Preliminary Topology Exam
August 22, 2014

1. (a) Let X be a compact topological space, let Y be a Hausdorff space, and suppose that $f : X \longrightarrow Y$ is a continuous bijection. Prove that f is a homeomorphism.

   (b) Show by example that if we do not require X to be compact in (a), then f is not necessarily a homeomorphism.

2. Let $C([0,\infty))$ denote the space of continuous real–valued functions on the interval $[0,\infty)$. Consider the topology generated by sets of the form

$$U(f,\delta) = \left\{ g \in C([0,\infty)) : \sum_{i=0}^{\infty} |f(i) - g(i)| < \delta \right\}.$$

   Prove or disprove that $C([0,\infty))$ is Hausdorff with this topology.

3. Let A be connected in a topological space X. Prove that the closure of A is connected.

4. Let $\mathbb{R}$ denote the real line *as a set*. Let $\mathcal{B}$ be the collection of subsets of $\mathbb{R}$ of the following two forms

   - sets of the form $(-b,-a) \cup (a,b)$, where $0 < a < b$ and (a,b) denotes the usual intervals in $\mathbb{R}$;
   - sets of the form $(-\infty,-c) \cup (-a,a) \cup (c,\infty)$, where $0 < a < c$.

   (a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$.

   (b) For $\mathbb{R}$ with this topology, decide what are the point(s) of convergence, if any, for the following sequences:

      i. $x_n = 1 - \frac{1}{n}$

      ii. $x_n = n$

5. Compute the homology of the Klein bottle.

6. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^2$. Compute $\pi_1(X)$, describe the universal cover $\widetilde{X}$, and describe the deck transformations of $\pi_1(X)$ on $\widetilde{X}$.

7. Let A be a strong deformation retract of X, and let $a_0 \in A$. Show that the inclusion $i : (A, a_0) \to (X, a_0)$ induces an isomorphism $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$.

8. Let S be a surface (i.e. a compact connected 2-dimensional manifold without boundary), and let S' be S with an open disk removed. Provide examples of surfaces showing that sometimes $H_1(S) \cong H_1(S')$ is true and sometimes it is false. Support your claims with proofs.