehat{\imath} + y\widehat{\jmath} + 4\widehat{k}\right)\cdot\widehat{k}\ r\ dr\ d\theta \\
&= 4\int_0^{2\pi}\int_0^2 1\ r\ dr\ d\theta \\
&= 4\left(\pi(2)^2\right) = 16\pi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int\int_S \left(y\widehat{\imath} + y\widehat{\jmath} + z\widehat{k}\right)\cdot\widehat{n}\ dS &= \int\int_{S\cup P} \left(y\widehat{\imath} + y\widehat{\jmath} + z\widehat{k}\right)\cdot\widehat{n}\ dS \\
&\quad - \int\int_P \left(y\widehat{\imath} + y\widehat{\jmath} + z\widehat{k}\right)\cdot\widehat{n}\ dS \\
&= 16\pi - 16\pi = 0.
\end{aligned}$$

(Method 2)

We parametrize the surface using the coordinate chart $\Phi(u,v) = (u, v, u^2+v^2)$ with u, v in the disk of radius 2. Then the outward normal is the downward normal to the surface $z = x^2 + y^2$, i.e. the vector in the direction of ∇f, where $f(x,y,z) = x^2 + y^2 - z$, so $\nabla f = (2x, 2y, -1) = (2u, 2v, -1)$. (Note we chose the sign so that the z-component would be negative so that the vector points downward.) Then the unit normal is $\widehat{n} = \frac{\nabla f}{\|\nabla f\|}$, and $\|\nabla f\| = \sqrt{4u^2 + 4v^2 + 1}$. The area form dS is the same as $\|\Phi_u \times \Phi_v\|\, du\ dv$, and $\Phi_u = (1, 0, 2u)$ and $\Phi_v = (0, 1, 2v)$, so that

$$\begin{aligned}
\|\Phi_u \times \Phi_v\| &= \left\| \begin{vmatrix} \widehat{\imath} & \widehat{\jmath} & \widehat{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} \right\| \\
&= \left\|(-2u)\widehat{\imath} - (2v)\widehat{\jmath} + (1)\widehat{k}\right\| \\
&= \sqrt{4u^2 + 4v^2 + 1} = \|\nabla f\|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int\int_S \left(y\widehat{i} + y\widehat{j} + z\widehat{k}\right) \cdot \widehat{n}\ dS \\
= &\int\int_{\text{disk}} \left(v, v, u^2 + v^2\right) \cdot \frac{\nabla f}{\|\nabla f\|} \|\nabla f\|\ du\ dv \\
= &\int\int_{\text{disk}} \left(v, v, u^2 + v^2\right) \cdot \nabla f\ du\ dv \\
= &\int\int_{\text{disk}} \left(v, v, u^2 + v^2\right) \cdot (2u, 2v, -1)\ du\ dv \\
= &\int\int_{\text{disk}} 2uv + 2v^2 - u^2 - v^2\ du\ dv \\
= &\int\int_{\text{disk}} 2uv + v^2 - u^2\ du\ dv \\
= &\int_{r=0}^{2} \int_{\theta=0}^{2\pi} 2r^2 \cos\theta \sin\theta + r^2 \left(\sin^2\theta - \cos^2\theta\right)\ d\theta\ r\ dr \\
= &\int_{r=0}^{2} r^3 \int_{\theta=0}^{2\pi} (\sin 2\theta - \cos(2\theta))\ d\theta\ r\ dr = 0.
\end{aligned}$$

□

2013S Real Exam #4

(a) Let f be a nonnegative, continuous function on the nonnegative reals. For a positive integer n, let $I_n = \displaystyle\int_0^n f(x)\ dx$. Prove that $\displaystyle\int_0^\infty f(x)\ dx$ converges if and only if the sequence (I_n) converges.
(b) Show that the conclusion in (a) may be false if the hypothesis that f is nonnegative is dropped.

Proof. **(a)**
First, suppose that f is a nonnegative continuous function such that the improper integral $\displaystyle\int_0^\infty f(x)\ dx$ exists.

Then, if for all $b > 0$, we define

$$I_b = \int_0^b f(x)\ dx,$$

Then $\lim_{b\to\infty} I_b$ exists and is a nonnegative real number ℓ. Then, for all $\epsilon > 0$, there exists $M > 0$ such that $|I_b - \ell| < \epsilon$, whenever $b > M$. So, if n is a positive integer and $n > M$, then

$$|I_n - \ell| < \epsilon.$$

Thus, $\lim_{n\to\infty} I_n = \ell$.

Conversely, suppose that f is a nonnegative continuous function and suppose that the sequence (I_n) for $n = \{1, 2, 3, \ldots\}$ converges. For all $b > 0$, if $n = \lfloor b \rfloor$, then

$$n - 1 < b \leq n.$$

This implies
$$I_{n-1} < I_b \leq I_n,$$
since f is nonnegative. By the squeeze theorem, $\lim_{b\to\infty} I_b$ exists. □

Proof. **(b)**
Let $f(x) = \cos(2\pi x)$. The improper integral $\int_0^\infty f(x)\,dx$ does not exist, since
$$\int_0^\infty \cos(2\pi x)\,dx = \lim_{b\to\infty} \frac{1}{2\pi}\sin(2\pi b),$$
which does not exist. However, for n a positive integer,
$$I_n = \int_0^n \cos(2\pi x)\,dx = \frac{1}{2\pi}\sin(2\pi n) = 0.$$
so that $\lim_{n\to\infty} I_n = 0$. □

2013S Real Exam #5
Let f be twice continuously differentiable. Prove that, given x and h, there exists θ such that
$$f(x+h) - 2f(x) + f(x-h) = f''(\theta)h^2.$$

Proof. By Taylor's Theorem,
$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(c_1) \\ f(x-h) &= f(x) - hf'(x) + \frac{h^2}{2}f''(c_2) \end{aligned}$$
for some c_1 between x and $x+h$ and some c_2 between $x-h$ and x. Adding the two equations, we have
$$f(x+h) - 2f(x) + f(x-h) = \frac{h^2}{2}\left(f''(c_1) + f''(c_2)\right)$$
Since f'' is continuous, by the intermediate value theorem, there exists θ between c_1 and c_2 such that
$$f''(\theta) = \frac{f''(c_1) + f''(c_2)}{2},$$
because $\frac{f''(c_1)+f''(c_2)}{2}$ is between $f''(c_1)$ and $f''(c_2)$, inclusive. □

2013S Real Exam #6
Let $f(x)$ be infinitely differentiable and odd. Suppose the Fourier series
$$a_0 + \sum_{n=1}^\infty (a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$
converges to $f(x)$ on $(-1, 0)$ and the Fourier series
$$c_0 + \sum_{n=1}^\infty (c_n \cos 2\pi nx + d_n \sin 2\pi nx)$$
converges to $f(x)$ on $(0, 1)$. Express the Fourier series of period 2 that converges to $f(x)$ on $(-1, 1)$ in terms of $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$.

Proof. By the formulas for the Fourier series coefficients, since

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos 2\pi n x + b_n \sin 2\pi n x)$$

for $x \in (-1, 0)$, we have

$$\begin{aligned} a_0 &= \int_{-1}^{0} f(x)\, dx \\ a_n &= 2\int_{-1}^{0} f(x) \cos 2\pi n x\, dx \\ b_n &= 2\int_{-1}^{0} f(x) \sin 2\pi n x\, dx \end{aligned}$$

Similarly, for the Fourier series valid on $(0, 1)$, the coefficients must satisfy

$$\begin{aligned} c_0 &= \int_{0}^{1} f(x)\, dx \\ c_n &= 2\int_{0}^{1} f(x) \cos 2\pi n x\, dx \\ d_n &= 2\int_{0}^{1} f(x) \sin 2\pi n x\, dx \end{aligned}$$

We desire the coefficients A_k and B_k such that

$$f(x) = A_0 + \sum_{k=1}^{\infty} (A_k \cos \pi k x + B_k \sin \pi k x) \tag{5}$$

for $x \in (-1, 1)$, and so that

$$\begin{aligned} A_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\, dx \\ A_k &= \int_{-1}^{1} f(x) \cos \pi k x\, dx \\ B_k &= \int_{-1}^{1} f(x) \sin \pi k x\, dx \end{aligned}$$

