Topology Preliminary Exam
January, 2011
Directions: All problems require justification. The time limit is 4 hours.

1. *Determine if the closed interval $[0,1]$ is homeomorphic to the open interval $(0,1)$, using the standard topologies.*

2. *Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) : x \in X\} \subset X \times X$ is closed when $X \times X$ is given the product topology.*

3.

- **a.** *Let $f : X \to Y$ be a continuous function between two topological spaces. Suppose that the sequence (x_n) converges to x. Prove that $(f(x_n))$ converges to $(f(x))$.*
- **b.** *Let $g : X \to Y$ be a function between two topological spaces. Suppose that whenever (x_n) converges to x, one has that $(g(x_n))$ converges to $g(x)$.*
  - **i.** *Give an example to show that it is possible that g is not continuous.*
  - **ii.** *Suppose further that X and Y are metric spaces. Prove that g is continuous.*

4. *Suppose X and Y are sequentially compact topological spaces. Show that $X \times Y$ is sequentially compact.*

5. *Show that $X \times Y$ is path connected if and only if X and Y are each path connected.*

6. *Let a and b be in the same path component of a topological space X. Prove that the fundamental groups $\pi_1(X,a)$ and $\pi_1(X,b)$ are isomorphic.*

7. *Suppose that a continuous map $F : S^3 \times S^3 \to \mathbf{R}P^3$ is not surjective. Prove that it is homotopic to a constant function (a map to a point).*

8. *Let C be the cylinder $[0,1] \times S^1$. Let $X = C\#C$, the connected sum of two copies of C.*

- **a.** *Compute all nonzero homology groups of X.*
- **b.** *Compute the fundamental group of X.*
- **c.** *Compute the Euler characteristic of X.*