Topology Preliminary Exam

August 2016

INSTRUCTIONS: Work all eight problems. Justify your work. Each problem is worth 10 points. You have four hours to complete the exam.

1. Let $\mathcal{D}$ be the collection of subsets of $\mathbb{Z}_+$ satisfying the following property: a subset U is in $\mathcal{D}$ if and only if $n \in U$ implies that all the divisors of n are in U.

   (a) Let $\mathcal{T}_1 = \mathcal{D} \cup \emptyset$. Prove that $\mathcal{T}_1$ is a topology on $\mathbb{Z}_+$.

   (b) Let $\mathcal{T}_2$ be the topology $\mathbb{Z}_+$ inherits from $\mathbb{R}$ as a subspace. Does $\mathcal{T}_1 = \mathcal{T}_2$? If not, is one of these topologies finer than the other one?

2. Prove that every compact Hausdorff space is regular.

3. Call a subset U of a topological space Y *regularly open* if U is equal to the interior of its closure in Y.

   (a) Show that the intersection of two regularly open sets is regularly open.

   (b) Give an example to show that the union of two regularly open sets is not necessarily regularly open.

4. (a) Prove that if a topological space is path connected, then it is connected.

   (b) Prove or disprove: if a topological space is connected, then it is path connected.

5. Let x_0 be a point in a topological space X.

   (a) Define the fundamental group $\pi_1(X, x_0)$.

   (b) Prove that the group operation on $\pi_1(X, x_0)$ is well-defined.

6. Let X be the topological space obtained from $\mathbb{R}^3$ by removing the x-axis and the y-axis. Compute the homology groups of X.

7. Fix a point x on a compact surface S. Compute the relative homology groups $H_*(S, S - \{x\})$.

8. Let K be the Klein bottle, and let T be the two-dimensional torus.

   (a) Prove or disprove: there is a covering map from K to T.

   (b) Prove or disprove: there is a covering map from T to K.