REAL ANALYSIS PRELIMINARY EXAMINATION

JANUARY 2024

Work all 8 problems, which are worth 10 points each. Justification is required for all statements.

(1) Estimate the numerical value of the double integral

$$\int_0^1 \int_0^1 e^{x^2y^2}\,dxdy$$

as a sum of fractions accurate to within $1/100$. The final answer does not have to be simplified.

(2) Let S be the part of the paraboloid $z = x^2 + y^2$ where $z \leq 1$. Evaluate the surface integral over S:

$$\int\int_S (3x + 4y + 5z)\,dA.$$

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be the function of period 2π such that $f(x) = x^3$ for $-\pi \leq x < \pi$.

(a) Prove that the Fourier series for f has the form

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

and write an integral formula for b_n (do not evaluate it).

(b) Explain why the Fourier series converges pointwise for all x and find the numerical value of the Fourier series at $x = 4$.

(c) Evaluate

$$\sum_{n=1}^{\infty} b_n^2.$$

(4) Suppose that $f_n \in C([a,b])$ is a sequence of functions converging uniformly to a function f.

(a) Show that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

(b) Give an example to show that the pointwise convergence of continuous functions f_n to a continuous function f does not imply convergence of the corresponding integrals.

(5) Let f be differentiable on $(0,\infty)$. Assume that there is a sequence x_n with $x_n \to \infty$ such that $f(x_n) \to 0$. Prove that there exists a sequence y_n with $y_n \to \infty$ such that $f'(y_n) \to 0$.

(6) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x^{4/3}\sin(y/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Find all points where f is differentiable.

(7) Let $E \subset \mathbb{R}$. Show that if every continuous function $f : E \to \mathbb{R}$ attains its maximum on E, then E is compact.

(8) (a) Prove that $\cos x = x$ has a unique real solution r.

Given $x_0 \in \mathbb{R}$, consider the sequence defined by $x_{n+1} = \cos x_n$, $n = 0, 1, 2, \ldots$.

(b) Prove that $(x_n)_{n\geq 0}$ converges to r.