Since f is an odd function,

$$A_k = 0 \text{ for all } k \geq 0. \tag{6}$$

For any $n \in \mathbb{Z}_{>0}$, then we see that

$$B_{2n} = \frac{1}{2}(b_n + d_n). \tag{7}$$

On the other hand, for any $n \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} B_{2n+1} &= \int_{-1}^{1} f(x) \sin \pi(2n+1)x\, dx \\ &= 2\int_{0}^{1} f(x) \sin \pi(2n+1)x\, dx. \end{aligned}$$

Substituting the expansion valid on $(0,1)$, we see that

$$\begin{aligned}
B_{2n+1} &= 2c_0\int_0^1 \sin\pi(2n+1)x\,dx \\
&\quad +\sum_{j=1}^{\infty}\left(2c_j\int_0^1\cos 2j\pi x\sin\pi(2n+1)x\,dx+2d_j\int_0^1\sin 2j\pi x\sin\pi(2n+1)x\,dx\right)\\
&= \frac{4c_0}{(2n+1)\pi}\\
&\quad +\sum_{j=1}^{\infty}\Bigg(c_j\int_0^1(\sin\pi(2n+1+2j)x+\sin\pi(2n+1-2j)x)\,dx\\
&\qquad +d_j\int_0^1(\cos\pi(2n+1-2j)x-\cos\pi(2n+1+2j)x)\,dx\Bigg)\\
&= \frac{4c_0}{(2n+1)\pi}\\
&\quad +\sum_{j=1}^{\infty}\left(c_j\left(\frac{2}{\pi(2n+1+2j)}+\frac{2}{\pi(2n+1-2j)}\right)+0\right)\\
&= \frac{4c_0}{(2n+1)\pi}+\sum_{j=1}^{\infty}\frac{4(2n+1)c_j}{\pi((2n+1)^2-4j^2)}=\frac{4}{\pi}\sum_{j=0}^{\infty}\frac{(2n+1)c_j}{(2n+1)^2-4j^2}. \qquad (8)
\end{aligned}$$

Note that since f is smooth, the constants c_j are rapidly decreasing, so the sum definitely converges (quickly). Substituting (6), (7), and (8) into (5), we have the Fourier expansion of f on $(-1,1)$. □

8. 2013F Real Exam

2013F Real Exam #1
Let C be the curve parametrized by

$$\vec{r}(t)=e^{\sqrt{t}}\,\vec{i}+\arctan(t^3)\,\vec{j},\quad 0\le t\le 1.$$

Evaluate

$$\int_C (6xy+2)\,dx+(3x^2+8y)\,dy.$$

Proof. Observe that $d(3xy^2+2x+4y^2)=(6xy+2)\,dx+(3x^2+8y)\,dy$. Let $f(x,y)=3xy^2+2x+4y^2$. Then, by the Fundamental Theorem of Calculus for Line Integrals,

$$\begin{aligned}
\int_C(6xy+2)\,dx+(3x^2+8y)\,dy &= f(\overrightarrow{r}(1))-f(\overrightarrow{r}(0))\\
&= 3e^2\frac{\pi}{4}+2e+4\frac{\pi^2}{16}-0-2-0\\
&= 3e^2\frac{\pi}{4}+2e+\frac{\pi^2}{4}-2
\end{aligned}$$

□

2013F Real Exam #2

Determine the maximum and minimum values of the quantity $xy + 4z$ on the half ellipsoid $x^2 + 4y^2 + 2z^2 = 64, \quad z \geq 0$.

Proof. Let $f(x, y, z) = x^2 + 4y^2 + 2z^2$, $g(x, y, z) = xy + 4z$. Using the method of Lagrange multipliers, we obtain critical points of g in the upper part where $z > 0$:

$$\begin{aligned} y &= 2\lambda x \\ x &= 8\lambda y \\ 4 &= 4\lambda z \\ x^2 + 4y^2 + 2z^2 &= 64 \end{aligned}$$

The first two equations imply that $0 = 16\lambda^2 y - y = y(16\lambda^2 - 1)$. Thus, $y = 0$ or $\lambda = \frac{1}{4}$ or $\lambda = -\frac{1}{4}$. If $y = 0$ then $x = 0$ and $z = \sqrt{32}$. If $\lambda = \pm\frac{1}{4}$, then $y = \pm\frac{x}{2}$, $z = \pm 4$, and then the surface equation yields $x^2 = 16$, so $x = \pm 4$, $y = \pm 2$. So far, we have critical points $(0, 0, \sqrt{32}), (4, 2, 4), (4, -2, 4), (-4, 2, 4), (-4, -2, 4)$.

Next, consider the boundary $x^2 + 4y^2 = 64$, and the function is $g(x, y) = xy$, so the Lagrange multipliers method gives the equations $y = 2\lambda x, x = 8\lambda y$, and we get again $y = 0$ or $\lambda = \pm\frac{1}{4}$. If $y = 0$, $x = 0$, which is not on the ellipse. If $\lambda = \pm\frac{1}{4}$, $y = \pm\frac{x}{2}$, and then we get $(x, y) = (\pm 4\sqrt{2}, 2\sqrt{2})$ or $(\pm 4\sqrt{2}, -2\sqrt{2})$ are the critical points on the elliptical boundary.

After comparing the values of $g(x, y, z)$ on all these points, we get $g(x, y, z)$ obtains the maximum value $g(-4, -2, 4) = g(-4, -2, 4) = 24$, and it obtains the minimum value $g(4\sqrt{2}, -2\sqrt{2}, 0) = g(-4\sqrt{2}, 2\sqrt{2}, 0) = -16$. □

2013F Real Exam #3

Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by the formula

$$f(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \dfrac{xy^2}{x^2 + y^2} & (x, y) \neq (0, 0). \end{cases}$$

(a) Prove that f is continuous at $(0, 0)$.
(b) Prove that if $\vec{u} = a\,\vec{i} + b\,\vec{j}$ is a unit vector, then the directional derivative of f at $(0, 0)$ in the direction of $\vec{u}$ exists, and compute its value.
(c) Is f is differentiable at $(0, 0)$?

Proof. (a) For any $\epsilon > 0$, let $\delta = \epsilon$. Then $\|(x, y)\| < \delta$ implies that $\sqrt{x^2 + y^2} < \delta$, which implies that $|x| < \delta$, $|y| < \delta$. When $y = 0$ and $x \neq 0$, we have $f(x, y) = 0$, so certainly $|f(x, y) - 0| = 0 < \epsilon$. When y is nonzero,

$$|f(x, y) - f(0, 0)| = \frac{|xy^2|}{|x^2 + y^2|} \leq \frac{|x|y^2}{y^2} = |x| < \delta = \epsilon.$$

So in all cases, $|f(x, y) - f(0, 0)| < \epsilon$ whenever $\|(x, y)\| \leq \delta$, so f is continuous at $(0, 0)$.

(b) For $u = a\hat{i} + b\hat{j}$, $\|u\| = 1$, we have

$$D_u f(0, 0) = \lim_{t \to 0} \frac{f(tu) - f(0, 0)}{t} = \lim_{t \to 0} \frac{tatb^2}{t^2a^2 + t^2b^2} = \lim_{t \to 0} \frac{tab^2}{a^2 + b^2} = 0,$$

since $a^2 + b^2 \neq 0$.

(c) From the above, the derivative matrix at $(0,0)$ is $(f_x, f_y) = (00)$. Then if f were differentiable at $(0,0)$, then

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - 0}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}} = 0.$$

But the limit above is definitely not zero (if it exists), because along $y = x$, the limit approaches

$$\lim_{y\to 0} \frac{y^3}{(y^2+y^2)^{3/2}} = 2^{-3/2}.$$

□

2013F Real Exam #4
Let $\{f_n\}$ be a sequence of continuous functions on $[0,1]$.

(1)
(a) Suppose $\{f_n\}$ converges uniformly to a function f. Prove that f is continuous.
(b) Give an example where $\{f_n\}$ converges pointwise to a function f that is not continuous.

