PRELIMINARY TOPOLOGY EXAM
AUGUST 16, 2012

(1) A space is *countably compact* if every countable cover has a finite subcover. Show that if X is countably compact, then every continuous function from X to $\mathbb{R}$ is bounded.

(2) Let Y, Z, and A be subsets of a topological space X. Consider the following two statements:
  (a) $A \cap (Y \cup Z)$ is open in the subspace topology of $Y \cup Z$
  (b) $A \cap Y$ is open in the subspace topology of Y and $A \cap Z$ is open in the subspace topology of Z.

  Is it true that (2a) implies (2b)? Is it true that (2b) implies (2a)? Prove or give counterexamples.

(3) Let X be Hausdorff and let $A \subset X$ be a compact set. Show that A is closed.

(4) Let $\mathbb{R}$ be the real line in the standard topology. Prove or disprove that $\mathbb{R}/(0,1)$ with the quotient topology is homeomorphic to $\mathbb{R}$.

(5) Find a space X whose fundamental group is $\mathbb{Z}_2 * \mathbb{Z}$. Describe the universal cover $\tilde{X}$ of X.

(6) Suppose X is a connected cell complex such that $\pi_1(X)$ is presented by $\langle x, y, z \mid xyx, yz^2y^{-1} \rangle$. What is $H_1(X)$?

(7) Suppose A, B, U, V are subspaces of a space X such that $X = A \cup B$ and $X = U \cup V$. Suppose also $A \subset U$ and $B \subset V$ and that the inclusions $A \hookrightarrow U$ and $B \hookrightarrow V$ induce isomorphisms $H_*(A) \cong H_*(U)$, $H_*(B) \cong H_*(V)$. Show that $H_*(A \cap B) \cong H_*(U \cap V)$.

(8) Let X be a path-connected cell complex, and let $p : X \to X$ be a covering map such that p is not injective. Prove that $\pi_1(X)$ is infinite.