Proof. (a) Let $\epsilon > 0$. Since $\{f_n\}$ converges uniformly to f on $[0,1]$, there exists $N \in \mathbb{N}$ such that for every $x \in [0,1]$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $n > N$. Choose some $n > N$. Since f_n is continuous on $[0,1]$, for all $x \in [0,1]$, there exists $\delta > 0$ such that if $y \in [0,1]$ and $|x-y| < \delta$, then $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Now pick any $x \in [0,1]$. Suppose $y \in [0,1]$ such that $|x-y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \end{aligned}$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Therefore, f is continuous on $[0,1]$.
(b) Let $f_n(x) = x^n$. Then

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

□

2013F Real Exam #5
Suppose that $\{a_n\}$ be a decreasing sequence of positive real numbers. Prove that the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the infinite series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

2013F Real Exam #6
Suppose that $h : [0,1] \longrightarrow \mathbb{R}$ is bounded and has the property that h is Riemann integrable on $[\epsilon, 1]$ for every $0 < \epsilon < 1$. Using the definition of the Riemann integral, prove that h is Riemann integrable on the interval $[0,1]$.

Proof. Given $\epsilon > 0$, we need to find a partition P that is $0 = x_0 < \ldots < x_n = 1$ such that $U(f,P) - L(f,P) < \epsilon$, where U and L denote the upper and lower Riemann sums, respectively. Given $\epsilon > 0$, we proceed as follows. Since h is is integrable on $[\delta, 1]$ for

$0 < \delta < 1$, we have a partition P_δ such that $U(h, P_\delta) - L(h, P_\delta) < \frac{\epsilon}{2}$. Let $\widetilde{P_\delta} = \{0\} \cup P_\delta$. Then $\widetilde{P_\delta}$ is a partition of $[0, 1]$. Let M and m be the maximum and minimum of h on $[0, 1]$. Then $U(h, \widetilde{P_\delta}) - L(h, \widetilde{P_\delta}) \leq 2M\delta + \epsilon/2$. If we choose $\delta < \frac{\epsilon}{4M}$, then choose P_δ accordingly, then $U(h, \widetilde{P_\delta}) - L(h, \widetilde{P_\delta}) \leq \epsilon$. ☐

2013F Real Exam #7

Let h be a real-valued function and differentiable function on $[0, \infty)$ such that $h(0) = 1$ and $3 \leq h'(x) \leq 4$ for all $x \geq 0$. Prove that there exists a constant c such that

$$1 \leq \frac{h(x)}{\sqrt{9x^2 + 1}} \leq c$$

for all $x \geq 0$.

2013F Real Exam #8

Let f be a continuous function of period 2π, and suppose that

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of f.

(a) Prove that the sum $\sum_{n=1}^{\infty} |a_n|^2$ converges.

(b) Suppose also that $\sum_{n=1}^{\infty} n \max\{|a_n|, |b_n|\}$ converges. Prove that f is differentiable and that the integral $\int_{-\pi}^{\pi} (f'(x))^2\, dx$ is finite.

Proof. We assume the function is $\mathbb{R}$-valued. **(a)** Since f is continuous and 2π-periodic, $|f|^2$ is continuous and bounded so that $\int_0^{2\pi} |f|^2$ is finite, and the Fourier series converges. We have

$$\begin{aligned}
\int_0^{2\pi} f\ dx &= \int_0^{2\pi} a_0\ dx = 2\pi a_0, \text{ so } a_0 = \frac{1}{2\pi} \int_0^{2\pi} f\ dx \\
\int_0^{2\pi} f(x) \cos(mx)\ dx &= \int_0^{2\pi} a_m \cos^2(mx)\ dx = \pi a_n\ , \text{ so } a_m = \frac{1}{\pi} \int_0^{2\pi} f\ (x) \cos(mx)\, dx. \\
\int_0^{2\pi} f(x) \sin(mx)\ dx &= \int_0^{2\pi} b_m \sin^2(mx)\ dx = \pi b_m\ , \text{ so } b_m = \frac{1}{\pi} \int_0^{2\pi} f\ (x) \sin(mx)\, dx.
\end{aligned}$$

We have, letting $S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$,

$$\begin{aligned}
\int S_N(x)^2 &= \int_0^{2\pi} \left(a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \right) \left(a_0 + \sum_{j=1}^{N} (a_j \cos jx + b_j \sin jx) \right) \, dx \\
&= \int_0^{2\pi} a_0^2 + 2a_0 \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) + \sum_{n=1}^{N} \sum_{j=1}^{N} (a_n \cos nx + b_n \sin nx)(a_j \cos jx + b_j \sin j \\
&= \int_0^{2\pi} a_0^2 + 2a_0 \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) + \sum_{n=1}^{N} \sum_{j=1}^{N} (a_n \cos nx + b_n \sin nx)(a_j \cos jx + b_j \sin j \\
&= \int_0^{2\pi} a_0^2 + 2a_0 \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) + \sum_{n=1}^{N} \sum_{j=1}^{N} a_n a_j \cos nx \cos jx + a_j b_n \sin nx \cos jx + b_j \\
&= a_0^2 2\pi + 2a_0 \sum_{n=1}^{\infty} \int (a_n \cos nx + b_n \sin nx) \, dx + \sum_{n=1}^{N} \sum_{j=1}^{N} a_n a_j \int \cos nx \cos jx \, dx + a_j b_n \int \sin n \\
&= a_0^2 2\pi + 2a_0 \cdot 0 + \sum_{n=1}^{N} \sum_{j=1}^{N} a_n a_j \delta_{jn} \pi + a_j b_n \cdot 0 + b_j a_n \cdot 0 + \sum_{n=1}^{N} \sum_{j=1}^{N} b_n b_j \delta_{jn} \pi \\
&= 2\pi a_0^2 + \pi \sum_{n=1}^{N} a_n^2 + \pi \sum_{n=1}^{N} b_n^2.
\end{aligned}$$

We know that $\int |f - S_N|^2 \to 0$ as $N \to \infty$.

$$|S_N(x)| = |S_N(x) - f(x) + f(x)| \le |S_N(x) - f(x)| + |f(x)|,$$

by the triangle inequality, so

$$|S_N(x)|^2 \le |S_N(x) - f(x)|^2 + |f(x)|^2 + 2|S_N(x) - f(x)||f(x)|.$$

Integrating,

$$\begin{aligned}
\int |S_N(x)|^2 \, dx &\le \int |S_N(x) - f(x)|^2 + \int |f(x)|^2 + 2\int |S_N(x) - f(x)||f(x)| \\
&\le \int |S_N(x) - f(x)|^2 + \int |f(x)|^2 + 2\sqrt{\int |S_N(x) - f(x)|^2} \sqrt{\int |f(x)|^2}
\end{aligned}$$

Since $\int |f(x)|^2$ is a finite number and $\int |S_N(x) - f(x)|^2$ is bounded independent of N, $\int |S_N(x)|^2 \, dx$ is bounded independent of N. Thus,

$$\lim_{N \to \infty} \left(2\pi a_0^2 + \pi \sum_{n=1}^{N} a_n^2 + \pi \sum_{n=1}^{N} b_n^2. \right)$$

is bounded, and thus $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) Suppose also that $\displaystyle\sum_{n=1}^{\infty} n \max\{|a_n|, |b_n|\}$ converges.

First we check differentiability of the Fourier series.
Since

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

we know that $f'(x)$ exists and equals the differentiated series if $\sum_{n=1}^{\infty} \left| \frac{d}{dx} (a_n \cos nx + b_n \sin nx) \right|$ converges. We check that

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \frac{d}{dx} (a_n \cos nx + b_n \sin nx) \right| &= \sum_{n=1}^{\infty} \left| (-na_n \sin nx + nb_n \cos nx) \right| \\ &\leq \sum_{n=1}^{\infty} (n|a_n| |\sin nx| + n|b_n| |\cos nx|) \\ &\leq \sum_{n=1}^{\infty} (n|a_n| + n|b_n|) \leq 2\sum_{n=1}^{\infty} n \max\{|a_n|, |b_n|\} < \infty. \end{aligned}$$

Thus, $f'(x)$ exists and satisfies

$$f(x) = \sum_{n=1}^{\infty} (-na_n \sin nx + nb_n \cos nx).$$

Let

$$T_N(x) = \sum_{n=1}^{N} (nb_n \cos nx - na_n \sin nx).$$

By the last calculation, if $\sum n^2 |b_n|^2 + n^2 |a_n|^2$ converges we know that $T_N(x)$ converges. But

$$\sum n^2 |b_n|^2 + n^2 |a_n|^2 \leq 2 \sum (n \max\{|a_n|, |b_n|\})^2.$$

But if $\sum |x_n| < \infty$, $\sum |x_n|^2 < \infty$ (since only a finite number of x_n have modulus > 1). So we have

$$\sum n^2 |b_n|^2 + n^2 |a_n|^2 < \infty,$$

so by the first part,

$$\int |f'(x)|^2 \leq \pi \sum_{n=1}^{\infty} n^2 |b_n|^2 + n^2 |a_n|^2 < \infty.$$

□

9. 2014S Real Exam

2014S Real Exam #1 Let f be the function of period 2 that equals $|x|$ on $[-1, 1]$.
(a) Find the Fourier series of f.
(b) Use it to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Proof. Let

$$f(x) = |x| = \frac{a_0}{2} + \sum a_n \cos(n\pi x) + \sum b_n \sin(n\pi x).$$

Then

$$\begin{aligned}
a_0 &= \int_{-1}^{1} |x|\, dx = 1 \\
a_n &= \int_{-1}^{1} |x| \cos(n\pi x)\ dx = 2\int_0^1 x\cos(n\pi x)\ dx \\
&= 2\left(\frac{x}{\pi n}\sin(\pi n x)\Big|_0^1 - \frac{1}{\pi n}\int_0^1 \sin(\pi n x)\, dx\right) \\
&= 2\left(\frac{1}{(\pi n)^2}\cos(\pi n x)\right)\Big|_0^1 = \frac{2}{\pi^2 n^2}(\cos\pi n - 1) = \begin{cases} -\frac{4}{\pi^2 n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\
b_n &= 0 \text{ since } |x| \text{ is even}
\end{aligned}$$

Thus,

$$f(x) = \frac{1}{2} + \sum_{n \text{ odd}} -\frac{4}{\pi^2 n^2}\cos(n\pi x).$$

(b) Letting $x = 0$, we get

$$0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^2}.$$

Thus,

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Now,

$$\begin{aligned}
\sum_n \frac{1}{n^2} &= \sum_{n \text{ odd}} \frac{1}{n^2} + \sum_{n \text{ even}} \frac{1}{n^2} \\
\sum_n \frac{1}{n^2} &= \sum_{n \text{ odd}} \frac{1}{n^2} + \frac{1}{4}\sum_n \frac{1}{n^2}
\end{aligned}$$

So

$$\frac{3}{4}\sum_n \frac{1}{n^2} = \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Thus,

$$\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}.$$

□

2014S Real Exam #4

Suppose that $\sum_{n=0}^{\infty} a_n$ converges.

(a) Prove or disprove that $\sum_{n=0}^{\infty} a_n^2$ converges.

(b) Prove or disprove that $\sum_{n=0}^{\infty} a_n^3$ converges.

Proof. **(a)** Let $a_n = (-1)^n (n+1)^{-1/2}$. Then $\sum_{n=0}^{\infty} a_n$ converges by the alternating series test ($|a_n| \to 0$, $(|a_n|)$ is decreasing, and (a_n) is alternating.) But $\sum a_n^2 = \sum \frac{1}{n+1}$ diverges.

(b) □

2014S Real Exam #7

(a) Prove or disprove that there exists a surjective continuous function $F : B \to H$, where B is an open ball in $\mathbb{R}^2$ and $H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 + 1 = 0\}$.

(b) Prove or disprove that there exists a surjective continuous function $F : H \to B$, with H and B as in **(a)**.

Proof. **(a)** There does not exist such a map, because B is connected and H is a hyperboloid of two sheets and is thus disconnected. (The continuous image of a connected set is connected.)

(b) First, let B be be the unit disk in $\mathbb{R}^2$ centered at 0. Let $p : H \to \diagdown^2$ be $p(x, y, z) = (x, y)$. Clearly, p is surjective and continuous. Let $q : \mathbb{R}^2 \to B$ be defined by

$$q(x, y) = \frac{1}{1 + \sqrt{x^2 + y^2}} (x, y).$$

We see that $\|q(x, y)\| < 1$ for each (x, y), so the image of q is contained in B. Since the function $r \mapsto \frac{r}{1+r}$ maps 0 to itself, $\mathbb{R}_{>0}$ to itself, and since it is increasing and $\lim\limits_{r\to\infty} \frac{r}{1+r} = 1$, it maps $[0, \infty)$ to $[0, 1)$ bijectively. Then q maps each ray $\{r(\cos\theta, \sin\theta) : 0 \le r < \infty\}$ to $\{r(\cos\theta, \sin\theta) : 0 \le r < 1\}$ bijectively, so that q is a bijective continuous map from $\mathbb{R}^2$ to B^2. Thus, $q \circ p$ is a continuous surjective map from H to B. Finally, this map may be composed with a bijective translation and dilation (ie homothety) that maps B to any open ball in $\mathbb{R}^2$. □

10. 2010F Algebra Exam

2010F Algebra #1

Let G be a group, let N be a normal subgroup of G of finite index. Suppose that H is a finite subgroup of G and that the order of H is relatively prime to the index of N in G. Prove that H is contained in N.

Proof. Let $[G : N] = n$, $|H| = m$ with $n, m \in \mathbb{N}$ and $\gcd(n, m) = 1$. Since N is normal, G/N is a group of order n. Since m is relatively prime to n, there exist integers s, t such that $ns + mt = 1$ (Bézout's Identity). Let $h \in H$. Note that

$$\begin{aligned} (hN)^1 &= (hN)^{ns+mt} \\ &= [(hN)^n]^s [(hN)^m]^t \\ &= N^s [h^m N]^t \end{aligned}$$

by Lagrange's Theorem applied to G/N. Then

$$hN = N^s N^t = N.$$

Thus, $h \in N$. □

2010F Algebra #4

Let G be a group and S a subset of G. For all $g_1, g_2 \in G$, suppose that either $Sg_1 = Sg_2$ or $Sg_1 \cap Sg_2 = \emptyset$. Prove that $S = Hg$ for some subgroup H and some $g \in G$.

Proof. Let r, s be arbitrary elements of S. Then $1 = rr^{-1} \in S_{r^{-1}}$ and similarly $1 \in S_{s^{-1}}$. So $S_{r^{-1}} \cap S_{s^{-1}} \neq \emptyset$. By the given, $S_{r^{-1}} = S_{s^{-1}}$ for any elements $r, s \in S$.

We now let $H = Sr^{-1}$ for a fixed $r \in S$. For any $h = s_0 r^{-1} \in Sr^{-1}$ with $s_0 \in S$, $h^{-1} = rs_0^{-1} \in Ss_0^{-1} = Sr^{-1} = H$, so H is closed under inverses. For any $h_1 = s_1 r^{-1}$ and

$h_2 = s_2 r^{-1}$ in $Sr^{-1}$with $s_1, s_2 \in S$. Since $Sr^{-1} = Ss_2^{-1}$, there exists $s_3 \in S$ such that $s_1 r^{-1} = s_3 s_2^{-1}$. Then $h_1 h_2 = s_1 r^{-1} s_2 r^{-1} = s_3 s_2^{-1} s_2 r^{-1} = s_3 r^{-1} \in Sr^{-1} = H$. Thus, H is closed under multiplication and is thus a subgroup of G. Then $Hr = (Sr^{-1})\, r = S$. ☐

2010F Algebra #7

Find all groups of order 4, and prove that your list is complete.

Let G be a group of order 4. If there exists an element of order 4, then G is cyclic and is $\mathbb{Z}_4$. Otherwise, all non-identity elements of G have order 2. In that case, let $a, b \in G$ be distinct such that $a^2 = b^2 = e$, the identity. Consider ab. Since $a^{-1} = a$, $ab \neq e$. Since $a \neq e$ and $b \neq e$, $ab \neq b$ and $ab \neq a$. Thus, ab is the other element of the group. Similarly, ba is that same element. Thus G is abelian and $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by $a \mapsto (1, 0)$, $b \mapsto (0, 1)$.

2010F Algebra #9

Suppose that $\det(A + xB) = x^5 + 10x + 5$ for some 5×5 matrices A and B with complex number entries and all $x \in \mathbb{C}$. Prove that B is an invertible matrix.

Proof. We have $\det A = 5$, so A is invertible. Then, multiplying by $\frac{1}{5} = \det(A^{-1})$, we have

$$\begin{aligned}
\det\left(A^{-1}(A + xB)\right) &= \frac{1}{5}x^5 + 2x + 1 \\
\det\left(I + A^{-1}xB\right) &= \frac{1}{5}x^5 + 2x + 1 \\
\det\left(I + xA^{-1}B\right) &= \frac{1}{5}x^5 + 2x + 1
\end{aligned}$$

For $x \neq 0$,

$$\begin{aligned}
x^5 \det\left(x^{-1}I + A^{-1}B\right) &= \frac{1}{5}x^5 + 2x + 1, \text{ or} \\
\det\left(A^{-1}B + x^{-1}I\right) &= \frac{1}{5} + 2x^{-4} + x^{-5}.
\end{aligned}$$

Letting $\lambda = -x^{-1}$, we have for $\lambda \neq 0$,

$$\det\left(A^{-1}B - \lambda I\right) = \frac{1}{5} + 2\lambda^4 - \lambda^5.$$

By continuity, this equation is true for all λ in $\mathbb{C}$, so $\det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{1}{5}$, and so $\det(B) \neq 0$. Thus B is invertible. ☐

11. 2011S Algebra Exam

2011S Algebra #1

Let A be a commutative ring with identity, and let I be a proper (2-sided) ideal. Prove that A/I is an integral domain if and only if whenever $ab \in I$, $a \in I$ or $b \in I$.

Proof. Note that A/I is always a ring if I is an ideal; if A is commutative, then it is easy to show that A/I is commutative as well: for any $a, b \in A$, $(a + I)(b + I) = ab + I = ba + I = (b + I)(a + I)$. Further I is the additive identity (zero) in A/I.
($\Longrightarrow$) Assume that A/I is an integral domain. Suppose that $a, b \in A$ such that $ab \in I$. Then

$$(a + I)(b + I) = ab + I = I.$$

Because $A \diagup I$ has no zero divisors, $a + I = I$ or $b + I = I$; thus $a \in I$ or $b \in I$.
($\Longleftarrow$) Assume that for all $a, b \in A$ such that $ab \in I$, either $a \in I$ or $b \in I$. Then if $(x + I)$, $(y + I)$ are any two elements of $A \diagup I$ such that $(x + I)(y + I) = I$, then $xy + I = I$, so that $xy \in I$. By the hypothesis, $x \in I$ or $y \in I$, meaning that either $x + I = I$ or $y + I = I$. Thus, $A \diagup I$ has no zero divisors and is thus an integral domain. $\square$

2011S Algebra #7
Find the Galois group of $f(x) = (x^2 - 2)(x^3 - 3)$
(a) over $\mathbb{Q}$.
(b) over $\mathbb{F}_7$, the finite field of order 7.

Proof. **(a)** Over $\mathbb{Q}$, $x^2 - 2$ does not factor (rational roots test, or Eisenstein criterion with $p = 2$), and similarly, $x^3 - 3$ does not have a root and thus does not reduce further. We see that

$$f(x) = \left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)\left(x - \sqrt[3]{3}\right)\left(x - \omega\sqrt[3]{3}\right)\left(x - \omega^2\sqrt[3]{3}\right),$$

where $\omega = e^{2\pi i/3}$ is a third root of unity, and therefore satisfies $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1) = 0$. The splitting field of f is $K = \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}, \omega\right)$, and

$$[K : \mathbb{Q}] = \left[\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}, \omega\right), \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}\right)\right]\left[\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}\right), \mathbb{Q}\left(\sqrt{2}\right)\right]\left[\mathbb{Q}\left(\sqrt{2}\right), \mathbb{Q}\right].$$

We have

$$\begin{aligned}\left[\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}, \omega\right), \mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}\right)\right] &= 2,\\ \left[\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}\right), \mathbb{Q}\left(\sqrt{2}\right)\right] &= 3,\\ \left[\mathbb{Q}\left(\sqrt{2}\right), \mathbb{Q}\right] &= 2,\end{aligned}$$

since the minimial polynomials of ω, $\sqrt[3]{3}$, $\sqrt{2}$, respectively, over $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{3}\right)$, $\mathbb{Q}\left(\sqrt{2}\right)$, and $\mathbb{Q}$, respectively, are $x^2 + x + 1$, $x^3 - 3$, and $x^2 - 2$, respectively. Thus, $[K : \mathbb{Q}] = 12$. The Galois group of K over Q is determined by its permuting action on the roots of these irreducible polynomials. We define the elements α, β, γ of the Galois group $G(K, \mathbb{Q})$ by

$$\alpha\begin{pmatrix}\sqrt{2}\\ \sqrt[3]{3}\\ \omega\end{pmatrix} = \begin{pmatrix}-\sqrt{2}\\ \sqrt[3]{3}\\ \omega\end{pmatrix},\ \beta\begin{pmatrix}\sqrt{2}\\ \sqrt[3]{3}\\ \omega\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\ \omega\sqrt[3]{3}\\ \omega\end{pmatrix},\ \gamma\begin{pmatrix}\sqrt{2}\\ \sqrt[3]{3}\\ \omega\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\ \omega\sqrt[3]{3}\\ \omega^2\end{pmatrix}.$$

Then $\alpha^2 = 1$ (identity). Also, $\beta^2\left(\sqrt[3]{3}\right) = \beta\left(\omega\sqrt[3]{3}\right) = \omega\omega\sqrt[3]{3} = \omega^2\sqrt[3]{3}$, and $\beta^3\left(\sqrt[3]{3}\right) = \omega^2\omega\sqrt[3]{3} = \sqrt[3]{3}$. Also, $\gamma^2\left(\sqrt[3]{3}\right) = \gamma\left(\omega\sqrt[3]{3}\right) = \omega^2\omega\sqrt[3]{3} = \sqrt[3]{3}$ and $\gamma^2(\omega) = \gamma(\omega^2) = \omega^4 = \omega$. Thus, β is a 3-cycle and γ is a 2-cycle when restricted to the Galois group of $x^3 - 3$, and so they generate all of S_3, the symmetric group of permutations of the roots of $x^3 - 3$, and they fix $\sqrt{2}$. The automorphism α generates the Galois group $\mathbb{Z}_2$ of $\mathbb{Q}\left(\sqrt{2}\right)$ and fixes the roots of $x^3 - 3$.Thus, α commutes with the group generated by β and γ, and thus, the whole Galois group is isomorphic to $\mathbb{Z}_2 \times S_3$.

(b) By inspection we see that $(x - 3)(x - 4) = x^2 - 7x + 12 = x^2 - 2$ in $\mathbb{F}_7$. Also, we check by substituting $x = 0, 1, 2, 3, 4, 5, 6$ into $x^3 - 3$ to see that it has no root in $\mathbb{F}_7$ and thus does not factor. So the Galois group is the Galois group of the splitting field of the irreducible polynomial $x^3 - 3$. Thus it has order 3 and is thus the cyclic group $\mathbb{Z}_3$ generated by the Frobenius automorphism $y \mapsto y^3$, and the splitting field is $\mathbb{Z}_{27}$. $\square$

12. 2011F Algebra Exam

2011F Algebra #1

Let K and M be subgroups of the group G. Define the relation $\sim$ on G by $x \sim y$ if there exists $k \in K$, $m \in M$ such that $x = kym$.

(a) Prove that $\sim$ is an equivalence relation.

(b) For K, M finite, prove that the cardinality $|[x]|$ of the equivalence class $[x]$ is $\frac{|K||M|}{|x^{-1}Kx \cap M|}$.

Proof. **(a)** Reflexivity: $x \sim x$ since for the identity e, $x = exe$, for all $x \in G$.

Symmetry: Assume $x \sim y$ for some $x, y \in G$, which means $x = kym$ for some $k \in K$, $m \in M$. Then $y = k^{-1}xm^{-1}$, and since $k^{-1} \in K$ and $m^{-1} \in M$, we have $y \sim x$.

Transitivity: Assume that $x \sim y$ and $y \sim z$, for some $x, y, z \in G$. Then $x = k_1 y m_1$ amd $y = k_2 z m_2$ for some $k_1, k_2 \in K$ and $m_1, m_2 \in M$. Then $x = k_1 (k_2 z m_2) m_1 = (k_1 k_2) z (m_2 m_1)$ by associativity, and since $k_1 k_2 \in K$ and $m_1 m_2 \in M$, $x \sim z$.

(b) Observe that

$$[x] = \{kxm : \text{ for some } k \in K, m \in M\}.$$

Let $\phi_x : K \times M \to [x]$ be defined by $\phi_x(k, m) = kxm$. Note that this is not a homomorphism (in particular, $[x]$ is not a group). Then for a given $k_0 x m_0 \in [x]$,

$$\begin{aligned} \phi_x^{-1}(k_0 x m_0) &= \{(k, m) : kxm = k_0 x m_0\} \\ &= \left\{(k, m) : x^{-1} k_0^{-1} k x = m_0 m^{-1}\right\}. \end{aligned}$$

Since $\left\{k_0^{-1} k : k \in K\right\} = K$ and $\{m_0 m^{-1} : m \in M\} = M$, the cardinality of this set is the same as

$$\left\{(k', m') : x^{-1} k' x = m'\right\}.$$

Since m' determines k' and k' determines m' in the equation above,

$$\begin{aligned} \left|\phi_x^{-1}(k_0 x m_0)\right| &= \left|\left\{(k', m') : x^{-1} k' x = m'\right\}\right| \\ &= \left|x^{-1} K x \cap M\right|. \end{aligned}$$

Thus, $\phi_x : K \times M \to [x]$ is a $|x^{-1}Kx \cap M|$-to-1 map, so that

$$|[x]| = \frac{|K| \cdot |M|}{|x^{-1}Kx \cap M|}.$$

□

2011F Algebra #2

A ring with multiplicative identity is called a *local ring* if it has exactly one maximal ideal. Show that a commutative ring with multiplicative identity is a local ring if and only if its set of non-units is an ideal.

Proof. First, observe that no proper ideal of such a ring R contains a unit: Suppose that an ideal U contains a unit u. Then u^{-1} exists, so $1 = u^{-1}u \in U$, so that $r \cdot 1 = r \in U$ for all $r \in R$, so $U = R$.

($\Longleftarrow$) Next, suppose that the set W of nonunits of R is an ideal. Let V be an ideal such that $V \not\subset W$. Then V contains a unit, so by the argument above, $V = R$. Therefore, W is the unique maximal ideal in R.

($\Longrightarrow$) Suppose that R has only one maximal ideal V. By the argument above, V contains no units. Suppose that there exists $x \in R \setminus V$ such that x is not a unit. If the ideal

$\langle x \rangle$ generated by x is proper, then it must be a subset of V, a contradiction showing that $\langle x \rangle = M$. Thus, $\langle x \rangle = M$, so $1 \in \langle x \rangle$ implies that x is a unit, a contradiction to the assumption. Therefore, V is the set of all non-units in R, and this set is an ideal. □

2011F Algebra #3

Let L be an algebraic extension of the field F. Show that any ring homomorphism $g : L \to L$ fixing F is an automorphism.

Proof. Note that g is automatically $1-1$.

[If $g(\beta) - g(\gamma) = g(\beta - \gamma) = 0$ and $\beta \neq \gamma$, then $(\beta - \gamma)^{-1}$ exists, and $1 = g(1) = g\left((\beta-\gamma)^{-1}(\beta-\gamma)\right) = g\left((\beta-\gamma)^{-1}\right) g(\beta-\gamma) = 0$, a contradiction.]

(onto) For any $\alpha \in L$. Since L is algebraic, there exists a minimal polynomial $p(x)$ with F coefficients such that $p(\alpha) = 0$. Let $p(x) = a_0 + a_1 x + ... + a_n x^n$. Then

$$0 = a_0 + a_1\alpha + ... + a_n\alpha^n.$$

Then

$$\begin{aligned} g(0) &= g(a_0 + a_1\alpha + ... + a_n\alpha^n) \\ &= a_0 + a_1 g(\alpha) + ... + a_n g(\alpha)^n \\ &= p(g(\alpha)). \end{aligned}$$

since g is a ring homomorphism and g fixes F. Now g maps the splitting field K of $p(x)$ over F to itself, and K is a finite-dimensional vector space over L. Since g is a ring homomorphism, it is a linear transformation from K to K (as finite-dimensional vector spaces over F), and thus since it is an isomorphism since it is $1-1$. Thus, there exists $\beta \in K$ such that $g(\beta) = \alpha$. □

13. 2013S Algebra Exam

2013S Algebra Exam #6

Let G be a group of order 56. Prove that G has a nontrivial normal subgroup.

Proof. We have $|G| = 56 = 2^3 \cdot 7$. Let n_2 be the number of Sylow 2-subgroups (of order 8), and let n_7 be the number of Sylow 7-subgroups. By the third Sylow theorem ($n_p \equiv 1 \bmod p$ and $n_p | \frac{|G|}{p^n}$), n_2 is an odd number that divides 7 (and thus is 1 or 7) , and $n_7 \equiv 1 \bmod 7$ and divides 8 (and thus is 1 or 8). If $n_7 = 1$, the single Sylow 7-subgroup is equal to all of its conjugates and thus is normal, and we are done. Suppose instead that $n_7 = 8$. Then there are $6 \cdot 8 = 48$ elements of order 7. The Sylow 2-subgroup contributes 8 additional elements, each of whose orders divides 8. If also G had more than one Sylow 2-subgroup (i.e. 7 of them), then there would be more than $48 + 8 = 56$ elements, a contradiction. Thus, if $n_7 = 8$, $n_2 = 1$, in which case the Sylow 2-subgroup would have to be normal. □

14. 2013F Algebra Exam

2013F Algebra Exam #1

Let G be a group, and for each g in G, define a function $\phi_g : G \longrightarrow G$ by the formula $\phi_g(x) = gxg^{-1}$ for every x in G.

(a) Prove that the set $\mathrm{Inn}(G) = \{\phi_g : g \in G\}$ is a group under function composition.

(b) Let $Z(G)$ denote the center of G. Prove that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.

Proof. **(a)** If 1 is the identity in G, observe that ϕ_1 is the identity, because

$$\phi_1 \circ \phi_g(x) = \phi_1(gxg^{-1}) = 1gxg^{-1}1 = gxg^{-1} = \phi_g(x)$$

for every $x \in G$. Similarly $\phi_g \circ \phi_1 = \phi_g$. Thus ϕ_1 is the identity in $\mathrm{Inn}(G)$.

Next, for all $g \in G$, observe that for all $x \in G$,

$$\phi_g \circ \phi_{g^{-1}}(x) = \phi_g(g^{-1}xg) = g^{-1}gxg^{-1}g = 1x1 = \phi_1(x),$$

so inverses exist.

Also, for $g_1, g_2 \in G$,

$$\phi_{g_1} \circ \phi_{g_2}(x) = \phi_{g_1}(g_2xg_2^{-1}) = g_1g_2xg_2^{-1}g_1^{-1} = (g_1g_2)x(g_1g_2)^{-1} = \phi_{g_1g_2}(x)$$

Thus, the set $\mathrm{Inn}(G)$ is closed under multiplication.

Finally, the associative property also holds: for any $g_1, g_2, g_3 \in G$, we have by the multiplication formula above and the associative property in G,

$$(\phi_{g_1} \circ \phi_{g_2}) \circ \phi_{g_3} = \phi_{g_1g_2} \circ \phi_{g_3} = \phi_{(g_1g_2)g_3} = \phi_{g_1(g_2g_3)} = \phi_{g_1} \circ \phi_{g_2g_3} = \phi_{g_1} \circ (\phi_{g_2} \circ \phi_{g_3}).$$

(b) Define the surjective homomorphism $\psi : G \to \mathrm{Inn}(G)$ defined by $\psi(g) = \phi_g$. We showed above that $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1g_2}$ for all $g_1, g_2 \in G$, proving that it is indeed a homomorphism, and it is clearly surjective. Observe that

$$\begin{aligned} \ker\psi &= \{z \in G : \psi(z)(g) = g \text{ for all } g \in G\} \\ &= \{z \in G : zgz^{-1} = g \text{ for all } g \in G\} \\ &= \{z \in G : zg = gz \text{ for all } g \in G\} = Z(G). \end{aligned}$$

Then, by the first isomorphism theorem for groups,

$$\mathrm{Inn}(G) \cong G/Z(G).$$

□

2013F Algebra Exam #2

Let H be a finite subgroup of G. Prove that the double coset

$$HxH \stackrel{\text{def}}{=} \{h_1xh_2 : h_1, h_2 \in H\}$$

has cardinality $|H|$ for all x if and only if H is a normal subgroup of G.

Proof. Suppose H is normal and $x \in G$. This implies that $xH = Hx$. Take any element $h_1xh_2 \in HxH$. Then there exists an $h_3 \in H$ such that $h_1x = xh_3$, so that $h_1xh_2 = xh_3h_2 \in xH$. Thus, $HxH \subseteq xH$. Also, for any $xh_4 \in xH$, $xh_4 = exh_4 \in HxH$, where e is the identity. Thus $xH \subseteq HxH$. Thus, $xH = HxH$, so $|xH| = |HxH|$. Also, $|xH| = |H|$, by the following argument. Define $\phi : H \to xH$ by $\phi(h) = xh$. Note that ϕ is clearly onto. Also, if $\phi(h_5) = \phi(h_6)$ for some $h_5, h_6 \in H$, we have $xh_5 = xh_6$, which implies $h_5 = x^{-1}xh_5 = x^{-1}xh_6 = h_6$, so ϕ is one-to-one and thus a bijection. Therefore, $|xH| = |H|$, and we are done with the first part.

Next, suppose that $|HxH| = |H|$ for all $x \in G$. Again, for any $x \in G$, $|H| = |xH| = |exH|$. Since $exH \subseteq HxH$, $|H| = |exH| = |HxH|$, so since $exH \subseteq HxH$ and HxH is finite, we must have $xH = HxH$. Similarly, $Hx = HxH$. Thus, $xH = Hx$ for every $x \in G$, and therefore H is normal. □

2013F Algebra Exam #3
Find a product of cyclic groups that is isomorphic to the factor group

$$(\mathbb{Z}_4 \times \mathbb{Z}_6) / \langle (2, 3) \rangle.$$

Proof. Note that $H = \langle (2, 3) \rangle = \{(0, 0), (2, 3)\}$. Then $|\mathbb{Z}_4 \times \mathbb{Z}_6 / \langle (2, 3) \rangle| = 24/2 = 12$, and the group is abelian. Observe that $(1, 1) + H$ has order 12 in $(\mathbb{Z}_4 \times \mathbb{Z}_6) / H$. Thus, $(\mathbb{Z}_4 \times \mathbb{Z}_6) / H \cong \mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$. □

2013F Algebra Exam #4
For an $n \times n$ matrix A and eigenvalue λ_0, prove the dimension of the eigenspace for λ_0 is at most its multiplicity as a root of the characteristic polynomial.

Proof. Recall that the eigenspace corresponding to λ_0 is

$$E_{\lambda_0} = \{v \in \mathbb{C}^n : (A - \lambda_0 I)v = 0\}.$$

Let $\{a_1, ..., a_\ell\}$ be a basis of E_{λ_0} . Choose the $b_1, ..., b_{n-\ell}$ so that $\{a_1, ..., a_\ell, b_1, ..., b_{n-\ell}\}$ is a basis of $\mathbb{C}^n$. Letting $B = C^{-1}AC$, where C is the matrix with these basis vectors as column vectors. Then B has the same characteristic polynomial as A, since $\det(A - \lambda I) = \det(C^{-1}(A - \lambda I)C) = \det(C^{-1}AC - C^{-1}\lambda IC) = \det(C^{-1}AC - C^{-1}\lambda IC) = \det(C^{-1}AC - \lambda I) = \det(B - \lambda I)$. Then we have $Aa_j = \lambda_0 a_j$ for $1 \leq j \leq \ell$. So B has the form

$$B = \begin{pmatrix} \lambda_0 I & D \\ 0 & G \end{pmatrix}$$

□

2013F Algebra Exam #5
Let I be a proper nontrivial prime ideal in a principal ideal domain D. Prove that I is a maximal ideal in D.

Proof. Since I is an ideal in a PID, I is principal, so $I =< a >$ for some nonzero a. Suppose that I is not maximal, i.e. that there exists another ideal B such that $I \subsetneq B \subsetneq D$. Then $B =< b >$, and $a \in B$ and $b \notin I$. Then $a = bd$ for some $d \in D$. But a is in a prime ideal, so either b or d is in I. So d is in I. Thus $d = ar$ for some $r \in D$. Then $a = bar$, so $a = abr$, or $a(1 - br) = 0$. Since D is an integral domain, and since $a \neq 0$, $br = 1$, so $< b >= B = D$. Contradiction. □

2013F Algebra Exam #6
Let V be an inner product space. Let W be a subspace of V and let $W^\perp$ denote its orthogonal complement.

(a) For V finite-dimensional, prove $\left(W^\perp\right)^\perp = W$.

(b) For $V = \mathbb{R}[x]$, $\langle p, q \rangle = \displaystyle\int_{-1}^{1} p(x)\, q(x)\, dx$, W the subspace of V with basis $\{x^2, x^4, x^6, \ldots\}$, find bases for $W^\perp$ and $\left(W^\perp\right)^\perp$, and show $W \subsetneq \left(W^\perp\right)^\perp$.

Proof. **(a)**

With notation as above, let $\{b_1, ..., b_m\}$ be an orthonormal basis for W, and complete this basis to an orthonormal basis of V: $\{b_1, ..., b_m, e_1, ..., e_r\}$; we can always do this with the

Gramm-Schmidt orthonormalization procedure. Then $W = \text{span}\{b_1, ..., b_m\}$. The element

$$v = \sum_{j=1}^{m} w_j b_j + \sum_{i=1}^{r} v_i e_i$$

is an element of $W^\perp$ if and only if $0 = \langle b_a, v \rangle$ for every a such that $1 \le a \le m$. This means

$$\begin{aligned} 0 &= \langle b_a, v \rangle = \left\langle b_a, \sum_{j=1}^{m} w_j b_j + \sum_{i=1}^{r} v_i e_i \right\rangle \\ &= w_a \end{aligned}$$

for $1 \le a \le m$, so that $W^\perp = \text{span}\{e_1, ..., e_r\}$.

Employing the same reasoning with the subspace $W^\perp$, we see that $\left(W^\perp\right)^\perp = \text{span}\{b_1, ..., b_m\} = W$.

(b)

For any

$$p(x) = \sum_{j=0}^{n} a_j x^j \in W^\perp,$$

for all $m \ge 1$, we have

$$\begin{aligned} 0 &= \int_{-1}^{1} p(x)\, x^{2m} dx \\ &= \sum_{j=0}^{n} a_j \int_{-1}^{1} x^{2m+j} dx = \sum_{j=0}^{n} a_j \frac{1}{2m+j+1}\left(1 - (-1)^{2m+j+1}\right) \\ &= \sum_{j \text{ even}} a_j \frac{2}{2m+j+1}, \end{aligned}$$

which implies only that $a_j = 0$ for j even. Thus, $\{x, x^3, x^5, ...\}$ is a basis for $W^\perp$.
Next, if p as above is in $\left(W^\perp\right)^\perp$, observe that for $m \ge 1$

$$\begin{aligned} 0 &= \int_{-1}^{1} p(x)\, x^{2m-1} dx \\ &= \sum_{j=0}^{n} a_j \int_{-1}^{1} x^{2m+j-1} dx = \sum_{j=0}^{n} a_j \frac{1}{2m+j}\left(1 - (-1)^{2m+j}\right) \\ &= \sum_{j \text{ odd}} a_j \frac{2}{2m+j}, \end{aligned}$$

which implies only that $a_j = 0$ for j odd. Thus, $\{1, x^2, x^4, x^6, ...\}$ is a basis for $\left(W^\perp\right)^\perp$.
Since $1 \in \left(W^\perp\right)^\perp$ and $1 \notin W$, we see that $W \subsetneqq \left(W^\perp\right)^\perp$. □

2013F Algebra Exam #7

Suppose K is a finite extension field of E and that E is a finite extension field of F. Prove that K is a finite extension field of F, and that $[K : F] = [K : E][E : F]$.

2013F Algebra Exam #8

Compute the Galois group of the splitting field of $x^4 + x + 1$ over $\mathbb{F}_2$ and over $\mathbb{F}_3$, the finite fields with 2 and 3 elements, respectively.

Proof. Consider $\mathbb{F}_2$. We first show that x^4+x+1 is irreducible over $\mathbb{F}_2$. We check that $x=0$ and $x=1$ are not zeros, so there are no linear factors. Next, if it factors into quadratics, then

$$\left(x^2+ax+1\right)\left(x^2+bx+1\right)=x^4+x+1$$

for some $a,b\in\{0,1\}$. This implies $a+b=0, a+b=1$, which is a contradiction.
It follows that the splitting field is $\mathbb{F}_{16}$ and that the Galois group is cyclic of order 4 (with generator $y\mapsto y^2$).

Consider $\mathbb{F}_3$. Then $x=1$ is a root, and we have

$$x^4+x+1=(x-1)\left(x^3+x^2+x+2\right),$$

but we can see that x^3+x^2+x+2 has no linear factors by plugging in $0,1,2$ and is thus irreducible. It follows that the splitting field is $\mathbb{F}_{27}$ and that the Galois group is cyclic of order 3 (with generator $y\mapsto y^3$). $\square$

Remark 1. *The Theorems we are using in the above are these:*

- *For p prime, the only extension fields of $\mathbb{F}_{p^k}$ are $\mathbb{F}_{p^n}$ for $n\geq k$. By definition, $\mathbb{F}_{p^k}$ is the splitting field of $x^{p^k}-x$ over $\mathbb{F}_p$ (note the polynomial is reducible).*
- *The Galois group of $\mathbb{F}_{p^{kn}}$ over the base field $\mathbb{F}_{p^k}$ is cyclic of order n, and it is generated by the Frobenius automorphism $x\mapsto x^{p^k}$.*
- *If $f(x)\in\mathbb{F}_p[x]$ is an irreducible polynomial of degree m, then the Galois group of $f(x)$ over $\mathbb{F}_p[x]$ is cyclic of degree m (generated by $y\mapsto y^p$), and $\mathbb{F}_{p^m}$ is the splitting field.*
  (The reason is that we know that the splitting field is $\mathbb{F}_{p^k}$ for some $k>1$, and we know the Galois group is cyclic and generated by the Frobenius automorphism $y\mapsto y^p$. This is a normal extension, so if α is a root of $f(x)$, then all the roots of f are in $\mathbb{F}_p(\alpha)$, so $\mathbb{F}_p(\alpha)$ is the splitting field. But then the degree of the splitting field is m, so the order of the Galois group is m; thus $k=m$.)
- *Same as the above is true if the prime p is replaced by p^j for some $j\geq 1$.*

15. 2014S Algebra Exam

2014S Algebra Exam #3

Every finite integral domain D is a field.

Proof. Let $D=\{0,a_0=1,a_1,...,a_n\}$. Let $a\in D$ such that $a\neq 0$. We need to show there exists $b\in D$ such that $ab=ba=1$. Consider the elements $aa_0,aa_1,...,aa_n$. Suppose $aa_i=aa_j$ for some $i\neq j$. Then $a(a_i-a_j)=0$. We assumed $a\neq 0$ and have that $a_i-a_j\neq 0$. But D has no zero-divisors, so this is a contradiction. Thus, multiplication by a is a 1-1 map of the finite set $D-\{0\}$ to itself and thus is a bijection. Therefore, there exists $b\in D$ such that $ab=a_0=1$. $\square$

2014S Algebra Exam #4

For matrices A and B where AB is defined, prove that $rank\,(AB)\leq\min\{rank\,(A)\,,rank\,(B)\}$.

Proof. Suppose A is an $m\times n$ matrix and B is an $n\times p$ matrix. Let $T_A:\mathbb{R}^n\to\mathbb{R}^m$ be the linear transformation defined by $T_A(x)=Ax$, and similarly define $T_B:\mathbb{R}^p\to\mathbb{R}^n$. Then $rank\,(AB)=\dim\left(T_A\circ T_B\left(\mathbb{R}^p\right)\right)\leq\dim\left(T_A\left(\mathbb{R}^n\right)\right)=rank\,(A)$. Next, we have the following lemma:

Lemma: For any linear transformation $T : V \to W$ of finite dimensional vector spaces, $\text{rank}(T) = \dim(T(V)) \leq \dim(V)$.

Proof of Lemma: Given a basis $b_1, ..., b_n$ of V, $\{T(b_1), ..., T(b_n)\}$ spans $T(V)$, so any basis of $T(V)$ has $\leq n$ elements. □

Thus, applying the lemma to $T_A : T_B(\mathbb{R}^p) \to \mathbb{R}^m$, $rank(AB) = \dim(T_A(T_B(\mathbb{R}^p))) \leq \dim(T_B(\mathbb{R}^p)) = rank(B)$. The result follows. □

2014S Algebra Exam #5

Prove that every finite multiplicative subgroup of a field is cyclic.

Proof. Let G be a finite multiplicative subgroup of a field $\mathbb{F}$; G is necessarily abelian. Let $m = \max\{|g| : g \in G\}$. (Here $|g|$ means the order of g.) Then $m|\ |G|$, so that $m \leq |G|$.

Claim: $g^m = 1$ for all $g \in G$.

Proof of Claim: By the Fundamental Theorem of Finitely Generated Abelian Groups, G is isomorphic to $\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times ... \times \mathbb{Z}_{p_k^{r_k}}$, where $p_1, ..., p_k$ are not necessarily distinct primes, and $r_1, ..., r_k$ are positive integers. The maximum order of an element of this group is the least common multiple of the integers $p_1^{r_1}, ..., p_k^{r_k}$; one can explicitly construct an element of that order, and further one can show that any element of the group has an order that is a factor of this maximum order. □

By the Claim, every $g \in G$ is a solution to the equation $x^m - 1$, but $x^m - 1$ has at most m solutions in $\mathbb{F}$. Therefore, $|G| \leq m$. Thus $|G| = m$, and there exists $h \in G$ such that $|h| = m$. Hence, G is cyclic. □

2014S Algebra Exam #8

Let G be a finite group, let $R^\times$ be the multiplicative group of units in a ring R, and let $\phi : G \to R^\times$ be a nontrivial homomorphism. Prove that $\sum_{g\in G} \phi(g)$ is either 0 or a zero divisor in R.

Proof. Consider $\left(\sum_{g\in G} \phi(g)\right) \phi(h)$, where $h \in G$ and $\phi(h) \neq 1$. Since ϕ is a ring homomorphism,

$$\begin{aligned}\left(\sum_{g\in G} \phi(g)\right) \phi(h) &= \sum_{g\in G} \phi(g)\phi(h) \\ &= \sum_{g\in G} \phi(gh) \\ &= \sum_{g'\in G} \phi(g'),\end{aligned}$$

since right multiplication $r_h : G \to G$ by h permutes the elements of G. (If $r_h(a) = r_h(b)$ for any $a, b \in G$, then $ah = bh$, so $a = b$ by right-multiplying by h^{-1}. So r_h is an injection and thus a bijection.) Thus,

$$\begin{aligned}0 &= \left(\sum_{g\in G} \phi(g)\right) \phi(h) - \left(\sum_{g\in G} \phi(g)\right) \\ &= \left(\sum_{g\in G} \phi(g)\right) (\phi(h) - 1).\end{aligned}$$

Thus, $\sum_{g\in G} \phi(g)$ is a zero divisor. $\square$